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Institutiones Calculi Differentialis Cum Eius Vsu In Analysi Finitorum Ac Doctrina Serierum

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INSTITUTIONES CALCULI DIFFERENTIALIS

CUM EIUS VSU

IN ANALYSI FINITORUM

AC

DOCTRINA SERIERUM

AUCTORE

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EIDGEN. POLYTECHN. SCHULE
BIBLIOTHEK

PRAEFATIO.

Quid ſit Calculus Differentialis, atque in genere Analyſis Infinitorum? iis qui nulla adhuc eius cognitione ſunt imbuti, vix explicari poteſt: neque hic, vti in aliis diſciplinis fieri ſolet, exordium tractationis a definitione commode ſumere licet. Non quod huius calculi nulla plane detur definitio; ſed quoniam ad eam intelligendam eiusmodi opus eſt notionibus, non ſolum in vita communi, verum etiam in ipſa Analyſi finitorum minus vſitatis, quae demum in Calculi Differentialis pertractatione euolui atque explicari ſolent: quo fit, vt eius definitio non ante percipi queat, quam eius principia iam ſatis dilucide fuerint perſpecta. Primum igitur hic calculus circa quantitates variabiles verſatur: etſi enim omnis quantitas ſua natura in infinitum augeri & diminui po-

 teſt;

teſt; tamen dum calculus ad certum quoddam inſtitutum dirigitur, aliae quantitates conſtanter eandem magnitudinem retinere concipiuntur, aliae vero per omnes gradus auctionis ac diminutionis variari: ad quam diſtinctionem notandam illae quantitates conſtantes, hae vero variabiles vocari ſolent; ita vt hoc diſcrimen non tam in rei natura, quam in quaeſtionis, ad quam calculus refertur, indole ſit poſitum. Quoniam haec differentia inter quantitates conſtantes & variabiles exemplo maxime illuſtrabitur, conſideremus iactum globi ex tormento bellico vi pulveris pyrii exploſi; ſiquidem hoc exemplum ad rem dilucidandam imprimis idoneum videtur. Plures igitur hic occurrunt quantitates, quarum ratio in iſta inueſtigatione eſt habenda: primo ſcilicet quantitas pulueris pyrii; tum eleuatio tormenti ſupra horizontem; tertio longitudo iactus ſuper plano horizontali; quarto tempus, quo globus exploſus in aere verſatur: ac niſi xperimenta eodem tormento inſtituantur, inſuper eius longitudo cum pondere globi in computum trahi deberet. Verum hic a varietate tormenti & globi animum remoueamus, ne in quaeſtiones nimium implicatas incidamus. Quodſi ergo ſeruata perpetuo eadem

pul-

pulueris pyrii quantitate, eleuatio tormenti continuo immutetur, iactusque longitudo cum tempore tranſitus globi per aerem requiratur; in hac quaeſtione copia pulueris ſeu vis impulſus erit quantitas conſtans, eleuatio autem tormenti cum longitudine iactus eiusque duratione ad quantitates variabiles referri debebunt; ſiquidem pro omnibus eleuationis gradibus has res definire velimus, vt inde innoteſcat, quantae mutationes in longitudine ac duratione iactus ab omnibus eleuationis variationibus oriantur. Alia autem erit quaeſtio, ſi ſeruata eadem tormenti eleuatione, quantitas pulueris pyrii continuo mutetur, & mutationes, quae inde in iactum redundant, definiri debeant: hic enim eleuatio tormenti erit quantitas conſtans, contra vero quantitas pulueris pyrii, & longitudo ac duratio iactus quantitates variabiles. Sic igitur patet, quomodo mutato quaeſtionis ſtatu eadem quantitas modo inter conſtantes, modo inter variabiles numerari queat: ſimul autem hinc intelligitur, ad quod in hoc negotio maxime eſt attendendum, quomodo quantitates variabiles aliae ab aliis ita pendeant, vt mutata vna reliquae neceſſario immutationes recipiant. Priori ſcilicet caſu, quo quantitas pulueris

pyrii eadem manebat, mutata tormenti eleuatione etiam longitudo & duratio iactus mutantur; ſuntque ergo longitudo & duratio iactus quantitates variabiles pendentes ab eleuatione tormenti, hacque mutata ſimul certas quasdam mutationes patientes: poſteriori vero caſu pendent a quantitate pulueris pyrii, cuius mutatio in illis certas mutationes producere debet. Quae autem quantitates hoc modo ab aliis pendent, vt his mutatis etiam ipſae mutationes ſubeant, eae harum functiones appellari ſolent; quae denominatio latiſſime patet, atque omnes modos, quibus vna quantitas per alias determinari poteſt, in ſe complectitur. Si igitur x *denotet quantitatem variabilem, omnes quantitates, quae vtcunque ab* x *pendent, ſeu per eam determinantur, eius functiones vocantur; cuiusmodi ſunt quadratum eius* xx, *aliaeue potentiae quaecunque, nec non quantitates ex his vtcunque compoſitae; quin etiam transcendentes, & in genere quaecunque ita ab* x *pendent, vt aucta vel diminuta* x *ipſae mutationes recipiant. Hinc iam naſcitur quaeſtio, qua quaeritur, ſi quantitas* x *data quantitate ſiue augeatur ſiue diminuatur, quantum inde quaeuis eius functiones immutentur,*

tur, *ſeu quantum incrementum decrementumue accipiant. Caſibus quidem ſimplicioribus haec quaeſtio facile reſoluitur: ſi enim quantitas* x *augeatur quantitate* ω, *eius quadratum* xx *hinc incrementum capiet* 2xω + ωω; *ſicque incrementum ipſius* x *ſe habebit ad incrementum ipſius* xx, *vt* ω *ad* 2xω + ωω, *hoc eſt*, *vt* 1 *ad* 2x + ω; *ſimilique modo in aliis caſibus ratio incrementi ipſius* x *ad incrementum*, *vel decrementum*, *quod quaeuis eius functio inde adipiſcitur*, *conſiderari ſolet. Eſt vero inueſtigatio rationis huiusmodi incrementorum ipſa non ſolum maximi momenti*, *ſed ei etiam vniuerſa Analyſis infinitorum innititur. Quod quo clarius appareat*, *ſumamus exemplum ſuperius quadrati* xx, *cuius incrementum* 2xω + ωω, *quod capit*, *dum ipſa quantitas* x *incremento* ω *augetur*, *vidimus ad hoc rationem tenere*, *vt* 2x + ω *ad* 1; *vnde perſpicuum eſt*, *quo minus ſumatur incrementum* ω, *eo propius iſtam rationem accedere ad rationem* 2x *ad* 1; *neque tamen ante prorſus in hanc rationem abit*, *quam incrementum illud* ω *plane euaneſcat. Hinc intelligimus*, *ſi quantitatis variabilis* x *incrementum* ω *in nihilum abeat*, *tum etiam quadrati eius* xx

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incrementum inde oriundum quidem euaneſcere, verumtamen ad id rationem tenere vt 2x *ad* 1; *& quod hic de quadrato eſt dictum, de omnibus aliis functionibus ipſius* x *eſt intelligendum; quippe quarum incrementa euaneſcentia, quae capiunt, dum ipſa quantitas* x *incrementum euaneſcens ſumit, ad hoc ipſum certam & aſſignabilem rationem tenebunt. Atque hoc modo ſumus deducti ad definitionem* Calculi Differentialis, *qui eſt* methodus determinandi rationem incrementorum euaneſcentium, quae functiones quaecunque accipiunt, dum quantitati variabili, cuius ſunt functiones, incrementum euaneſcens tribuitur: *hacque definitione veram indolem calculi differentialis contineri, atque adeo exhauriri, iis, qui in hoc genere non ſunt hoſpites, facile erit perſpicuum. Calculus igitur differentialis non tam in his ipſis incrementis euaneſcentibus, quippe quae ſunt nulla, exquirendis, quam in eorum ratione ac proportione mutua ſcrutanda occupatur: & cum hae rationes finitis quantitatibus exprimantur, etiam hic calculus circa quantitates finitas verſari eſt cenſendus. Quamuis enim praecepta, vti vulgo tradi ſolent, ad iſta incrementa euaneſcentia*

defi-

definienda videantur accommodata; nunquam tamen ex iis abſolute ſpectatis, ſed potius ſemper ex eorum ratione concluſiones deducuntur. Simili vero modo calculi integralis ratio eſt comparata, qui conuenientiſſime ita definitur, vt dicatur eſſe methodus ex cognita ratione incrementorum euaneſcentium ipſas illas functiones, quarum ſunt incrementa, inueniendi. *Quo autem facilius hae rationes colligi, atque in calculo repraeſentari poſſint, haec ipſa incrementa euaneſcentia, etiamſi ſint nulla, tamen certis ſignis denotari ſolent; quibus adhibitis nihil obſtat, quo minus iis certa nomina imponantur. Vocantur itaque differentialia, quae cum quantitate deſtituantur, infinite parua quoque dicuntur; quae igitur ſua natura ita ſunt interpretanda, vt omnino nulla ſeu nihilo aequalia reputentur. Ita ſi quantitati* x *incrementum tribuatur* ω, *vt abeat in* $x+\omega$, *eius quadratum* xx *abibit in* $xx+2x\omega+\omega\omega$, *ideoque incrementum capit* $2x\omega+\omega\omega$; *quare incrementum ipſius* x, *quod eſt* ω, *ſe habebit ad incrementum quadrati, quod eſt* $2x\omega+\omega\omega$, *vti* 1 *ad* $2x+\omega$; *quae ratio abit in* 1 *ad* $2x$, *tum demum, cum* ω *euaneſcit. Fiat igitur*

$\omega = 0$, *& ratio iſtorum incrementorum euaneſcentium, quae ſola in calculo differentiali ſpectatur, vtique eſt vt* 1 *ad* 2 x*; neque viciſſim haec ratio veritati eſſet conſentanea, niſi reuera illud incrementum* ω *euaneſceret, penitusque nihilo fieret aequale. Quodſi ergo hoc nihilum per* ω *indicatum referat incrementum quantitatis* x, *quia hoc ſe habet ad incrementum quadrati* xx *vt* 1 *ad* 2 x, *erit quadrati* xx *incrementum* $= 2x\omega$, *ideoque etiam nihilo aequale; vnde ſimul conſtat annihilationem horum incrementorum non obſtare, quominus eorum ratio, quae eſt vt* 1 *ad* 2 x *ſit determinata. Quod nihilum iam hic littera* ω *exhibetur, id in calculo differentiali, quia vt incrementum quantitatis* x *ſpectatur, ſigno* dx *repraeſentari, eiusque differentiale vocari ſolet; poſitoque* dx *loco* ω, *ipſius* xx *differentiale erit* $2x\,dx$. *Simili modo oſtenditur fore cubi* x^3 *differentiale* $= 3xx\,dx$, *& in genere cuiusque dignitatis* x^n *differentiale fore* $= nx^{n-1}dx$. *Quaecunque autem aliae functiones ipſius* x *proponantur, in calculo differentiali regulae traduntur eorum differentialia inueniendi: verum perpetuo tenendum eſt, cum haec differentialia abſolute ſint*

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nihila, ex iis nihil aliud concludi, niſi eorum rationes mutuas, quae vtique ad quantitates finitas reducuntur. Cum autem hoc modo, qui ſolus eſt rationi conſentaneus, principia Calculi differentialis ſtabiliuntur, omnes obtrectationes, quae contra hunc calculum proferri ſunt ſolitae, ſponte corruunt; quae tamen ſummam vim retinerent, ſi differentialia ſeu infinite parua non plane annihilarentur. Pluribus autem, qui Calculi differentialis praecepta tradidere, viſum eſt differentialia a nihilo abſoluto ſecernere, peculiaremque ordinem quantitatum infinite paruarum, quae non penitus euaneſcant, ſed quantitatem quandam, quae quidem eſſet omni aſſignabili minor, retineant, conſtituere: his igitur iure eſt obiectum, rigorem geometricum negligi, & concluſiones inde deductas, propterea quod huiusmodi infinite parua negligerentur, merito eſſe ſuſpectas: quantumuis enim exigua haec infinite parua concipiantur, tamen non ſolum ſingulis, ſed etiam pluribus atque adeo innumerabilibus ſimul rejiciendis, errorem tandem inde enormem reſultare poſſe. Quam obiectionem perperam eiusmodi exemplis, quibus per Calculum differentialem eaedem concluſiones ac per Geometriam

elementarem eliciuntur, infringere conantur: nam ſi ea infinite parua, quae in calculo negliguntur, non ſunt nihil, inde neceſſario error, isque eo maior, quo magis ea coacervantur, reſultare debet; hocque ſi minus eueniat, id potius vitio calculi, quo nonnunquam errores per alios errores compenſantur, eſſet tribuendum, quam ipſe calculus ab erroris ſuſpicione liberaretur. Quodſi autem nullo nouo errore huiusmodi compenſatio fiat, talibus exemplis luculenter id ipſum, quod volo, euincitur, ea quae fuerint neglecta, omnino & abſolute pro nihilo eſſe habenda; neque infinite parua, quae in calculo differentiali tractantur, a nihilo abſoluto diſcrepare. Minime etiam negotium conficitur, quando a nonnullis infinite parua ita deſcribuntur, vt inſtar puluisculorum reſpectu vaſti montis vel etiam totius globi terreſtris ſpectari debeant: etſi enim qui magnitudinem totius globi terreſtris calculo determinare ſuſceperit, ei error non vnius ſed plurium millium puluisculorum facile condonari ſoleat; tamen rigor geometricus etiam a tantillo errore abhorret, nimisque grauis eſſet haec obiectio, ſi vllam vim retineret. Deinde etiam difficile dictu eſt, quid lucri inde ſperent, qui in-

infinite parua a nihilo diſtingui volunt: metuunt autem, ne, ſi plane euaneſcant, etiam comparatio eorum, ad quam totum negotium perduci ſentiunt, tollatur: quomodo enim abſolute nihila inter ſe comparari queant, nullo modo concipi poſſe profitentur. Neceſſe ergo putant iis aliquam magnitudinem relinquere, quo habeant aliquid, in quo comparationem inſtituant: hanc tamen magnitudinem tam paruam admittere coguntur, vt quaſi eſſet nulla, ſpectari ac ſine errore in calculo negligi poſſit. Neque tamen certam ac definitam ipſi magnitudinem, licet incomprehenſibiliter paruam, aſſignare audent; ſemper enim ſi eam bis terue minorem aſſumerent, eodem modo comparationes ſe eſſent habiturae. Ex quo perſpicuum eſt, nihil plane ipſam magnitudinem ad comparationem inſtituendam conferre, hancque adeo non tolli, etiamſi illa magnitudo penitus euaneſcat. Ex dictis autem ſupra manifeſtum eſt, eam comparationem, quae in calculo differentiali ſpectatur, ne locum quidem habere, niſi illa incrementa prorſus euaneſcant: incrementum enim quantitatis x, *quod in genere indicauimus per* ω, *ad incrementum quadrati* xx, *quod eſt* $2x\omega + \omega\omega$, *rationem*

habet vt 1 *ad* 2 x + ω; *quae ſemper differt a ratione* 1 *ad* 2 x, *niſi ſit* ω = 0; *at ſi ſtatuamus eſſe* ω = 0, *tum demum vere affirmare poſſumus, hanc rationem fieri exacte vt* 1 *ad* 2 x. *Interim tamen perſpicitur, quo minus illud incrementum* ω *accipiatur, eo propius ad hanc rationem accedi; vnde non ſolum licet, ſed etiam naturae rei conuenit, haec incrementa primum vt finita conſiderare, atque etiam in figuris, ſi quibus opus eſt ad rem illuſtrandam, finite repraeſentare; deinde vero haec incrementa cogitatione continuo minora fieri concipiantur, ſicque eorum ratio continuo magis ad certum quendam limitem appropinquare reperietur, quem autem tum demum attingant, cum plane in nihilum abierint. Hic autem limes, qui quaſi rationem vltimam incrementorum illorum conſtituit, verum eſt objectum Calculi differentialis; cuius igitur prima fundamenta is ieciſſe exiſtimandus eſt, cui primum in mentem venit, has rationes vltimas, ad quas quantitatum variabilium incrementa, dum continuo magis diminuuntur, appropinquant, & cum euaneſcunt, tum demum attingunt, contemplari. Huius autem ſpeculationis veſtigia deprehendimus apud antiquiſſimos Aucto-*

tores, quibus idcirco idea quaedam leuisque cognitio Analyſis infinitorum abiudicari nequit. Paullatim deinde haec ſcientia maiora accepit incrementa, neque ſubito ad id faſtigium, in quo nunc cernitur, eſt euecta; etiamſi quidem in ea multo plura adhuc ſint occulta, quam in lucem protracta. Cum enim Calculus differentialis ad omnis generis functiones, vtcunque ſint compoſitae, extendatur, non repente methodus innotuit, omnium plane functionum incrementa euaneſcentia inter ſe comparandi; ſed ſenſim haec inuentio ad functiones continuo magis complicatas proceſſit. Quod ſcilicet ad functiones rationales attinet, ratio vltima, quam earum incrementa euaneſcentia inter ſe tenent, multo ante NEUTONI *ac* LEIBNIZII *tempora aſſignari potuit; ita vt Calculus differentialis, quatenus ad ſolas functiones rationales applicatur, diu ante haec tempora inuentus ſit cenſendus. Tum vero nullum eſt dubium, quin* NEUTONO *eam Calculi differentialis partem, quae circa functiones irrationales verſatur, acceptam referre debeamus; ad quam inſigni ſuo Theoremate de euolutione generali poteſtatum binomii feliciter eſt deductus, quo eximio inuento limites calculi*

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diſferentialis iam mirifice erant amplificati. LEIBNIZIO *autem non minus ſumus obſtricti, quod hunc calculum, antehac tantum velut ſingulare artificium ſpectatum, in formam diſciplinae redegerit, eiusque praecepta tanquam in ſyſtema collegerit, ac dilucide explicauerit. Hinc enim maxima ſubſidia ſuggerebantur, ad hunc calculum vlterius excolendum, & ea, quae adhuc deſiderabantur, ex certis principiis elicienda. Mox igitur ſtudio cum Ipſius* LEIBNIZII, *tum* BERNOUILLIORUM *ad hoc ab eo incitatorum, fines Calculi diſferentialis etiam ad functiones transcendentes, quae pars adhuc fuerat inculta, ſunt promoti, tum vero etiam ſolidiſſima fundamenta Calculi integralis conſtituta; quibus inſiſtentes, qui deinceps in hoc genere elaborarunt, continuo maiora incrementa addiderunt.* NEUTONUS *vero etiam ampliſſima dederat ſpecimina Calculi integralis, cuius prima inuentio, cum a prima origine calculi diſferentialis vix ſeparari queat, non ita abſolute conſtitui poteſt; & quoniam maxima eius pars adhuc excolenda reſtat, hic calculus ne nunc quidem pro abſolute inuento haberi poteſt; ſed potius quantum cuique pro viribus ad eius perfectionem conferre*

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contigerit, id grata mente agnoſcere debemus. Atque haec de gloria inuentionis huius calculi tenenda eſſe iudico, de qua quidem antehac tantopere eſt diſceptatum. Quod autem ad varia nomina, quae iſti calculo a diuerſarum nationum Mathematicis imponi ſolent, attinet, ea omnia huc redeunt, vt cum data hic definitione egregie conſentiant: ſiue enim incrementa illa euaneſcentia, quorum ratio conſideratur, differentialia vocentur, ſive fluxiones, ea ſemper nihilo aequalia ſunt intelligenda; in quo vera notio infinite paruorum conſtitui debet. Hinc vero etiam omnia, quae de differentialibus ſecundi & altiorum ordinum curioſe magis quam vtiliter ſunt diſputata, reddentur planiſſima, cum omnia per ſe aeque euaneſcant, neque ea vnquam per ſe, ſed potius eorum relatio mutua ſpectari ſoleat. Cum enim ratio, quam duarum functionum incrementa euaneſcentia tenent, iterum per functionem quandam exprimatur, ſi & huius functionis incrementum euaneſcens cum aliis conferatur, res ad differentialia ſecunda referri eſt cenſenda; ſicque porro progreſſio ad differentialia altiorum graduum intelligi debet, ita vt ſemper quantitates finitae reuera animo ob-

verſentur, ſignaque differentialium tantum ad eas commode repraeſendandas adhibeantur. Primo quidem intuitu iſta Analyſis infinitorum deſcriptio plerisque leuis ac nimis ſterilis videatur, etſi ſpecies illa arcana infinite parvorum re haud plus polliceatur: verum ſi rationes, quae inter incrementa euaneſcentia functionum quarumuis intercedunt, probe cognoſcamus, haec cognitio ſaepenumero per ſe maximi eſt momenti; tum vero in plerisque iisque maxime arduis inueſtigationibus ita eſt neceſſaria, vt ſine eius adminiculo nihil plane intelligi poſſit. Veluti ſi quaeſtio ſit de motu globi ex tormento exploſi, ſimulque ratio reſiſtentiae aeris haberi debeat, quomodo motus per ſpatium finitum ſit futurus, nullo modo ſtatim definire licet, dum tam directio ſemitae, in qua globus incedit, quam ipſius celeritas, a qua reſiſtentia pendet, quouis momento immutatur. Quo minus autem ſpatium, per quod motus fiat, conſideremus, eo minor erit illa variabilitas, eoque facilius ad cognitionem veri pertingere licebit; quodſi autem illud ſpatium plane euaneſcens reddamus, quia iam omnis inaequalitas tam in directione viae quam in celeritate tollitur, effectum reſiſtentiae per regulas motus accu-

curate definire, motusque mutationem puncto temporis productam aſſignare licebit. Cognitis autem his mutationibus momentaneis, ſeu potius cum ipſae ſint nullae, earum relatione mutua, iam plurimum ſumus lucrati; atque calculi integralis opus eſt, exinde motum per ſpatium finitum variatum concludere. Minime autem neceſſe eſſe arbitror vſum Calculi differentialis atque Analyſeos infinitorum in genere pluribus oſtendere; cum nunc quidem ſatis ſit exploratum, ſi vel leuiſſimam inueſtigationem, in quam motus corporum tam ſolidorum quam fluidorum ingrediatur, accuratius inſtituere velimus, id non ſolum non ſine Analyſi infinitorum praeſtari poſſe, ſed hanc ipſam ſcientiam ſaepe nondum ſatis excultam eſſe, vt rem penitus explicare valeamus. Per omnes ſcilicet Matheſeos partes vſus huius Analyſeos ſublimioris vsque adeo diffunditur, vt omnia, quae ſine eius interuentu adhuc expedire licuit, pro nihilo propemodum ſint habenda.

Conſtitui igitur in hoc libro vniuerſum Calculum differentialem ex veris principiis deriuare, atque ita copioſe pertractare, vt nihil praetermitterem eorum, quae quidem adhuc eo pertinentia ſunt inuenta. In duas opus

diuiſi partes, in quarum priori iactis calculi differentialis fundamentis methodum expoſui omnis generis functiones differentiandi, neque tantum differentialia primi ordinis, ſed etiam ſuperiorum ordinum inueniendi; ſiue functiones vnicam variabilem ſiue duas pluresue inuoluant. In altera autem parte ampliſſimum huius calculi vſum in ipſa Analyſi finitorum ac doctrina ſerierum expoſui; vbi etiam imprimis Theoriam maximorum ac minimorum dilucide explicaui. De vſu autem huius calculi in Geometria linearum curuarum nihil adhuc affero, quod eo minus deſiderabitur, cum in aliis operibus haec pars ita copioſe ſit pertractata, vt adeo prima calculi differentialis principia quaſi ex Geometria ſint petita, ad hancque ſcientiam, cum vix ſatis eſſent euoluta, ſumma cura applicata. Hic autem omnia ita intra Analyſeos purae limites continentur, vt ne vlla quidem figura opus fuerit, ad omnia huius calculi praecepta explicanda.

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Errores Typographici.

pag. 509. lin 3. loco $z = xx$ lege $z = xx + x$

pag. 513. lin. 14. loco $\frac{(z' - z)}{a^x(a-1)^2}$ lege $\frac{(z' - az)}{a^x(a-1)^2}$.

INSTITUTIONUM CALCULI DIFFERENTIALIS

PARS PRIOR

CONTINENS

COMPLETAM HUIUS CALCULI EXPLICATIONEM.

CAPUT I.

DE DIFFERENTIIS FINITIS.

I.

Ex iis, quae in Libro ſuperiori de quantitatibus variabilibus atque functionibus ſunt expoſita, perſpicuum eſt, prout quantitas variabilis actu variatur, ita omnes eius functiones variationem pati. Sic, ſi quantitas variabilis x capiat incrementum ω, ita vt pro x ſcribatur $x + \omega$, omnes functiones ipſius x, cuiusmodi ſunt xx; x^3; $\frac{a+x}{xx+aa}$, alios induent valores: ſcilicet xx abibit in $xx + 2x\omega + \omega\omega$;

x^3 abibit in $x^3 + 3xx\omega + 3x\omega\omega + \omega^3$; & $\frac{a+x}{aa+xx}$ transmutabitur in $\frac{a+x+\omega}{aa+xx+2x\omega+\omega\omega}$. Huiusmodi ergo alteratio ſemper orietur, niſi functio ſpeciem tantum quantitatis variabilis mentiatur, reuera autem ſit quantitas conſtans, veluti x^0: quo caſu talis functio inuariata manet, vtcunque quantitas x immutetur.

2. Quae cum ſint ſatis expoſita, propius accedamus ad eas functionum affectiones, quibus vniuerſa analyſis infinitorum innititur. Sit igitur y functio quaecunque quantitatis variabilis x: pro qua ſucceſſiue valores in arithmetica progreſſione procedentes ſubſtituantur, ſcilicet: x; $x+\omega$; $x+2\omega$; $x+3\omega$; $x+4\omega$; &c. ac denotet y^{I} valorem quem functio y induit, ſi in ea loco x ſubſtituatur $x+\omega$; ſimili modo ſit y^{II} is ipſius y valor, ſi loco x ſcribatur $x+2\omega$; parique ratione denotent y^{III}; y^{IV}; y^{V}; &c. valores ipſius y, qui emergunt dum loco x ponuntur $x+3\omega$; $x+4\omega$; $x+5\omega$; &c. ita vt iſti diuerſi valores ipſarum x & y ſequenti modo ſibi reſpondeant:

$$\begin{array}{lllllll} x; & x+\omega; & x+2\omega; & x+3\omega; & x+4\omega; & x+5\omega; & \&c. \\ y; & y^{\text{I}}; & y^{\text{II}}; & y^{\text{III}}; & y^{\text{IV}}; & y^{\text{V}}; & \&c. \end{array}$$

3. Quemadmodum ſeries arithmetica x; $x+\omega$; $x+2\omega$; &c. in infinitum continuari poteſt, ita ſeries ex functione y orta y; y^{I}; y^{II}; &c. quoque in infinitum progredietur, eiusque natura pendebit ab indole functio-

tionis y. Sic, ſi fuerit $y = x$; vel $y = ax + b$; ſeries y; y^{I}; y^{II}; &c. quoque erit arithmetica: ſi fuerit $y = \frac{a}{bx + c}$, ſeries prodibit harmonica: ſint autem ſit $y = a^{x}$, habebitur ſeries geometrica. Neque vlla excogitari poteſt ſeries, quae non hoc modo ex certa functione ipſius y oriri queat; vocari autem ſolet huiusmodi functio ipſius x, ratione ſeriei, quae ex illa oritur, eius TERMINUS GENERALIS; quare, cum omnis ſeries certa lege formata habeat terminum generalem, ea viciſſim ex certa ipſius x functione oritur, vti in doctrina de ſeriebus fuſius explicari ſolet.

4. Hic autem potiſſimum ad differentias, quibus termini ſeriei y, y^{I}, y^{II}, y^{III}, &c. inter ſe diſcrepant, attendimus; quas vt ad differentialium naturam accommodemus, ſequentibus ſignis indicemus, vt ſit

$$y^{I} - y = \Delta y; \quad y^{II} - y^{I} = \Delta y^{I}; \quad y^{III} - y^{II} = \Delta y^{II}; \text{ \&c.}$$

Exprimet ergo Δy incrementum, quod functio y capit, ſi in ea loco x ponatur $x + \omega$, denotante ω numerum quemcunque pro lubitu aſſumtum. In doctrina quidem ſerierum ſumi ſolet $\omega = 1$; verum hic ad noſtrum inſtitutum expedit, valore generali vti, qui pro arbitrio augeri diminuiue queat. Vocari quoque ſolet hoc incrementum Δy functionis y eius DIFFERENTIA, qua ſequens valor y^{I} primum y ſuperat, atque perpetuo tanquam incrementum conſideratur; etiamſi ſaepius re vera decrementum exhibeat, id quod ex eius valore negatiuo agnoſcitur.

 5. Quo-

5. Quoniam y^{II} oritur ex y, ſi loco x ſcribatur $x+2\omega$; manifeſtum eſt eandem quantitatem eſſe orituram, ſi primum pro x ponatur $x+\omega$, tumque denuo $x+\omega$ loco x ſtatuatur. Hinc y^{II} orietur ex y^{I}, ſi in hoc loco x ſcribatur $x+\omega$; eritque ideo Δy^{I} incrementum ipſius y^{I} quod capit poſito $x+\omega$ loco x; ſicque Δy^{I} vocatur ſimili modo *Differentia* ipſius y^{I}. Pari ratione porro erit Δy^{II} differentia ipſius y^{II}, ſeu eius incrementum, quod accipit, ſi loco x ponatur $x+\omega$; atque Δy^{III} erit differentia, ſeu incrementum ipſius y^{III}, & ita porro. Hoc pacto ex ſerie valorum ipſius y, qui ſunt y; y^{I}; y^{II}; y^{III}; &c. obtinebitur ſeries differentiarum Δy; Δy^{I}; Δy^{II}; &c. quae inueniuntur, ſi quilibet terminus illius ſeriei a ſequente ſubtrahatur.

6. Inuenta ſerie differentiarum, ſi ex ea denuo differentiae capiantur, quamlibet a ſequente ſubtrahendo, orientur differentiae differentiarum, quae vocantur *Differentiae ſecundae;* hocque modo per characteres conuenientiſſime repraeſentantur, vt ſignificet:

$$\Delta\Delta y = \Delta y^{\text{I}} - \Delta y$$
$$\Delta\Delta y^{\text{I}} = \Delta y^{\text{II}} - \Delta y^{\text{I}}$$
$$\Delta\Delta y^{\text{II}} = \Delta y^{\text{III}} - \Delta y^{\text{II}}$$
$$\Delta\Delta y^{\text{III}} = \Delta y^{\text{IV}} - \Delta y^{\text{III}}$$

&c.

Vocatur itaque $\Delta\Delta y$ differentia ſecunda ipſius y; $\Delta\Delta y^{\text{I}}$ differentia ſecunda ipſius y^{I}, & ita porro. Simili autem modo ex differentiis ſecundis, ſi denuo earum differen-

tiae

tiae capiantur, prodibunt differentiae tertiae hoc modo ſcribendae $\Delta^3 y$; $\Delta^3 y^{I}$; &c. hincque porro differentiae quartae $\Delta^4 y$; $\Delta^4 y^{I}$; &c. ſicque vltra quousque libuerit.

5. Repraeſentemus ſingulas has differentiarum ſeries ita in ſchemate, quo earum nexus facilius in oculos incidat:

PROGRESSIO ARITHMETICA.

x ; $x+\omega$; $x+2\omega$; $x+3\omega$; $x+4\omega$; $x+5\omega$; &c.

VALORES FUNCTIONIS.

y ; y^{I} ; y^{II} ; y^{III} ; y^{IV} ; y^{V} ; &c.

DIFFERENTIAE PRIMAE.

Δy ; Δy^{I} ; Δy^{II} ; Δy^{III} ; Δy^{IV} ; &c.

DIFFERENTIAE SECUNDAE.

$\Delta\Delta y$; $\Delta\Delta y^{I}$; $\Delta\Delta y^{II}$; $\Delta\Delta y^{III}$; &c.

DIFFERENTIAE TERTIAE.

$\Delta^3 y$; $\Delta^3 y^{I}$; $\Delta^3 y^{II}$; &c.

DIFFERENTIAE QUARTAE.

$\Delta^4 y$; $\Delta^4 y^{I}$; &c.

DIFFERENTIAE QUINTAE.

$\Delta^5 y$; &c.

&c.

quarum quaelibet ex praecedente oritur, quosque terminos a ſequentibus ſubtrahendo. Quacunque ergo functione ipſius x loco y ſubſtituta, quoniam valores

y^{I}

y^{I}, y^{II}, y^{III}, &c. per notas compoſitiones facile formantur, ex iis ſine labore ſingulae differentiarum ſeries invenientur.

8. Ponamus eſſe $y = x$; eritque $y^{I} = x^{I} = x + \omega$; $y^{II} = x^{II} = x + 2\omega$: & ita porro. Vnde differentiis ſumendis erit $\Delta x = \omega$; $\Delta x^{I} = \omega$; $\Delta x^{II} = \omega$; &c. ideoque omnes differentiae primae ipſius x erunt conſtantes, ac proinde differentiae ſecundae omnes euaneſcent; pariterque differentiae tertiae, & ſequentium ordinum omnes. Cum igitur ſit $\Delta x = \omega$, ob analogiam loco litterae ω iſte character Δx commode adhibebitur. Quantitatis ergo variabilis x, cuius valores ſucceſſiui x, x^{I}, x^{II}, x^{III}, &c. arithmeticam progreſſionem conſtituere aſſumuntur, differentiae Δx, Δx^{I}, Δx^{II}, &c. erunt conſtantes atque inter ſe aequales; ac propterea erit $\Delta\Delta x = 0$, $\Delta^{3} x = 0$, $\Delta^{4} x = 0$, ſicque porro.

9. Pro valoribus ipſius x, qui ipſi ſucceſſiue tribuuntur, progreſſionem arithmeticam hic aſſumſimus, ita vt horum valorum differentiae primae ſint conſtantes, ſecundae ac reliquae omnes euaneſcant. Quod etſi ab arbitrio noſtro pendet, cum aliam quamcunque progreſſionem aeque adhibere potuiſſemus; tamen progreſſio arithmetica prae reliquis omnibus commodiſſime vſurpari ſolet, cum quod ſit ſimpliciſſima atque intellectu facillima, tum vero maxime, quod ad omnes omnino valores, quos quidem x induere poteſt, pateat. Tribuendo enim ipſi ω valores tam negatiuos quam affirmatiuos, in hac ſerie valorum ipſius x omnes omnino continentur quanti-

tita-

titates reales, quae in locum ipſius x ſubſtitui poſſunt: contra autem ſi ſeriem geometricam elegiſſemus, ad valores negatiuos nullus aditus patuiſſet. Hanc ob cauſam variabilitas functionum y ex valoribus ipſius x progreſſionem arithmeticam conſtituentibus aptiſſime diiudicatur.

10. Vti eſt $\Delta y = y^{I} - y$, ita differentiae vlteriores quoque ex terminis primae ſeriei y, y^{I}, y^{II}, y^{III}, &c. definiri poſſunt.

Cum enim ſit $\Delta y^{I} = y^{II} - y^{I}$

erit

$$\Delta\Delta y = y^{II} - 2y^{I} + y$$

&

$$\Delta\Delta y^{I} = y^{III} - 2y^{II} + y^{I}$$

ideoque

$$\Delta^{3} y = \Delta\Delta y^{I} - \Delta\Delta y = y^{III} - 3y^{II} + 3y^{I} - y$$

ſimili modo erit

$$\Delta^{4} y = y^{IV} - 4y^{III} + 6y^{II} - 4y^{I} + y$$

&

$$\Delta^{5} y = y^{V} - 5y^{IV} + 10y^{III} - 10y^{II} + 5y^{I} - y$$

quarum formularum coefficientes numerici eandem legem tenent, quae in poteſtatibus Binomii obſeruatur. Quemadmodum ergo differentia prima ex duobus terminis ſeriei y; y^{I}; y^{II}; y^{III}; &c. determinatur, ita differentia ſecunda determinatur ex tribus, tertia ex quatuor, & ita de ceteris. Cognitis autem differentiis cuiusque ordinis ipſius y, ſimili modo differentiae omnium ordinum ipſius y^{I}; y^{II}; &c. definientur.

11. Propoſita ergo quacunque Functione y ſingulae eius differentiae, tam prima quam ſequentes, quae quidem differentiae ω, qua valores ipſius x progrediuntur, reſpondent, poterunt inueniri. Neque vero ad hoc opus eſt, vt ſeries valorum ipſius y vlterius continuetur: quemadmodum enim differentia prima Δy reperitur, ſi in y loco x ſcribatur $x + \omega$, atque a valore orto y^{I} ipſa functio y ſubtrahatur; ita differentia ſecunda $\Delta\Delta y$ obtinebitur ſi in differentia prima Δy loco x ponatur $x + \omega$, vt oriatur Δy^{I}, atque Δy a Δy^{I} ſubtrahatur. Simili modo ſi differentiae ſecundae $\Delta\Delta y$ capiatur Differentia, eam ſubtrahendo a valore, quem induit, ſi loco x ponatur $x + \omega$, proueniet differentia tertia $\Delta^3 y$; hincque porro eodem modo differentia quarta $\Delta^4 y$, &c. Dummodo ergo quis nouerit differentiam primam cuiusque functionis inueſtigare, ſimul poterit differentiam ſecundam, tertiam, omnesque ſequentes inuenire: propterea quod differentia ſecunda ipſius y nil aliud eſt, niſi differentia prima ipſius Δy; & differentia tertia ipſius y nil aliud, niſi differentia prima ipſius $\Delta\Delta y$; ſicque porro de reliquis.

12. Si functio y fuerit ex duabus pluribusue partibus compoſita, vt ſit $y = p + q + r +$ &c.; tum, quia eſt $y^{\text{I}} = p^{\text{I}} + q^{\text{I}} + r^{\text{I}} +$ &c., erit differentia $\Delta y = \Delta p + \Delta q + \Delta r +$ &c., ſimilique modo porro $\Delta\Delta y = \Delta\Delta p + \Delta\Delta q + \Delta\Delta r +$ &c., vnde inuentio differentiarum, ſi functio propoſita ex partibus fuerit compoſita, non parum facilior redditur. Quod ſi vero functio y fuerit productum ex duabus functionibus p & q, nempe

$y =$

$y = pq$, quia erit $y^I = p^I q^I$, & $p^I = p + \Delta p$ atque $q^I = q + \Delta q$, fiet $p^I q^I = pq + p\Delta q + q\Delta p + \Delta p \Delta q$, hincque $\Delta y = p\Delta q + q\Delta p + \Delta p \Delta q$. Vnde, ſi ſit p quantitas conſtans $= a$, ob $\Delta a = o$; erit functionis $y = aq$, differentia prima $\Delta y = a\Delta q$, ſimilique modo differentia ſecunda $\Delta\Delta y = a\Delta\Delta q$, tertia $\Delta^3 q = a\Delta^3 q$, & ita porro.

13. Quoniam omnis functio rationalis integra eſt aggregatum ex aliquot poteſtatibus ipſius x; omnes differentias functionum rationalium integrarum inuenire poterimus, ſi differentias poteſtatum tantum exhibere nouerimus. Hancobrem ſingularum poteſtatum quantitatis variabilis x differentias inueſtigemus in ſequentibus exemplis.

Cum autem ſit $x^0 = 1$, erit $\Delta x^0 = 0$; propterea quod x^0 non variatur, etiamſi x abeat in $x + \omega$.

Tum vero vidimus eſſe $\Delta x = \omega$; & $\Delta\Delta x = 0$, ſimulque differentiae ſequentium ordinum euaneſcunt. Quae cum ſint manifeſta a Poteſtate ſecunda incipiamus:

EXEMPLUM I.

Inuenire differentias omnium ordinum poteſtatis x^2.

Cum hic ſit $y = x^2$, erit $y^I = (x + \omega)^2$; ideoque $\Delta y = 2\omega x + \omega\omega$; quae eſt differentia prima. Iam ob ω quantitatem conſtantem, erit $\Delta\Delta y = 2\omega\omega$, & $\Delta^3 y = 0$; $\Delta^4 y = 0$; &c.

EXEMPLUM II.

Inuenire differentias omnium ordinum poteſtatis x^3.

Ponatur $y = x^3$; &, cum ſit $y^I = (x + \omega)^3$,

 erit

erit

$$\Delta y = 3\omega xx + 3\omega^2 x + \omega^3$$

quae eſt differentia prima. Deinde ob

$$\Delta.\, xx = 2\omega x + \omega\omega$$

erit

$$\Delta.\ 3\omega xx = 6\omega\omega x + 3\omega^3$$

&

$$\Delta.\ 3\omega^2 x = 3\omega^3\ ;\quad \&\quad \Delta.\,\omega^3 = 0:$$

quibus collectis erit

$$\Delta\Delta y = 6\omega^2 x + 6\omega^3\ :\ \text{atque}\ \Delta^3 y = 6\omega^3:$$ §.10

Differentiae vero ſequentes euaneſcent.

EXEMPLUM III.

Inuenire differentias omnium ordinum poteſtatis x^4.

Poſito $y = x^4$; ob $y^I = (x + \omega)^4$

erit

$$\Delta y = 4\omega x^3 + 6\omega^2 x^2 + 4\omega^3 x + \omega^4\ ;$$

quae eſt differentia prima. Tum ex praecedentibus eſt:

$$\Delta.\ 4\omega x^3 = 12\omega^2 x^2 + 12\omega^3 x + 4\omega^4$$
$$\Delta.\ 6\omega^2 x^2 = \ldots\ 12\omega^3 x + 6\omega^4$$
$$\Delta.\ 4\omega^3 x = \ldots\ldots + 4\omega^4$$
$$\Delta.\ \omega^4 = \ldots\ldots\ 0$$

His colligendis erit differentia ſecunda:

$$\Delta\Delta y = 12\omega^2 x^2 + 24\omega^3 x + 14\omega^4\ :$$

Quia deinde porro eſt:

$$\Delta.\ 12\omega^2 x^2 = 24\omega^3 x + 12\omega^4$$
$$\Delta.\ 24\omega^3 x = \ldots\ 24\omega^4$$
$$\Delta.\ 14\omega^4 = \ldots\ 0$$

pro-

prodibit differentia tertia:

$$\Delta^3 y = 24\omega^3 x + 36\omega^4$$

atque tandem differentia quarta:

$$\Delta^4 y = 24\omega^4$$

quae cum ſit conſtans, differentiae ſequentium ordinum euaneſcent.

EXEMPLUM IV.

Inuenire differentias cuiusuis ordinis poteſtatis x^n.

Ponatur $y = x^n$; &, cum ſit $y^{\mathrm{I}} = (x+\omega)^n$; $y^{\mathrm{II}} = (x+2\omega)^n$; $y^{\mathrm{III}} = (x+3\omega)^n$; &c. Poteſtates euolutae dabunt:

$$y = x^n$$

$$y^{\mathrm{I}} = x^n + \frac{n}{1}\omega x^{n-1} + \frac{n(n-1)}{1.\,2}\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1.\,2.\,3}\omega^3 x^{n-3} + \&c.$$

$$y^{\mathrm{II}} = x^n + \frac{n}{1}2\omega x^{n-1} + \frac{n(n-1)}{1.\,2}4\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1.\,2.\,3}8\omega^3 x^{n-3} + \&c.$$

$$y^{\mathrm{III}} = x^n + \frac{n}{1}3\omega x^{n-1} + \frac{n(n-1)}{1.\,2}9\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1.\,2.\,3}27\omega^3 x^{n-3} + \&c.$$

$$y^{\mathrm{IV}} = x^n + \frac{n}{1}4\omega x^{n-1} + \frac{n(n-1)}{1.\,2}16\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1.\,2.\,3}64\omega^3 x^{n-3} + \&c.$$

 Hinc,

Hinc, differentiis ſumendis, prodibit:

$$\Delta y = \frac{n}{1}\omega x^{n-1} + \frac{n(n-1)}{1.\ 2}\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1.\ 2.\ 3}\omega^3 x^{n-3} + \&c.$$

$$\Delta y^{\mathrm{I}} = \frac{n}{1}\omega x^{n-1} + \frac{n(n-1)}{1.\ 2}3\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1.\ 2.\ 3}7\omega^3 x^{n-3} + \&c.$$

$$\Delta y^{\mathrm{II}} = \frac{n}{1}\omega x^{n-1} + \frac{n(n-1)}{1.\ 2}5\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1.\ 2.\ 3}19\omega^3 x^{n-3} + \&c.$$

$$\Delta y^{\mathrm{III}} = \frac{n}{1}\omega x^{n-1} + \frac{n(n-1)}{1.\ 2}7\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1.\ 2.\ 3}37\omega^3 x^{n-3} + \&c.$$

ſumantur denuo differentiae, atque obtinebitur:

$$\Delta\Delta y = n(n-1)\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1.\ 2.\ 3}6\omega^3 x^{n-3} + \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4}14\omega^4 x^{n-4} + \&c.$$

$$\Delta\Delta y^{\mathrm{I}} = n(n-1)\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1.\ 2.\ 3}12\omega^3 x^{n-3} + \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4}50\omega^4 x^{n-4} + \&c.$$

$$\Delta\Delta y^{\mathrm{II}} = n(n-1)\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1.\ 2.\ 3}18\omega^{3n-3} + \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4}110\omega^4 x^{n-4} + \&c.$$

Ex

Ex his per ſubtractionem vlterius eruitur:

$$\Delta^3 y = n(n-1)(n-2)\omega^3 x^{n-3} + \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4.} 36\omega^4 x^{n-4} +$$

&c.

$$\Delta^3 y^{I} = n(n-1)(n-2)\omega^3 x^{n-3} + \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4.} 60\omega^4 x^{n-4} +$$

&c.

atque porro:

$$\Delta^4 y = n(n-1)(n-2)(n-3)\omega^4 x^{n-4} +$$

&c.

14. Quo lex, ſecundum quam iſtae differentiae poteſtatis x^n progrediuntur, facilius perſpiciatur. Ponamus primo breuitatis ergo:

$$A = \frac{n}{1}$$

$$B = \frac{n(n-1)}{1.\ 2.}$$

$$C = \frac{n(n-1)(n-2)}{1.\ 2.\ 3.}$$

$$D = \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4}$$

$$E = \frac{n(n-1)(n-2)(n-3)(n-3)}{1.\ 2.\ 3.\ 4.\ 5}$$

&c.

De-

Deinde ſequens formetur Tabula, quae pro ſingulis differentiis inſeruiet:

y	1;	0;	0;	0;	0;	0;	0;	0;	0;	&c.
Δy	0;	1;	1;	1;	1;	1;	1;	1;	1;	&c.
$\Delta^2 y$	0;	0;	2;	6;	14;	30;	62;	126;	254;	&c.
$\Delta^3 y$	0;	0;	0;	6;	36;	150;	540;	1806;	5796;	&c.
$\Delta^4 y$	0;	0;	0;	0;	24;	240;	1560;	8400;	40824;	&c.
$\Delta^5 y$	0;	0;	0;	0;	0;	120;	1800;	16800;	126000;	&c.
$\Delta^6 y$	0;	0;	0;	0;	0;	0;	720;	15120;	191520;	&c.
$\Delta^7 y$	0;	0;	0;	0;	0;	0;	0;	5040;	141120;	&c.

in qua Tabula numerus cuiusuis ſeriei inuenitur, ſi eiusdem ſeriei praecedens ad numerum ſupra poſitum addatur, atque ſumma per indicem characteri Δ infixum multiplicetur. Sic, in ſerie differentiae $\Delta^5 y$ reſpondente, termintis 16800 inuenitur, ſi praecedens 1800 ad ſupra ſcriptum 1560 addatur, atque Summa 3360 per 5 multiplicetur.

15. Tabula ergo hac conſtituta, ſingulae differentiae Poteſtatis $x^n = y$ ſequenti modo ſe habebunt:

$$\Delta y = A\omega x^{n-1} + B\omega^2 x^{n-2} + C\omega^3 x^{n-3} + D\omega^4 x^{n-4} + \text{\&c.}$$

$$\Delta^2 y = 2B\omega^2 x^{n-2} + 6C\omega^3 x^{n-3} + 14D\omega^4 x^{n-4} + \text{\&c.}$$

$$\Delta^3 y = 6C\omega^3 x^{n-3} + 36D\omega^4 x^{n-4} + 150E\omega^5 x^{n-5} + \text{\&c.}$$

$$\Delta^4 y = 24D\omega^4 x^{n-4} + 240E\omega^5 x^{n-5} + 1560F\omega^6 x^{n-6} + \text{\&c.}$$

Ge-

Generatim autem poteſtatis x^n differentia ordinis m, ſeu $\Delta^m y$, ſequenti modo exprimetur.

Sit

$$I = \frac{n(n-1)(n-2)\ .\ .\ .\ .\ (n-m+1)}{1.\ 2.\ 3\ .\ .\ .\ .\ .\ m};$$

$$K = \frac{n-m}{m+1}\ I;$$

$$L = \frac{n-m-1}{m+2}\ K;$$

$$M = \frac{n-m-2}{m+3}\ L;$$

&c.

Deinde vero ſit:

$$\alpha = (m+1)^m - \frac{m}{1} m^m + \frac{m(m-1)}{1.\ 2}(m-1)^m - \frac{m(m-1)(m-2)}{1.\ 2.\ 3}(m-2)^m + \&c.$$

$$\beta = (m+1)^{m+1} - \frac{m}{1} m^{m+1} + \frac{m(m-1)}{1.\ 2}(m-1)^{m+1} - \frac{m(m-1)(m-2)}{1.\ 2.\ 3}(m-2)^{m+1} + \&c.$$

$$\gamma = (m+1)^{m+2} - \frac{m}{1} m^{m+2} + \frac{m(m-1)}{1.\ 2}(m-1)^{m+2} - \frac{m(m-1)(m-2)}{1.\ 2.\ 3}(m-2)^{m+2} + \&c.$$

quibus valoribus inuentis erit

$$\Delta^m y = \alpha I \omega^m x^{n-m} + \beta K \omega^{m+1} x^{n-m-1} + \gamma L \omega^{m+2} x^{n-m-2}\ \&c.$$

cuius expreſſionis ratio ex modo, quo ſingulae differentiae ex valoribus y, y', y'', y''', &c. eliciuntur, ſponte ſequitur.

 16. Ex

16. Ex his perſpicuum eſt, ſi exponens n fuerit numerus integer affirmatiuus, tandem ad differentias perueniri conſtantes, hisque vlteriores omnes eſſe $= o$. Sic erit

$$\Delta . x = \omega$$
$$\Delta^2 . x^2 = 2\omega^2$$
$$\Delta^3 . x^3 = 6\omega^3$$
$$\Delta^4 . x^4 = 24\omega^4 \quad \text{\& tandem}$$
$$\Delta^n . x^n = 1 . 2 . 3 . . . . n . \omega^n$$

Omnis ergo functio rationalis integra tandem ad differentias conſtantes deducetur. Scilicet, functio ipſius x primi gradus, $ax + b$ differentiam primam iam habet conſtantem $= a\omega$. Functio ſecundi gradus $axx + bx + c$ differentiam ſecundam habebit conſtantem $= 2a\omega\omega$. functionis autem tertii gradus differentia tertia erit conſtans; quarti quarta, & ita porro.

17. Modus autem, quo inuenimus differentias poteſtatis x^n, quoque latius patet, atque ad eas poteſtates, quarum exponens n eſt numerus negatiuus, vel fractus, vel adeo irrationalis, extenditur. Quod quo clarius appareat, differentias tantum primas praecipuarum huiusmodi poteſtatum exhibebimus, quoniam lex differentiarum ſecundarum ac ſequentium non tam facile cernitur: erit ergo,

$$\Delta . x = \omega$$
$$\Delta . x^2 = 2\omega x + \omega^2$$
$$\Delta . x^3 = 3\omega x^2 + 3\omega^2 x + \omega^3$$
$$\Delta . x^4 = 4\omega x^3 + 6\omega^2 x^2 + 4\omega^3 x + \omega^4 \text{ \&c.}$$

Si-

Similili modo vero erit

$$\Delta . x^{-1} = - \frac{\omega}{x^2} + \frac{\omega^2}{x^3} - \frac{\omega^3}{x^4} + \&c.$$

$$\Delta . x^{-2} = - \frac{2\omega}{x^3} + \frac{3\omega^2}{x^4} - \frac{4\omega^3}{x^5} + \&c.$$

$$\Delta . x^{-3} = - \frac{3\omega}{x^4} + \frac{6\omega^2}{x^5} - \frac{10\alpha^3}{x^6} + \&c.$$

$$\Delta . x^{-4} = - \frac{4\alpha}{x^5} + \frac{10\omega^2}{x^6} - \frac{20\alpha^3}{x^7} + \&c.$$

Et inde pro reliquis. Pariter erit

$$\Delta . x^{\frac{1}{2}} = \frac{\omega}{2x^{\frac{1}{2}}} - \frac{\alpha^2}{8x^{\frac{3}{2}}} + \frac{\omega^3}{16x^{\frac{5}{2}}} - \&c.$$

$$\Delta . x^{\frac{1}{3}} = \frac{\omega}{3x^{\frac{2}{3}}} - \frac{\alpha^2}{9x^{\frac{5}{3}}} + \frac{5\omega^3}{81x^{\frac{8}{3}}} - \&c.$$

$$\Delta . x^{-\frac{1}{2}} = - \frac{\omega}{2x^{\frac{3}{2}}} + \frac{3\alpha^2}{8x^{\frac{5}{2}}} - \frac{5\omega^3}{16x^{\frac{7}{2}}} + \&c.$$

$$\Delta . x^{-\frac{1}{3}} = - \frac{\omega}{3x^{\frac{4}{3}}} + \frac{2\omega^2}{9x^{\frac{7}{3}}} - \frac{14\omega^3}{81x^{\frac{10}{3}}} + \&c.$$

 18. Ap-

18. Apparet itaque has differentias, ſi exponens ipſius x non fuerit numerus integer affirmatiuus, in infinitum progredi, ſeu ex terminorum numero infinito conſtare. Interim tamen eaedem differentiae quoque per expreſſionem finitam exhiberi poſſunt. Cum enim, poſito $y = x^{-1} = \frac{1}{x}$, ſit $y^{I} = \frac{1}{x+\omega}$, erit $\Delta. x^{-1} = \Delta. \frac{1}{x} = \frac{1}{x+\alpha} - \frac{1}{x}$; vnde, ſi fractio $\frac{1}{x+\omega}$ in ſeriem conuertatur, prodit expreſſio ſuperior. Simili modo erit

$$\Delta. x^{-2} = \Delta. \frac{1}{xx} = \frac{1}{(x+\omega)^2} - \frac{1}{xx},$$

atque pro irrationalibus erit

$$\Delta. \sqrt{x} = \sqrt{(x+\omega)} - \sqrt{x}, \ \& \ \Delta. \frac{1}{\sqrt{x}} = \frac{1}{\sqrt{(x+\omega)}} - \frac{1}{\sqrt{x}};$$

quae formulae ſi more ſolito in ſeries explicentur, ſuperiores expreſſiones praebent.

19. Hoc vero modo quoque differentiae functionum, ſiue fractarum ſiue irrationalium, inueniri poſſunt: ſic, ſi quaeratur differentia prima fractionis. $\frac{1}{aa+xx}$ ponatur $y = \frac{1}{aa+xx}$; &, quia eſt $y^{I} = \frac{1}{aa+xx+2\omega x+\omega^2}$ erit $\Delta y = \Delta. \frac{1}{aa+xx} = \frac{1}{aa+xx+2\omega x+\omega\omega} - \frac{1}{aa+xx}$, quae expreſſio quoque in ſeriem infinitam conuerti poteſt.

Pona-

Ponatur $aa + xx = \text{P}$ & $2\omega x + \omega\omega = \text{Q}$; erit

$$\frac{1}{\text{P}+\text{Q}} = \frac{1}{\text{P}} - \frac{\text{Q}}{\text{P}^2} + \frac{\text{Q}^2}{\text{P}^3} - \frac{\text{Q}^3}{\text{P}^4} + \&c.$$

&

$$\Delta y = -\frac{\text{Q}}{\text{P}^2} + \frac{\text{Q}^2}{\text{P}^3} - \frac{\text{Q}^3}{\text{P}^4} + \&c.$$

Reſtitutis ergo loco P & Q valoribus erit:

$$\Delta y = \Delta.\frac{1}{aa+xx} = -\frac{2\omega x + \omega\omega}{(aa+xx)^2} + \frac{4\omega\omega xx + 4\omega^3 x + \omega^4}{(aa+xx)^3} - \frac{8\omega^3 x^3 + 12\omega^4 x^2 + 6\omega^5 x + \omega^6}{(aa+xx)^4} + \&c.$$

qui termini ſi ſecundum poteſtates ipſius ω ordinentur erit:

$$\Delta.\frac{1}{aa+xx} = -\frac{2\omega x}{(aa+xx)^2} + \frac{\omega^2(3xx-aa)}{(aa+xx)^3} - \frac{4\omega^3(x^3-aax)}{(aa+xx)^4} + \&c.$$

20. Similibus ſeriebus infinitis differentiae functionum irrationalium quoque exprimi poſſunt.

Sit propoſita iſta functio $y = \sqrt{(aa+xx)}$; &, cum ſit $y^{\text{I}} = \sqrt{(aa+xx+2\omega x+\omega\omega)}$, ponatur $aa+xx = \text{P}$ & $2\omega x + \omega\omega = \text{Q}$

erit $$\Delta y = \sqrt{(\text{P}+\text{Q})} - \sqrt{\text{P}} = \frac{\text{Q}}{2\sqrt{\text{P}}} - \frac{\text{QQ}}{8\text{P}\sqrt{\text{P}}} + \frac{\text{Q}^3}{16\text{P}^2\sqrt{\text{P}}} - \&c.$$

vnde fiet

$$\Delta y = \Delta.\,\mathcal{V}(aa+xx) = \frac{2\omega x+\omega\omega}{2\mathcal{V}(aa+xx)} - \frac{4\omega^2x^2-4\omega^3x-\omega^4}{8(aa+xx)\mathcal{V}(aa+xx)} + \&c.$$

vel

$$= \frac{\omega x}{\mathcal{V}(aa+xx)} + \frac{aa\omega^2}{2(aa+xx)\mathcal{V}(aa+xx)} - \frac{aa\omega^3 x}{2(aa+xx)^2\mathcal{V}(aa+xx)} \&c.$$

Hincque adeo colligimus functionis cuiuscunque ipſius x, quae ſit y, differentiam hac forma exprimi poſſe, vt ſit

$$\Delta y = - P\omega + Q\omega^2 + R\omega^3 + S\omega^4 + \&c.$$

exiſtentibus P, Q, R, S, &c. certis ipſius x functionibus, quae quouis caſu ex functione y definiri poſſunt.

21. Neque etiam ex hac forma differentiae functionum tranſcendentium excluduntur, id quod ex ſequentibus exemplis clarius apparebit.

EXEMPLUM I.

Inuenire differentiam primam logarithmi hyperbolici ipſius x.

Ponatur $y = lx$; & cum ſit $y^I = l(x+\omega)$, erit

$$\Delta y = y^I - y = l(x+\omega) - lx = l\left(1+\frac{\omega}{x}\right).$$

Huiumsodi autem logarithmum ſupra docuimus per ſeriem infinitam exprimere; qua adhibita, erit conf. Introd. I. cap. 7. § 124

$$\Delta y = \Delta.\,lx = \frac{\omega}{x} - \frac{\omega^2}{2xx} + \frac{\omega^3}{3x^3} - \frac{\omega^4}{4x^4} + \&c.$$

EX-

EXEMPLUM II.

Inuenire differentiam primam quantitatis exponentialis a^x.

Poſito $y = a^x$ erit $y^I = a^{x+\omega} = a^x . a^\omega$: at ſupra oſtendimus eſſe

$$a^\omega = 1 + \frac{\omega la}{1} + \frac{\omega^2 (la)^2}{1.\ 2} + \frac{\omega^3 (la)^3}{1.\ 2.\ 3} + \&c.$$ § 125 intr.

quo valore introducto erit

$$\Delta . a^x = y^I - y = \Delta y = \frac{a^x \omega la}{1} + \frac{a^x \omega^2 (la)^2}{1.\ 2} + \frac{a^x \omega^3 (la)^3}{1.\ 2.\ 3} + \&c.$$

EXEMPLUM III.

In circulo, cuius radius $= 1$, inuenire differentiam ſinus arcus x.

Sit $\text{ſin}\, x = y$, erit $y^I = \text{ſin}\,(x+\omega)$, vnde $\Delta y = y^I - y = \text{ſin}\,(x+\omega) - \text{ſin}\, x$.

At eſt $\text{ſin}\,(x+\omega) = \text{coſ}\,\omega .\, \text{ſin}\, x + \text{ſin}\,\omega .\, \text{coſ}\, x$, atque per ſeries infinitas oſtendimus eſſe,

$$\text{coſ}\,\omega = 1 - \frac{\omega^2}{1.\ 2} + \frac{\omega^4}{1.\ 2.\ 3.\ 4} - \frac{\omega^6}{1.\ 2.\ 3.\ 4.\ 5.\ 6} + \&c.$$ § 134 intr:

&

$$\text{ſin}\,\omega = \omega - \frac{\omega^3}{1.2.3} + \frac{\omega^5}{1.2.3.4.5} - \frac{\omega^7}{1.2.3.4.5.6.7} + \&c.$$ cit.

quibus ſeriebus ſubſtitutus erit:

$$\Delta .\, \text{ſin}\, x = \omega .\, \text{coſ}\, x - \frac{\omega^2}{2}\, \text{ſin}\, x - \frac{\omega^3}{6}\, \text{coſ}\, x + \frac{\omega^4}{24}\, \text{ſin}\, x + \frac{\omega^5}{120}\, \text{coſ}\, x - \&c.$$

EX-

EXEMPLUM IV.

In circulo cuius radius $=1$ inuenire differentiam coſinus arcus x.

Poſito $y = \operatorname{coſ} x$, ob $y^{I} = \operatorname{coſ}(x+\omega)$ erit $y^{I} = \operatorname{coſ}\omega . \operatorname{coſ} x - \operatorname{ſin}\omega . \operatorname{ſin} x$
& $\Delta y = \operatorname{coſ}\omega . \operatorname{coſ} x - \operatorname{ſin}\omega . \operatorname{ſin} x - \operatorname{coſ} x$

Seriebus ergo ante expoſitis adhibendis prodibit:

$$\Delta . \operatorname{coſ} x = -\omega \operatorname{ſin} x - \frac{\omega^2}{2} \operatorname{coſ} x + \frac{\omega^3}{6} \operatorname{ſin} x + \frac{\omega^4}{24} \operatorname{coſ} x - \frac{\omega^5}{120} \operatorname{ſin} x - \&c.$$

22. Cum igitur propoſita quacunque functione ipſius x, ſiue algebraica ſiue transcendente, quae ſit y, eius differentia prima eiusmodi habeat formam vt ſit:

$$\Delta y = P\omega + Q\omega^2 + R\omega^3 + S\omega^4 + \&c.$$

ſi huius differentia denuo capiatur, patebit differentiam ſecundam ipſius y huiusmodi formam eſſe habituram:

$$\Delta\Delta y = P\omega^2 + Q\omega^3 + R\omega^4 + \&c.$$

ſimilique modo differentia tertia ipſius y, erit huiusmodi

$$\Delta^3 y = P\omega^3 + Q\omega^4 + R\omega^5 + \&c.$$

ſicque porro.

Vbi notandum eſt litteras P, Q, R, &c. hic non pro valoribus determinatis adhiberi, neque eadem littera in diuerſis differentiis eandem functionem ipſius x denotari: ideo enim tantum iisdem litteris vtor, ne ſufficiens diuerſarum litterarum numerus deficiat.

Ceterum *iſtae differentiarum formae* probe ſunt *notandae*, cum in Analyſi infinitorum maximum vſum offerant.

23. Cum igitur modum expoſuerim, quo cuiusuis functionis differentia prima, ex eaque porro differentiae ſequentium ordinum inueniri queant; quippe quae ex valoribus functionis y ſucceſſiuis y^{I}, y^{II}, y^{III}, y^{IV}, &c. reperiuntur: viciſſim ex differentiis ipſius y cuiusque ordinis datis, iſti ipſi variati valores ipſius y elici poterunt. Erit enim

$$y^{I} = y + \Delta y$$

$$y^{II} = y + 2\Delta y + \Delta\Delta y$$

$$y^{III} = y + 3\Delta y + 3\Delta\Delta y + \Delta^3 y$$

$$y^{IV} = y + 4\Delta y + 6\Delta\Delta y + 4\Delta^3 y + \Delta^4 y$$

&c.

vbi coefficientes numerici iterum ex euolutione binomii naſcuntur. Quemadmodum ergo y^{I}, y^{II}, y^{III}, &c. ſunt valores ipſius y, qui oriuntur ſi loco x ſucceſſiue ponantur hi valores $x + \omega$, $x + 2\omega$, $x + 3\omega$, &c. ſtatim valorem ipſius $y^{(n)}$ aſſignare poterimus, qui prodit ſi loco x ſcribatur $x + n\omega$, erit ſcilicet iſte valor:

$$y + \frac{n}{1}\Delta y + \frac{n(n-1)}{1 \cdot 2}\Delta^2 y + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}\Delta^3 y + \&c.$$

Hincque adeo etiam valores ipſius y praeberi poſſunt ſi n fuerit numerus negatiuus. Sic, ſi loco x ponatur $x - \omega$, functio y abibit in hanc formam:

$$y - \Delta y + \Delta^2 y - \Delta^3 y + \Delta^4 y - \&c.$$

ſin autem loco x ponatur $x - 2\omega$, functio y tranſibit in:

$$y - 2\Delta y + 3\Delta^2 y - 4\Delta^3 y + 5\Delta^4 y - \&c.$$

24. Pauca quaedam addamus de methodo inuerſa, qua, ſi detur differentia, ex ea ipſa illa functio, cuius eſt differentia, inueſtigari debeat. Cum autem hoc ſit difficillimum atque ſaepe numero ipſam analyſin infinitorum requirat, caſus tantum quosdam faciliores euoluamus. Primum igitur, regrediendo, ſi functionis cuiuspiam differentiam inuenerimus, viciſſim hac differentia propoſita, ipſa illa functio, vnde eſt nata, exhiberi poterit. Sic, cum functionis $ax + b$ differentia ſit $a\omega$, ſi quaeratur cuiusnam functionis differentia ſit $a\omega$; reſponſio erit in promtu, eam functionem eſſe $ax + b$. In hac igitur reperitur quantitas conſtans b, quae in differentia non inerat, & quae propterea ab arbitrio noſtro pendet. Perpetuo autem ſi functionis cuiusuis P differentia fuerit Q, quoque functionis P + A, (denotante A quantitatem quamcunque conſtantem,) differentia erit Q. Hinc, ſi iſta differentia Q proponatur, functio, ex qua ea eſt orta, erit P + A, atque idcirco determinatum valorem non habet, cum conſtans A ab arbitrio pendeat.

25. Vocemus eam functionem quaeſitam cuius differentia proponitur, SUMMAM; quod nomen commode adhibetur, cum quod ſumma differentiae opponi ſolet, tum etiam, quod functio quaeſita reuera ſit ſumma omnium valorum praecedentium differentiae. Quemadmodum

dum enim eſt $y^{I} = y + \Delta y$, & $y^{II} = y + \Delta y + \Delta y^{I}$, ſi valores ipſius y retro continuentur; ita, ut is, qui valori $x - \omega$ reſpondet, ſcribatur y_{I}, huncque praedens y_{II}, & qui vltra praecedunt y_{III}, y_{IV}, y_{V}, &c. hincque ſeries formetur retrograda, cum ſuis differentiis:

$$y_{V}\ ;\ y_{IV}\ ;\ y_{III}\ ;\ y_{II}\ ;\ y_{I}\ ;\ y$$

&

$$\Delta y_{V}\ ;\ \Delta y_{IV}\ ;\ \Delta y_{III}\ ;\ \Delta y_{II}\ ;\ \Delta y_{I}$$

erit

$$y = \Delta y_{I} + y_{I}$$

&

ob $y_{I} = \Delta y_{II} + y_{II}$, porroque $y_{II} = \Delta y_{III} + y_{III}$

erit vtique

$$y = \Delta y_{I} + \Delta y_{II} + \Delta y_{III} + \Delta y_{IV} + \Delta y_{V}$$

&c.

ſicque erit functio y, cuius differentia eſt Δy, ſumma omnium valorum antecedentium differentiae Δy, qui oriuntur, ſi loco x ſcribantur valores antecedentes $x - \omega$; $x - 2\omega$; $x - 3\omega$; &c.

26. Quemadmodum ad differentiam denotandam vſi ſumus ſigno Δ, ita ſummam indicabimus ſigno Σ: ſcilicet, ſi functionis y differentia fuerit z, erit $z = \Delta y$; vnde, ſi y detur, differentiam z inuenire ante docuimus. Quodſi autem data ſit differentia z, eiusque ſumma y reperiri debeat, fiet $y = \Sigma z$; atque adeo, ex aequatione $z = \Delta y$ regrediendo, formabitur haec aequatio $y = \Sigma z$; vbi conſtans quantitas quaecunque adiici poterit ob ratio-

 nes

nes ſupra datas; ex quo, aequatio $z = \Delta y$, ſi inuertatur, dabit quoque $y = \Sigma z + C$. Deinde, cum quantitatis ay differentia ſit $a\Delta y = az$, erit $\Sigma az = ay$, ſi quidem a ſit quantitas conſtans. Quia ergo eſt $\Delta x = \omega$; erit $\Sigma\omega = x + C$ & $\Sigma a\omega = ax + C$; atque ob ω quantitatem conſtantem, erit $\Sigma\omega^2 = \omega x + C$; $\Sigma\omega^3 = \omega^2 x + C$; & ita porro.

27. Si igitur differentias poteſtatum ipſius x ſupra inuentas inuertamus, erit § 13.

$$\Sigma\omega = x\ ;\quad \text{hincque}\quad \Sigma 1 = \frac{x}{\omega}\ ;$$

Deinde habemus

$$\Sigma\,(2\omega x + \omega^2) = x^2\ ;$$

vnde ſit

$$\Sigma x = \frac{x^2}{2\omega} - \Sigma\frac{\omega}{2} = \frac{x^2}{2\omega} - \frac{x}{2}.$$

Porro eſt

$$\Sigma\,(3\omega xx + 3\omega^2 x + \omega^3) = x^3$$

ſeu

$$3\omega\,\Sigma x^2 + 3\omega^2\,\Sigma x + \omega^3\,\Sigma 1 = x^3$$

ergo

$$\Sigma x^2 = \frac{x^3}{3\omega} - \omega\,\Sigma x - \frac{\omega^2}{3}\,\Sigma 1$$

ſeu

$$\Sigma x^2 = \frac{x^3}{3\omega} - \frac{x^2}{2} + \frac{\omega x}{6}.$$

ſimili modo erit

$$\Sigma x^3 = \frac{x^4}{4\omega} - \frac{3\omega}{2}\,\Sigma x^2 - \omega^2\,\Sigma x - \frac{\omega^3}{4}\,\Sigma 1$$

vbi

vbi, ſi loco Σx^2, Σx & $\Sigma 1$ valores ante inuenti ſubſtituantur, reperietur:

$$\Sigma x^3 = \frac{x^4}{4\omega} - \frac{x^3}{2} + \frac{\omega xx}{4}.$$

Deinde, cum ſit

$$\Sigma x^4 = \frac{x^5}{5\omega} - 2\omega\Sigma x^3 - 2\omega^2\Sigma x^2 - \omega^3\Sigma x - \frac{\omega^4}{5}\Sigma 1$$

erit, adhibendis ſubſtitutionibus:

$$\Sigma x^4 = \frac{x^5}{5\omega} - \frac{1}{2}x^4 + \frac{1}{3}\omega x^3 - \frac{1}{30}\omega^3 x$$

ſimili modo vlterius progrediendo reperietur

$$\Sigma x^5 = \frac{x^6}{6\omega} - \frac{1}{2}x^5 + \frac{5}{12}\omega x^4 - \frac{1}{12}\omega^3 x^2$$

&

$$\Sigma x^6 = \frac{x^7}{7\omega} - \frac{1}{2}x^6 + \frac{1}{2}\omega x^5 - \frac{1}{6}\omega^3 x^3 + \frac{1}{42}\omega^5 x$$

quas expreſſiones infra facilius inuenire docebimus.

28. Si ergo differentia propoſita fuerit functio rationalis integra ipſius x, eius ſumma, (ſeu ea functio, cuius ea eſt differentia) ex his formulis facile inuenitur. Quia enim differentia ex aliquot poteſtatibus ipſius x conſtabit, quaeratur vniuscuiusque termini ſumma, omnesque iſtae ſummae colligantur.

EXEMPLUM I.

Quaeratur functio, cuius differentia ſit $= axx + bx + c$.

Quaerantur ſingulorum terminorum ſummae ope formularum ante inuentarum, erit

 Σaxx

$$\Sigma axx = \frac{ax^3}{3\omega} - \frac{axx}{2} + \frac{a\omega x}{6}$$

&

$$\Sigma bx = \;.\;.\; \frac{bxx}{2\omega} - \frac{bx}{2}$$

atque

$$\Sigma c = \;.\;.\;.\;.\;.\;.\;.\; \frac{cx}{\omega}$$

Hinc colligendo has ſummas erit

$$\Sigma(axx+bx+c) = \frac{a}{3\omega}x^3 - \frac{(a\omega-b)}{2\omega}x^2 + \frac{(a\omega^2-3b\omega+6c)}{6\omega}x + C$$

quae eſt functio quaeſita, cuius differentia eſt $axx+bx+c$.

EXEMPLUM II.

Quaeratur functio, cuius differentia eſt $x^4-2\omega^2xx+\omega^4$.

Operationem ſimili modo inſtituendo habebitur.

$$\Sigma x^4 = \frac{1}{5\omega}x^5 - \frac{1}{2}x^4 + \frac{1}{3}\omega x^3 - \frac{1}{30}\omega x^3$$

&

$$-\Sigma 2\omega^2x^2 = \;.\; -\frac{2\omega}{3}x^3 + \omega^2x^2 - \frac{\omega^3}{3}x$$

atque

$$+\Sigma\omega^4 = \;.\;.\;.\;.\;.\;.\;.\;.\;.\; +\omega^3x$$

vnde functio quaeſita erit:

$$\frac{1}{5\omega}x^5 - \frac{1}{2}x^4 - \frac{1}{3}\omega x^3 + \omega^2x^2 + \frac{19}{30}\omega^3x + C$$

Si enim hic loco x ponatur $x+\omega$, atque a quantitate reſultante ſubtrahatur iſta inuenta, remanebit propoſita differentia $x^4 - 2\omega^2x^2 + \omega^4$.

29. Si

29. Si ſummas, quas pro poteſtatibus ipſius x invenimus, attentius inſpiciamus, in terminis primis, ſecundis, ac tertiis mox quidem legem obſeruabimus, qua illi ſecundum ſingulas poteſtates progrediuntur: reliquorum autem terminorum lex non ita eſt perſpicua, vt ſummam poteſtatis x^n in genere inde colligere liceat. Interim tamen in ſequentibus docebitur eſſe:

$$\Sigma x^n =$$

$$\frac{x^{n+1}}{(n+1)\omega} - \frac{1}{2}x^n + \frac{1}{2}\cdot\frac{n\omega}{2.3}x^{n-1} - \frac{1}{6}\cdot\frac{n(n-1)(n-2)\omega^3}{2.\ 3.\ 4.\ 5}x^{n-3}$$

$$+\frac{1}{6}\cdot\frac{n(n-1)(n-2)(n-3)(n-4)\omega^5}{2.\ 3.\ 4.\ 5.\ 6.\ 7.}x^{n-5}$$

$$-\frac{3}{10}\cdot\frac{n(n-1)\ \ldots\ldots\ (n-6)\omega^7}{2.\ 3\ \ldots\ldots\ 8.\ 9}x^{n-7}$$

$$+\frac{5}{6}\cdot\frac{n(n-1)\ \ldots\ldots\ (n-8)\omega^9}{2.\ 3\ \ldots\ldots\ 10.\ 11}x^{n-9}$$

$$-\frac{691}{210}\cdot\frac{n(n-1)\ \ldots\ldots\ (n-10)\omega^{11}}{2.\ 3\ \ldots\ldots\ 12.\ 13}x^{n-11}$$

$$+\frac{35}{2}\cdot\frac{n(n-1)\ \ldots\ \ n-12)\omega^{13}}{2.\ 3\ \ldots\ldots\ 14.\ 15}x^{n-13}$$

$$-\frac{3617}{30}\cdot\frac{n(n-1)\ \ldots\ (n-14)\omega^{15}}{2.\ 3\ \ldots\ 16.\ 17}x^{n-15}$$

$$+\frac{43867}{42}\cdot\frac{n(n-1)\ \ldots\ldots\ (n-16)\omega^{17}}{2.\ 3\ \ldots\ldots\ 18.\ 19}x^{n-17}$$

$$-\frac{1222277}{110}\cdot\frac{n(n-1)\ \ldots\ldots\ (n-18)\omega^{19}}{2.\ 3\ \ldots\ldots\ 20.\ 21}x^{n-19}$$

$$+\frac{854513}{6}\cdot\frac{n(n-1)\ \ldots\ldots\ (n-20)\omega^{21}}{2.\ 3\ \ldots\ldots\ 22.\ 23}x^{n-21}$$

—

$$-\frac{1181820455}{546}\cdot\frac{n(n-1)\ldots(n-22)\omega^{23}}{2.\,3\ldots\ldots 24.\,25}x^{n-23}$$

$$+\frac{76977927}{2}\cdot\frac{n(n-1)\ldots\ldots(n-24)\omega^{25}}{2.\,3\ldots\ldots\ldots 26.\,27}x^{n-25}$$

$$-\frac{23749461029}{30}\cdot\frac{n(n-1)\ldots(n-26)\omega^{27}}{2.\,3\ldots\ldots 28.\,29}x^{n-27}$$

$$+\frac{8615841276005}{462}\cdot\frac{n(n-1)\ldots(n-28)\omega^{29}}{2.\,3\ldots\ldots 30.\,31}x^{n-29}$$

$$\&c. + C$$

cuius progreſſionis praecipuum momentum in coefficientibus mere numericis eſt ſitum, qui quemadmodum formentur, hic locus nondum eſt, vbi exponi queat.

30. Apparet autem niſi n ſit numerus integer affirmatiuus, hanc ſummae expreſſionem in infinitum progredi, neque hoc modo ſummam in forma finita exhiberi poſſe. Ceterum hic notandum eſt, non omnes poteſtates ipſius x propoſita x^n inferiores occurrere; deſunt enim termini x^{n-2}, x^{n-4}, x^{n-6}, x^{n-8}, &c. quippe quorum coefficientes ſunt $=0$, etiamſi termini ſecundi x^n coefficiens hanc legem non ſequatur, ſed ſit $=-\frac{1}{2}$. Poterunt ergo huius expreſſionis ope ſummae poteſtatum, quarum exponentes ſunt vel negatiui vel fracti in forma infinita exhiberi ſolo excepto caſu quo $n=-1$, quia tum fit terminus $\frac{x^{n+1}}{(n+1)\omega}$ ob $n+1=0$ infinitus. Sic, poſito $n=-2$; erit

$\Sigma\frac{1}{xx}$

$$\Sigma \frac{1}{xx} = C - \frac{1}{\omega x} - \frac{1}{2xx} - \tfrac{1}{2}.\frac{\omega}{3x^3} + \tfrac{1}{6}.\frac{\omega^3}{5x^5}$$

$$- \tfrac{1}{6}.\frac{\omega^5}{7x^7} + \tfrac{3}{10}.\frac{\omega^7}{9x^9} - \tfrac{5}{6}.\frac{\omega^9}{11x^{11}} + \frac{691}{210}.\frac{\omega^{11}}{13x^{13}}$$

$$- \tfrac{35}{2}.\frac{\omega^{13}}{15x^{15}} + \frac{3617}{30}.\frac{\omega^{15}}{17x^{17}} - \&c.$$

31. Si ergo differentia propoſita fuerit poteſtas ipſius x quaecunque, eius ſumma hinc perpetuo aſſignari, ſeu functio, cuius ea ſit differentia, exhiberi poterit. Sin autem differentia propoſita aliam habeat formam, vt in poteſtates ipſius x, tanquam partes, diſtribui nequeat, tum ſumma difficillime ac ſaepenumero prorſus non inueniri poteſt: niſi forte pateat, eam ex quapiam functione eſſe ortam. Hanc ob cauſam conueniet plurium functionum differentias inueſtigare easque probe notare, vt ſi quando huiusmodi differentia proponatur, eius ſumma, ſeu functio vnde eſt orta, ſtatim exhiberi queat. Interim tamen methodus infinitorum plures regulas ſuppeditabit, quarum ope inuentio ſummarum mirifice ſubleuabitur.

32. Facilius autem ſaepe ſumma quaeſita reperitur, ſi differentia propoſita ex factoribus ſimplicibus conſtet, qui progreſſionem arithmeticam conſtituant, cuius differentia ſit ipſa quantitas ω. Sic, ſi propoſita fuerit functio $(x+\omega)(x+2\omega)$, cuius differentia quaeratur: quia, poſito $x+\omega$ loco x, haec functio abit in $(x+2\omega)$

E (x+

$(x + 3\,\omega)$, eius differentia erit $2\,\omega\,(x + 2\,\omega)$. Quare viciſſim, ſi proponatur differentia $2\,\omega\,(x + 2\,\omega)$, eius ſumma erit $(x + \omega)\,(x + 2\,\omega)$, hinc ergo erit

$$\Sigma\,(x + 2\,\omega) = \frac{1}{2\,\omega}\,(x + \omega)\,(x + 2\,\omega).$$

Simili modo, ſi proponatur functio $(x + n\omega)(x + (n+1)\omega)$, cum ſit eius differentia $2\,\omega\,(x + (n + 1)\,\omega)$ erit

$$\Sigma\,(x + (n + 1)\,\omega) = \frac{1}{2\,\omega}\,(x + n\omega)\,(x + (n + 1)\omega)$$

&

$$\Sigma\,(x + n\,\omega) = \frac{1}{2\,\omega}\,(x + (n - 1)\,\omega)\,(x + n\,\omega).$$

33. Si functio ex pluribus factoribus conſtet, vt ſit $y = (x + (n - 1)\,\omega)\;(x + n\,\omega)\;(n + 1)\,\omega)$, cum ſit

$$y^{\mathrm{I}} = (x + n\,\omega)\,(x + (n + 1)\,\omega)\,(x + (n + 2)\,\omega),$$

erit

$$\Delta\,y = 3\,\omega\,(x + n\,\omega)\;(x + (n + 1)\,\omega)$$

ac propterea

$$\Sigma(x + n\omega)(x + (n + 1)\omega) = \frac{1}{3\omega}(x + (n - 1)\omega)(x + n\omega)(x + (n + 1)\omega)$$

Pari modo reperietur eſſe:

$$\Sigma\,(x + n\,\omega)\,(x + (n + 1)\,\omega)\,(x + (n + 2)\,\omega) =$$
$$\frac{1}{4\omega}\,(x + (n - 1)\omega)\,(x + n\omega)\,(x + (n + 1)\omega)\,(x + (n + 2)\omega).$$

vnde lex inueniendi ſummas, ſi differentia ex pluribus huiusmodi factoribus conſtet, ſponte patet. Quamuis au-

autem hae differentiae ſint functiones rationales integrae, tamen earum ſummae hoc modo facilius reperiuntur, quam per methodum praecedentem.

34. Hinc quoque via patet ad differentiarum fractarum ſummas inueniendas. Sit enim propoſita fractio

$$y = \frac{1}{x + n\omega}; \text{ quia erit } y^{1} = \frac{1}{x + (n+1)\omega}$$

erit

$$\Delta y = \frac{1}{x + (n+1)\omega} - \frac{1}{x + n\omega} = \frac{-\omega}{(x + n\omega)(x + (n+1)\omega)}$$

ac propterea

$$\Sigma \frac{1}{(x + n\omega)(x + (n+1)\omega)} = -\frac{1}{\omega}.\ \frac{1}{x + n\omega}$$

Sit porro

$$y = \frac{1}{(x + n\omega)(x + (n+1)\omega)}$$

ob $y^{1} = \dfrac{1}{(x + (n+1)\omega)(x + (n+2)\omega)}$

erit

$$\Delta y = \frac{-2\omega}{(x + n\omega)(x + (n+1)\omega)(x + (n+2)\omega)}$$

Hinc ideo fiet

$$\Sigma \frac{1}{(x + n\omega)(x + (n+1)\omega)(x + (n+2)\omega)}$$

$$= \frac{-1}{2\omega}.\ \frac{1}{(x + n\omega)(x + (n+1)\omega)}.$$

 Simili

Simili modo erit porro

$$\Sigma \frac{1}{(x+n\omega)(x+(n+1)\omega)(x+(n+2)\omega)(x+(n+3)\omega)}$$
$$= \frac{-1}{3\omega}\cdot\frac{1}{(x+n\omega)(x+(n+1)\omega)(x+(n+2)\omega)}.$$

35. Modus iſte ſummandi probe eſt tenendus, quia huiusmodi differentiarum ſummae per praecedentem methodum inueniri non poſſunt. Quodſi autem differentia inſuper habeat numeratorem, vel factores denominatoris non in arithmetica progreſſione procedant, tum tutiſſimus modus inueſtigandi ſummas eſt, vt differentia propoſita in ſuas fractiones ſimplices reſoluatur, quarum ſingulae etſi ſummari nequeunt, tamen binis coniungendis toties ſumma inueniri poteſt, quoties id quidem fieri licet; tantum enim erit diſpiciendum, vtrum ſumma ope huius formulae inueniri queat:

$$\Sigma \frac{1}{x+(n+1)\omega} - \Sigma \frac{1}{x+n\omega} = \frac{1}{x+n\omega}$$

etſi enim neutra harum ſummarum per ſe exhiberi poteſt, tamen earum differentia cognoſcitur.

36. His igitur caſibus negotium redit ad reſolutionem cuiusque fractionis in fractiones ſuas ſimplices, quae in ſuperiori libro fuſius eſt oſtenſa. Quemadmodum ergo eius beneficio ſummae inueniri queant, aliquot exemplis docebimus.

EXEM-

EXEMPLUM I.

Quaeratur ſumma, cuius differentia ſit

$$\frac{3x+2\omega}{x(x+\omega)(x+2\omega)}.$$

Reſoluatur haec differentia propoſita in ſuas fractiones ſimplices, quae erunt

$$\frac{1}{\omega}\cdot\frac{1}{x}+\frac{1}{\omega}\cdot\frac{1}{x+\omega}-\frac{2}{\omega}\cdot\frac{1}{x+2\omega}.$$

Cum iam ſit ex ſuperiori formula:

$$\Sigma\frac{1}{x+n\omega}=\Sigma\frac{1}{x+(n+1)\omega}-\frac{1}{x+n\omega}$$

erit

$$\Sigma\frac{1}{x}=\Sigma\frac{1}{x+\omega}-\frac{1}{x}.$$

Hinc erit ſumma quaeſita

$$\frac{1}{\omega}\Sigma\frac{1}{x}+\frac{1}{\omega}\Sigma\frac{1}{x+\omega}-\frac{2}{\omega}\Sigma\frac{1}{x+2\omega}=$$

$$\frac{2}{\omega}\Sigma\frac{1}{x+\omega}-\frac{2}{\omega}\Sigma\frac{1}{x+2\omega}-\frac{1}{\omega x},$$

at eſt

$$\Sigma\frac{1}{x+\omega}=\Sigma\frac{1}{x+2\omega}-\frac{1}{x+\omega};$$

vnde ſumma quaeſita erit

$$-\frac{1}{\omega x}-\frac{2}{\omega(x+\omega)}=\frac{-3x-\omega}{\omega x(x+\omega)}.$$

EXEMPLUM II.

Quaeratur ſumma, cuius differentia eſt

$$\frac{3\omega}{x(x+3\omega)}.$$

Poſita hac differentia $= z$, erit $z = \frac{1}{x} - \frac{1}{x+3\omega}$ ideoque

$$\Sigma z = \Sigma\frac{1}{x} - \Sigma\frac{1}{x+3\omega} = \Sigma\frac{1}{x+\omega} -$$

$$\Sigma\frac{1}{x+3\omega} - \frac{1}{x} = \Sigma\frac{1}{x+2\omega} - \Sigma\frac{1}{x+3\omega} - \frac{1}{x} - \frac{1}{x+\omega}$$

$$= -\frac{1}{x} - \frac{1}{x+\omega} - \frac{1}{x+2\omega}.$$

quae eſt ſumma quaeſita. Quoties ergo hoc modo ſigna ſummatoria Σ ſeſe tandem tollunt, toties differentiae propoſitae ſumma exhiberi poterit; ſin autem haec deſtructio non ſuccedat, ſignum hoc eſt, ſummam inueniri non poſſe.

CAPUT

CAPUT II.

DE VSU DIFFERENTIARUM IN DOCTRINA SERIERUM.

37.

Naturam ſerierum per differentias maxime illuſtrari, ex primis rudimentis ſatis eſt notum. Progreſſionis enim arithmeticae, quae primum conſiderari ſolet, praecipua proprietas in hoc verſatur, vt eius differentiae primae ſint inter ſe aequales; hinc differentiae ſecundae ac reliquae omnes erunt cyphrae. Dantur deinde ſeries, quarum differentiae ſecundae demum ſunt aequales, quae hanc ob rem *ſecundi ordinis* commode appellantur, dum progreſſiones arithmeticae ſeries *primi ordinis* vocantur. Porro igitur ſeries *tertii ordinis* erunt, quarum differentiae tertiae ſunt conſtantes; atque ad *quartum ordinem* & ſequentes eae referentur ſeries, quarum differentiae quartae, & vlteriores demum ſunt conſtantes.

38. In hac diuiſione infinita ſerierum genera comprehenduntur, neque tamen omnes ſeries ad haec genera reuocare licet. Occurrunt enim innumerabiles ſeries, quae, differentiis ſumendis, nunquam ad terminos conſtantes deducunt: cuiusmodi, praeter innumeras alias ſunt progreſſiones geometricae, quae nunquam praebent differentias conſtantes, vti ex hoc exemplo videre licet.

1, 2, 4, 8, 16, 32, 64, 128, &c.
1, 2, 4, 8, 16, 32, 64, &c.
1, 2, 4, 8, 16, 32, &c.

Cum enim ſeries differentiarum cuiusque ordinis aequalis ſit ipſi ſeriei propoſitae, aequalitas differentiarum prorſus excluditur. Quocirca plures ſerierum claſſes conſtitui debebunt, quarum una tantum in hos ordines, qui tandem ad differentias conſtantes revocantur, ſubdiuiditur; quam claſſem in hoc capite potiſſimum conſiderabimus.

39. Duae autem res ad naturam ſerierum cognoſcendam imprimis requiri ſolent, Terminus generalis atque Summa ſeu Terminus ſummatorius. Terminus generalis eſt expreſſio indefinita, quae vnumquemque ſeriei terminum complectitur, atque eiusmodi propterea eſt functio quantitatis variabilis x, quae, poſito $x = 1$, terminum ſeriei primum exhibet; ſecundum vero poſito $x = 2$; tertium poſito $x = 3$; quartum poſito $x = 4$; & ita porro. Cognito ergo termino generali, quotuscunque ſeriei terminus inuenietur, etiamſi lex, qua ſinguli termini cohaerent, non reſpiciatur. Sic verbi gratia ponendo $x = 1000$, ſtatim terminus milleſimus cognoſcetur. Ita huius ſeriei

1, 6, 15, 28, 45, 66, 91, 120, &c.

Terminus generalis eſt $2xx - x$; poſito enim $x = 1$, haec formula dat terminum primum 1; poſito $x = 2$, oritur terminus ſecundus 6; ſi ponatur $x = 3$, oritur tertius 15; &c. vnde patet huius ſeriei terminum centeſimum, poſito $x = 100$ fore $= 2.10000 - 100 = 19900$.

40. In-

40. Indices ſeu exponentes in qualibet ſerie vocantur numeri, qui indicant quotus quisque terminus ſit in ordine: ſic, termini primi index erit 1, ſecundi 2, tertii 3, & ita porro. Hinc indices ſingulis cuiusque ſeriei terminis inſcribi ſolent, hoc modo

INDICES.

1, 2, 3, 4, 5, 6, 7, &c.

TERMINI.

A, B, C, D, E, F, G, &c.

vnde ſtatim patet G eſſe ſeriei propoſitae terminum ſeptimum, cum eius index ſit 7. Hinc terminus generalis nil aliud erit, niſi terminus ſeriei, cuius index vel exponens eſt numerus indefinitus x. Quemadmodum ergo in quolibet ſerierum ordine, quarum differentiae vel primae, vel ſecundae, vel aliae ſequentes ſunt conſtantes, terminum generalem inueniri oporteat, primum docebimus: tum vero ad inueſtigationem ſummae ſumus progreſſuri.

41. Incipiamus ab ordine primo, qui continet progreſſiones arithmeticas, quarum differentiae primae ſunt conſtantes; ſitque a terminus ſeriei primus, & b terminus primus ſeriei differentiarum, cui ſequentes omnes ſunt aequales: vnde ſeries ita erit comparata.

INDICES.

1, 2, 3, 4, 5, 6,

TERMINI.

a, $a+b$, $a+2b$, $a+3b$, $a+4b$, $a+5b$, &c.

DIFFERENTIAE.

b, b, b, b, b, &c.

F Ex

Ex qua ſtatim patet, terminum, cuius index ſit $= x$, fore $a + (x-1)b$, eritque ergo terminus generalis $= bx + a - b$, qui ex terminis primis cum ipſius ſeriei, tum ſeriei differentiarum componitur. Quodſi autem terminus ſecundus ſeriei $a+b$ vocetur a^{I}, ob $b = a^{\mathrm{I}} - a$, erit terminus generalis $= (a^{\mathrm{I}}-a)x + 2a - a^{\mathrm{I}} = a^{\mathrm{I}}(x-1) - a(x-2)$ vnde, ex cognitis terminis primo & ſecundo progreſſionis arithmeticae, eius terminus generalis formabitur.

42. Sint in ſerie ſecundi ordinis termini primi, ipſius ſeriei $= a$; differentiarum primarum $= b$; differentiarum ſecundarum $= c$; eritque ipſa ſeries cum ſuis differentiis ita comparata.

INDICES.

1, 2, 3, 4, 5, 6, 7,

TERMINI.

$a; a+b; a+2b+c; a+3b+3c; a+4b+6c; a+5b+10c; a+6b+15c;$ &c.

DIFFER. I.

$b;\ b+c;\ b+2c;\ b+3c;\ b+4c;\ b+5c;$ &c.

DIFFER. II.

$c,\ c,\ c,\ c,\ c,$

ex cuius inſpectione liquet terminum, cuius index $= x$ fore $= a + (x-1)b + \frac{(x-1)(x-2)}{1.\ 2}c$; qui ergo eſt terminus generalis ſeriei propoſitae. Ponatur autem ipſius ſeriei terminus ſecundus $= a^{\mathrm{I}}$, terminus tertius $= a^{\mathrm{II}}$, cum ſit $b = a^{\mathrm{I}} - a$; & $c = a^{\mathrm{II}} - 2a^{\mathrm{I}} + a$; vti ex natura

tura differentiarum (§. 10.) intelligitur, erit terminus generalis

$$a + (x-1)(a^{I} - a) + \frac{(x-1)(x-2)}{1.\ 2}(a^{II} - 2a^{I} + a)$$

qui reducitur ad hanc formam

$$\frac{a^{II}(x-1)(x-2)}{1.\ 2} - \frac{2a^{I}(x-1)(x-3)}{1.\ 2} + \frac{a(x-2)(x-3)}{1.\ 2}$$

vel etiam ad hanc

$$\frac{a^{II}}{2}(x-1)(x-2) - \frac{2a^{I}}{2}(x-1)(x-3) + \frac{a}{2}(x-2)(x-3)$$

aut denique ad hanc

$$\tfrac{1}{2}(x-1)(x-2)(x-3)\left(\frac{a^{II}}{x-3} - \frac{2a^{I}}{x-2} + \frac{a}{x-1}\right);$$

ideoque ex tribus terminis ipſius ſeriei definitur.

43. Sit ſeries tertii ordinis a, a^{I}, a^{II}, a^{III}, a^{IV}, &c. eius differentiae primae b, b^{I}, b^{II}, b^{III}, &c. & differentiae ſecundae c, c^{I}, c^{II}, c^{III}, &c. & tertiae d, d, d, &c. quippe quae ſunt conſtantes.

INDICES.

1, 2, 3, 4, 5, 6,

TERMINI.

a, a^{I}, a^{II}, a^{III}, a^{IV}, a^{V}, &c.

DIFFER. I.

b, b^{I}, b^{II}, b^{III}, b^{IV}, &c.

DIFFER. II.

c, c^{I}, c^{II}, c^{III}, &c.

DIFFER. III.

d, d, d, &c.

 Quia

Quia eſt $a^{\text{I}} = a + b$; $a^{\text{II}} = a + 2b + c$; $a^{\text{III}} = a + 3b + 3c + d$; $a^{\text{IV}} = a + 4b + 6c + 4d$; &c.; erit terminus generalis, ſeu is cuius index eſt x,

$$a + \frac{(x-1)}{1} b + \frac{(x-1)(x-2)}{1.\ 2} c + \frac{(x-1)(x-2)(x-3)}{1.\ 2.\ 3} d$$

ſicque terminus generalis ex differentiis formabitur.

Cum autem porro ſit

$$b = a^{\text{I}} - a;\ c = a^{\text{II}} - 2a^{\text{I}} + a;\ d = a^{\text{III}} - 3a^{\text{II}} + 3a^{\text{I}} - a$$

ſi hi valores ſubſtituantur erit terminus generalis

$$a^{\text{III}} \frac{(x-1)(x-2)(x-3)}{1.\ 2.\ 3} - 3a^{\text{II}} \frac{(x-1)(x-2)(x-4)}{1.\ 2.\ 3}$$

$$+ 3a^{\text{I}} \frac{(x-1)(x-3)(x-4)}{1.\ 2.\ 3} - a \frac{(x-2)(x-3)(x-4)}{1.\ 2.\ 3}$$

qui etiam hoc modo exprimetur, vt ſit

$$= \frac{(x-1)(x-2)(x-3)(x-4)}{1.\ 2.\ 3} \left(\frac{a^{\text{III}}}{x-4} - \frac{3a^{\text{II}}}{x-3} + \frac{3a^{\text{I}}}{x-2} - \frac{a}{x-1} \right).$$

44. Sit nunc ſeries cuiuscunque ordinis propoſita:

INDICES

1, 2, 3, 4, 5, 6,

TERMINI.

a, a^{I}, a^{II}, a^{III}, a^{IV}, a^{V}, &c.

DIFFER. I.

b, b^{I}, b^{II}, b^{III}, b^{IV}, &c.

DIFFER. II.

c, c^{I}, c^{II}, c^{III}, &c.

DIF-

DIFFER. III.

d, d^{I}, d^{II}, &c.

DIFFER. IV.

e, e^{I}, &c.

DIFFER. V.

f, &c.

ex ipſius ſeriei termino primo, atque ex differentiarum terminis primis b, c, d, e, f, &c. terminus generalis ita exprimetur, vt ſit:

$$a + \frac{(x-1)}{1} b + \frac{(x-1)(x-2)}{1.\ 2} c + \frac{(x-1)(x-2)(x-3)}{1.\ 2.\ 3} d + \frac{(x-1)(x-2)(x-3)(x-4)}{1.\ 2.\ 3.\ 4} e + \&c.$$

donec ad differentias conſtantes perueniatur. Ex quo patet, ſi nunquam prodeant differentiae conſtantes, terminum generalem per expreſſionem infinitam exhiberi.

45. Quia differentiae ex ipſis terminis ſeriei formantur, ſi earum valores ſubſtituantur, prodibit terminus generalis in eiusmodi forma expreſſus, cuiusmodi pro ſeriebus primi, ſecundi, & tertii ordinis exhibuimus. Scilicet, pro ſeriebus ordinis quarti, erit terminus generalis

$$\frac{(x-1)(x-2)(x-3)(x-4)(x-5)}{1.\ 2.\ 3.\ 4} \times \left(\frac{a^{IV}}{x-5} - \frac{4a^{III}}{x-4} + \frac{6a^{II}}{x-3} - \frac{4a^{I}}{x-2} + \frac{a}{x-1}\right)$$

 vnde

vnde lex, qua ſequentium ordinum termini generales componuntur, facile perſpicitur. Ex his autem patet pro quouis ordine terminum generalem fore functionem ipſius x rationalem integram, in qua maxima ipſius x dimenſio congruat cum ordine, ad quem ſeries refertur. Ita ſerierum primi ordinis erit terminus generalis functio primi gradus, ſecundi ordinis ſecundi gradus, & ita porro.

46. Differentiae autem, vti ſupra vidimus, ex ipſis terminis ſeriei ita reſultant, vt ſit

$$b = a^{\text{I}} - a$$
$$b^{\text{I}} = a^{\text{II}} - a^{\text{I}}$$
$$b^{\text{II}} = a^{\text{III}} - a^{\text{II}}$$
&c.
$$c = a^{\text{II}} - 2a^{\text{I}} + a$$
$$c^{\text{I}} = a^{\text{III}} - 2a^{\text{II}} + a^{\text{I}}$$
$$c^{\text{II}} = a^{\text{IV}} - 2a^{\text{III}} + a^{\text{II}}$$
&c.
$$d = a^{\text{III}} - 3a^{\text{II}} + 3a^{\text{I}} - a$$
$$d^{\text{I}} = a^{\text{IV}} - 3a^{\text{III}} + 3a^{\text{II}} - a^{\text{I}}$$
$$d^{\text{II}} = a^{\text{V}} - 3a^{\text{IV}} + 3a^{\text{III}} - a^{\text{II}}$$
&c.

Quare, cum in ſeriebus primi ordinis ſint omnes valores ipſius $c = 0$; erit

$a^{\text{II}} = 2a^{\text{I}} - a$; $a^{\text{III}} = 2a^{\text{II}} - a^{\text{I}}$; $a^{\text{IV}} = 2a^{\text{III}} - a^{\text{II}}$; &c.

vnde patet has ſeries ſimul eſſe recurrentes, & ſcalam relationis eſſe $2, -1$. Deinde, cum in ſeriebus ſecundi ordi-

ordinis ſint omnes valores ipſius $d = 0$, erit

$$a^{III} = 3a^{II} - 3a^{I} + a;\ a^{IV} = 3a^{III} - 3a^{II} + a^{I};\ \&c.$$

ideoque & hae erunt recurrentes ſcala relatione exiſtente

$$3, -3, +1.$$

Simili modo apparebit omnes huius claſſis ſeries, cuiuscunque ſint ordinis, ſimul ad claſſem ſerierum recurrentium pertinere, atque ita quidem, vt ſcala relationis conſtet ex coefficientibus poteſtatis binomii, vno gradu ſuperioris, quam eſt ordo, ad quem ſeries refertur.

47. Quia vero pro ſeriebus primi ordinis quoque omnes valores ipſius d & e, & ſequentium differentiarum omnium ſunt $= 0$, erit quoque in his

$$a^{III} = 3a^{II} - 3a^{I} + a$$
$$a^{IV} = 3a^{III} - 3a^{II} + a^{I}$$
&c.

aut

$$a^{IV} = 4a^{III} - 6a^{II} + 4a^{I} - a$$
$$a^{V} = 4a^{IV} - 6a^{III} + 4a^{II} - a^{I}$$
&c.

Pertinebunt ergo & hinc ad ſeries recurrentes idque infinitis modis, cum ſcalae relationis eſſe queant:

$$3, -3, +1;\ 4, -6, +4, -1;\ 5, -10, +10, -5, +1;$$
&c.

Similique modo intelligitur vnamquamque ſeriem huius, quam tractamus, claſſis ſimul eſſe ſeriem recurrentem innumeris modis: ſcala enim relationis erit

$$\frac{n}{1}, -\frac{n(n-1)}{1.\ 2}, +\frac{n(n-1)(n-2)}{1.\ 2.\ 3}, -\frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4},$$
&c.

dum-

dummodo n ſit numerus integer maior, quam numerus quo ordo indicatur. Orietur ergo haec ſeries quoque ex euolutione fractionis, cuius denominator eſt $(1-y)^n$, prouti in ſuperiori libro de ſeriebus recurrentibus fuſius eſt oſtenſum.

48. Quemadmodum vidimus, omnium huius claſſis ſerierum, cuiuscunque ſint ordinis, terminos generales eſſe functiones ipſius x rationales integras, ita viciſſim apparebit omnes ſeries, quarum termini generales ſint huiusmodi functiones ipſius x, ad hanc claſſem pertinere, atque tandem ad differentias conſtantes perduci. Et quidem, ſi terminus generalis fuerit functio primi gradus $ax+b$, dum ſeries inde orta erit primi ordinis ſeu arithmetica, differentias primas habebit conſtantes. Sin autem terminus generalis fuerit functio ſecundi gradus in hac forma $axx+bx+c$ contenta, tum ſeries ex eo oriunda, dum loco x ſucceſſiue numeri 1, 2, 3, 4, 5, &c. ſubſtituuntur, erit ordinis ſecundi, atque differentias ſecundas habebit conſtantes: ſimili modo, terminus generalis tertii gradus ax^3+bx^2+cx+d dabit ſeriem tertii ordinis atque ita porro.

49. Ex termino enim generali non ſolum omnes ſeriei termini inueniuntur, ſed etiam ſeries differentiarum tam primarum quam ſequentium deduci poſſunt. Cum enim, ſi ſeriei terminus primus ſubtrahatur a ſecundo, prodeat ſeriei differentiarum terminus primus: ſecundus autem, ſi ipſius ſeriei terminus ſecundus a tertio auferatur, ita ſeriei differentiarum in obtinebitur terminus,

minus, cuius index eſt x; ſi ipſius ſeriei terminus, cuius index eſt x, ſubtrahatur a ſequente cuius index eſt $x+1$. Quare ſi in termino ſeriei generali loco x ponatur $x+1$, ab hocque valore terminus generalis ſubtrahatur, remanebit terminus generalis ſeriei differentiarum: ſi igitur X fuerit ſeriei terminus generalis, erit eius differentia ΔX, (quae modo in praecedente capite oſtenſo inuenietur, ſi ſtatuatur ibi $\omega = 1$,) terminus generalis ſeriei differentiarum primarum. Simili igitur modo erit $\Delta\Delta X$ terminus generalis ſeriei differentiarum ſecundarum; $\Delta^3 X$ tertiarum, ſicque deinceps.

50. Quodſi autem terminus generalis X fuerit functio rationalis integra, in qua maximus exponens poteſtatis ipſius x ſit n; ex capite praecedente colligitur, eius differentiam ΔX fore functionem vno gradu inferiorem, nempe gradus $n-1$. Hincque porro $\Delta\Delta X$ erit functio gradus $n-2$, & $\Delta^3 X$ functio gradus $n-3$, & ita porro. Quare, ſi X fuerit functio primi gradus, vti $ax+b$, tum eius differentia ΔX erit conſtans $=a$; quae cum ſit terminus generalis ſeriei primarum differentiarum, perſpicitur ſeriem, cuius terminus generalis X ſit functio primi gradus, fore arithmeticam ſeu primi ordinis. Simili modo ſi terminus generalis X fuerit functio ſecundi gradus ob $\Delta\Delta X$ conſtantem, ſeries inde orta differentias ſecundas habebit conſtantes, eritque propterea ordinis ſecundi; ſicque perpetuo, cuius gradus fuerit functio X terminum generalem conſtituens, eiusdem ordinis erit ſeries ex eo nata.

51. Hanc ob rem ſeries poteſtatum numerorum naturalium ad differentias conſtantes perueniunt, vti ex ſequenti ſchemate fit manifeſtum.

POTEST. I.

1, 2, 3, 4, 5, 6, 7, 8, &c.

DIFFER. I.

1, 1, 1, 1, 1, 1, 1, &c.

POTEST. II.

1, 4, 9, 16, 25, 36, 49, 64, &c.

DIFFER. I.

3, 5, 7, 9, 11, 13, 15, &c.

DIFFER. II.

2, 2, 2, 2, 2, 2, &c.

POTEST. III.

1, 8, 27, 64, 125, 216, 343, &c.

DIFFER. I.

7, 19, 37, 61, 91, 127, &c.

DIFFER. II.

12, 18, 24, 30, 36, &c.

DIFFER. III.

6, 6, 6, 6, &c.

POTEST. IV.

1, 16, 81, 256, 625, 1296, 2401, &c.

DIFFER. I.

15, 65, 175, 369, 671, 1105, &c.

DIFFER. II.

50, 110, 194, 302, 434, &c.

DIFFER. III.

60, 84, 108, 132, &c.

DIFFER. IV.

24, 24, 24, &c.

Quae

Quae igitur in capite praecedente de differentiis cuiusque ordinis inueniendis ſunt praecepta, ea hic inſeruient ad terminos generales differentiarum quarumuis, quae ex ſeriebus naſcuntur, inueniendos.

52. Si terminus generalis cuiusquam ſeriei fuerit cognitus, eius ope non ſolum omnes eius termini in infinitum inueniri, ſed etiam ſeries retro continuari, eiusque termini, quorum exponentes ſint numeri negatiui, exhiberi poterunt, loco x numeros negatiuos ſubſtituendo: ſic, ſi terminus generalis fuerit $\frac{xx+3x}{2}$, ponendo loco x tam negatiuos quam affirmatiuos indices, ſeries vtrinque continuata erit huiusmodi.

INDICES.

&c. −5, −4, −3, −2, −1, 0, 1, 2, 3, 4, 5, 6, &c.

SERIES.

&c. +5, +2, 0, −1, −1, 0, 2, 5, 9, 14, 20, 27, &c.

DIFFER. I.

−3, −2, −1, 0, 1, 2, 3, 4, 5, 6, 7, &c.

DIFFER. II.

1, 1, 1, 1, 1, 1, 1, 1, 1, 1 &c.

Cum igitur ex differentiis terminus generalis formetur, quaeque ſeries ex differentiis retro continuari poterit; ita quidem, vt, ſi differentiae tandem fiant conſtantes, hi termini finite exhiberi, contra vero per expreſſionem infinitam aſſignari queant. Quin etiam ex termino gene-

rali ii termini, quorum indices ſunt fracti, definientur, in quo ſerierum INTERPOLATIO continetur.

53. His de termino ſerierum generali monitis, progrediamur ad ſummam, ſeu terminum ſummatorium ſerierum cuiusque ordinis inueſtigandum. Propoſita autem quacunque ſerie, TERMINUS *ſummatorius* eſt functio ipſius x, quae aequalis eſt ſummae tot terminorum ſeriei, quot vnitates continet numerus x. Ita ergo terminus ſummatorius erit comparatus, vt ſi ponatur $x = 1$, prodeat terminus primus ſeriei; ſin autem ponatur $x = 2$, vt prodeat ſumma primi & ſecundi; facto autem $x = 3$, ſumma primi, ſecundi ac tertii; ſicque deinceps. Hinc, ſi ex ſerie propoſita noua ſeries formetur, cuius primus terminus aequalis ſit primo illius, ſecundus aequalis ſummae duorum, tertius aequalis ſummae trium, atque ita porro, haec noua ſeries vocatur illius *ſummatrix*, huiusque ſeriei ſummatricis terminus generalis erit terminus ſummatorius ſeriei propoſitae: ex quo inuentio termini ſummatorii ad inuentionem termini generalis reuocatur.

54. Sit ergo ſeries propoſita haec

$$a,\quad a^{\mathrm{I}},\quad a^{\mathrm{II}},\quad a^{\mathrm{III}},\quad a^{\mathrm{IV}},\quad a^{\mathrm{V}},\quad \&c.$$

huiusque ſeriei ſummatrix ſit

$$A,\quad A^{\mathrm{I}},\quad A^{\mathrm{II}},\quad A^{\mathrm{III}},\quad A^{\mathrm{IV}},\quad A^{\mathrm{V}},\quad \&c.$$

erit ex eius natura modo expoſita:

A =

$$A = a$$
$$A^{I} = a + a^{I}$$
$$A^{II} = a + a^{I} + a^{II}$$
$$A^{III} = a + a^{I} + a^{II} + a^{III}$$
$$A^{IV} = a + a^{I} + a^{II} + a^{III} + a^{IV}$$
&c.

Hinc ſeriei ſummatricis differentiae erunt:

$A^{I} - A = a^{I}$; $A^{II} - A^{I} = a^{II}$; $A^{III} - A^{II} = a^{III}$; &c.

vnde ſeries propoſita termino primo minuta erit ſeries differentiarum primarum ſeriei ſummatricis. Quodſi igitur ſeriei ſummatrici praefigatur terminus, $= 0$ vt habeatur:

$0, A, A^{I}, A^{II}, A^{III}, A^{IV}, A^{V},$ &c.

huius ſeries primarum differentiarum erit ipſa ſeries propoſita:

$a, a^{I}, a^{II}, a^{III}, a^{IV}, a^{V},$ &c.

55. Hanc ob rem ſeriei propoſitae differentiae primae, erunt differentiae ſecundae ſummatricis, atque differentiae ſecundae illius erunt differentiae tertiae huius, tertiae autem illius quartae huius, atque ita porro. Quare, ſi ſeries propoſita tandem habeat differentias conſtantes, tunc etiam eius ſummatrix ad differentias conſtantes deducetur, eritque igitur ſeries eiusdem naturae, at vno ordine ſuperior. Huiusmodi ergo ſerierum perpetuo terminus ſummatorius exhiberi poterit per expreſſionem finitam. Namque terminus generalis ſeriei:

$0, A, A^{I}, A^{II}, A^{III}, A^{IV},$ &c.

ſeu is, qui indici x conuenit exhibebit ſummam $x - 1$

terminorum ſeriei huius a, a^{I}, a^{II}, a^{III}, a^{IV}, &c. atque ſi tum loco x ſcribatur $x+1$, orietur ſumma x terminorum, ipſeque terminus ſummatorius.

56. Sit igitur Seriei propoſitae

a, a^{I}, a^{II}, a^{III}, a^{IV}, a^{V}, a^{VI}, &c.

Series differentiarum primarum

b, b^{I}, b^{II}, b^{III}, b^{IV}, b^{V}, b^{VI}, &c.

Series differentiarum ſecundarum

c, c^{I}, c^{II}, c^{III}, c^{IV}, c^{V}, c^{VI}, &c.

Series differentiarum tertiarum

d, d^{I}, d^{II}, d^{III}, d^{IV}, d^{V}, d^{VI}, &c.

ſicque porro donec ad differentias conſtantes perueniatur. Deinde formetur ſeries ſummatrix, quae cum praefixa o in locum termini primi, cum ſuis differentiis continuis ſe habebit ſequenti modo:

INDICES.

1, 2, 3, 4, 5, 6, 7, &c.

SUMMATRIX.

o, A, A^{I}, A^{II}, A^{III}, A^{IV}, A^{V}, &c.

SERIES PROPOSITA.

a, a^{I}, a^{II}, a^{III}, a^{IV}, a^{V}, a^{VI}, &c.

DIFFER. I.

b, b^{I}, b^{II}, b^{III}, b^{IV}, b^{V}, b^{VI}, &c.

DIFFER. II.

c, c^{I}, c^{II}, c^{III}, c^{IV}, c^{V}, c^{VI}, &c.

DIFFER. III.

d, d^{I}, d^{II}, d^{III}, d^{IV}, d^{V}, d^{VI}, &c.

erit

erit ſeriei ſummatricis terminus generalis, ſeu qui indici x reſpondet

$$0 + (x-1)a + \frac{(x-1)(x-2)}{1.\ 2}b + \frac{(x-1)(x-2)(x-3)}{1.\ 2.\ 3}c + \&c.$$

qui ſimul exhibet ſummam $x - 1$ terminorum ſeriei propoſitae, a, a^{I}, a^{II}, a^{III}, a^{IV}, &c.

57. Quod ſi ergo in hac ſumma loco $x - 1$ ſcribatur x, prodibit ſeriei propoſitae terminus ſummatorius ſummam x terminorum complectens

$$= xa + \frac{x(x-1)}{1.\ 2}b + \frac{x(x-1)(x-2)}{1.\ 2.\ 3}c + \frac{x(x-1)(x-2)(x-3)}{1.\ 2.\ 3.\ 4}d + \&c.$$

Hinc, ſi litterae b, c, d, e, valores ipſis aſſignatos retineant, erit

SERIEI.

a, a^{I}, a^{II}, a^{III}, a^{IV}, a^{V}, &c.

TERMINUS GENERALIS.

$$a + (x-1)b + \frac{(x-1)(x-2)}{1.\ 2}c + \frac{(x-1)(x-2)(x-3)}{1.\ 2.\ 3}d + \frac{(x-1)(x-2)(x-3)(x-4)}{1.\ 2.\ 3.\ 4}e + \&c.$$

ET TERMINUS SUMMATORIUS.

$$xa + \frac{x(x-1)}{1.\ 2}b + \frac{x(x-1)(x-2)}{1.\ 2.\ 3}c + \frac{x(x-1)(x-2)(x-3)}{1.\ 2.\ 3.\ 4}d + \&c.$$

Inuento ergo ſeriei cuiusuis ordinis hoc, quem oſtendimus, modo termino generali, non difficulter ex eo terminus ſummatorius reperietur, quippe qui ex iisdem differentiis conflatur.

58. Hic

58. Hic modus terminum ſummatorium per differentias ſeriei inueniendi imprimis ad eiusmodi ſeries, quae tandem ad differentias conſtantes deducunt, eſt accommodatus; in aliis enim caſibus expreſſio finita non reperitur. Quodſi autem ea, quae ante de indole termini ſummatorii ſunt expoſita, attentius perpendamus, alius modus ſe offert terminum ſummatorium immediate ex termino generali inueniendi, qui multo latius patet, atque in infinitis caſibus ad expreſſiones finitas deducit, quibus prior modus infinitas exhibet. Sit enim propoſita ſeries quaecunque.

$$a,\ b,\ c,\ d,\ e,\ f,\ \&c.$$

cuius terminus generalis, ſeu indici x reſpondens ſit $= X$; terminus autem ſummatorius ſit $= S$, qui cum ſummam tot terminorum ab initio exhibeat, quot numerus x continet vnitates, erit ſumma $x - 1$ terminorum $= S - X$; eritque adeo X differentia expreſſionis $S - X$, cum relinquatur, ſi haec a ſequente S ſubtrahatur.

59. Cum igitur ſit $X = \Delta(S - X)$ differentia eo modo ſumta, quem capite praecedente docuimus, hoc tantum diſcrimine, vt quantitas illa conſtans ω hic nobis ſit $= 1$. Quare, ſi ad ſummas regrediamur, erit $\Sigma X = S - X$, ideoque terminus ſummatorius quaeſitus

$$S = \Sigma X + X + C.$$

Quaeri ergo debet ſumma functionis X methodo ante tradita, ad eamque addi ipſe terminus generalis X, eritque aggregatum terminus ſummatorius. Quoniam autem

in

in ſummis ſumendis inuoluitur quantitas conſtans, ſiue addenda ſiue ſubtrahenda; ea ad praeſentem caſum accommodari debebit. Manifeſtum autem eſt, ſi ponatur $x = 0$, quo caſu numerus terminorum ſummandorum eſt nullus, ſummam quoque fore nullam; ex quo quantitas illa conſtans C ita determinari debebit, vt poſito $x = 0$, fiat quoque $S = 0$. Poſitis ergo in illa aequatione $S = \Sigma X + X + C$ tam $S = 0$ quam $x = 0$, valor ipſius C invenietur.

60 Quoniam ergo hic totum negotium ad ſummationem functionum ſupra monſtratam reducitur, ponendo $\omega = 1$, exinde depromamus ſummationes traditas; ac primo quidem pro poteſtatibus ipſius x erit

$$\Sigma x^0 = \Sigma 1 = x$$

$$\Sigma x = \tfrac{1}{2}x^2 - \tfrac{1}{2}x$$

$$\Sigma x^2 = \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + \tfrac{1}{6}x$$

$$\Sigma x^3 = \tfrac{1}{4}x^4 - \tfrac{1}{2}x^3 + \tfrac{1}{4}x^2$$

$$\Sigma x^4 = \tfrac{1}{5}x^5 - \tfrac{1}{2}x^4 + \tfrac{1}{3}x^3 - \tfrac{1}{30}x$$

$$\Sigma x^5 = \tfrac{1}{6}x^6 - \tfrac{1}{2}x^5 + \tfrac{5}{12}x^4 - \tfrac{1}{12}x^2$$

$$\Sigma x^6 = \tfrac{1}{7}x^7 - \tfrac{1}{2}x^6 + \tfrac{1}{2}x^5 - \tfrac{1}{6}x^3 + \tfrac{1}{42}x$$

quibus accenſeatur ſummatio generalis poteſtatis x^n §. 29. tradita, dummodo ibi vbique loco ω vnitas ſcribatur. Harum ergo formularum ope omnium ſerierum, quarum termini generales ſunt functiones rationales integrae ipſius x, termini ſummatorii expedite inueniri poterunt.

 61. De-

61. Denotet S.X terminum ſummatorium ſeriei, cuius terminus generalis eſt $= X$; eritque, vt vidimus,

$$S.X = \Sigma X + X + C$$

dummodo conſtans C ita aſſumatur, vt terminus ſummatorius S.X euaneſcat poſito $x = 0$. Hinc igitur terminos ſummatorios ſerierum poteſtatum, ſeu quarum termini generales comprehenduntur in hac forma x^n exprimamus. Poſito itaque

$$S.x^n = 1 + 2^n + 3^n + 4^n + \ldots + x^n$$

erit

$$S.x^n = \frac{1}{n+1}x^{n+1} + \tfrac{1}{2}x^n + \tfrac{1}{2}\cdot\frac{n}{2.3}x^{n-1} - \tfrac{1}{6}\cdot\frac{n(n-1)(n-2)}{2.\,3.\,4.\,5}x^{n-3}$$

$$+ \tfrac{1}{6}\cdot\frac{n(n-1)(n-2)(n-3)(n-4)}{2.\,3.\,4.\,5.\,6.\,7}x^{n-5} - \tfrac{3}{10}\cdot\frac{n(n-1)\ \ldots\ (n-6)}{2.\,3\ \ldots\ 8.\,9}x^{n-7}$$

$$+ \tfrac{5}{6}\cdot\frac{n(n-1)\ \ldots\ (n-8)}{2.\,3.\,4\ \ldots\ 10.11}x^{n-9} - \tfrac{691}{210}\cdot\frac{n(n-1)\ \ldots\ (n-10)}{2.3\ \ldots\ 12.13}x^{n-11}$$

$$+ \tfrac{35}{2}\cdot\frac{n(n-1)\ \ldots\ (n-12)}{2.3\ \ldots\ 14.15}x^{n-13} - \tfrac{3617}{30}\cdot\frac{n(n-1)\ \ldots\ (n-14)}{2.3\ \ldots\ 16.17}x^{n-15}$$

$$+ \frac{43867}{42}\cdot\frac{n(n-1)\ \ldots\ldots\ (n-16)}{2.\,3\ \ldots\ldots\ 18.19}x^{n-17}$$

$$- \frac{1222277}{110}\cdot\frac{n(n-1)\ \ldots\ldots\ (n-18)}{2.\,3\ \ldots\ldots\ 20.21}x^{n-19}$$

$$+ \frac{854513}{6}\cdot\frac{n(n-1)\ \ldots\ldots\ (n-20)}{2.\,3\ \ldots\ldots\ 22.23}x^{n-21}$$

$$- \frac{1181820455}{546}\cdot\frac{n(n-1)\ \ldots\ldots\ (n-22)}{2.\,3\ \ldots\ldots\ 24.25}x^{n-23}$$

$$+ \frac{76977927}{2}\cdot\frac{n(n-1)\ \ldots\ldots\ (n-24)}{2.\,3\ \ldots\ldots\ 26.\,27}x^{n-25}$$

&c.

62.

62. Hinc ergo ſummae pro variis ipſius n valoribus ita ſe habebunt:

$$S.x^0 = x$$

$$S.x^1 = \tfrac{1}{2}x^2 + \tfrac{1}{2}x$$

$$S.x^2 = \tfrac{1}{3}x^3 + \tfrac{1}{2}xx + \tfrac{1}{6}x$$

$$S.x^3 = \tfrac{1}{4}x^4 + \tfrac{1}{2}x^3 + \tfrac{1}{4}x^2$$

$$S.x^4 = \tfrac{1}{5}x^5 + \tfrac{1}{2}x^4 + \tfrac{1}{3}x^3 - \tfrac{1}{30}x$$

$$S.x^5 = \tfrac{1}{6}x^6 + \tfrac{1}{2}x^5 + \tfrac{5}{12}x^4 - \tfrac{1}{12}x^2$$

$$S.x^6 = \tfrac{1}{7}x^7 + \tfrac{1}{2}x^6 + \tfrac{1}{2}x^5 - \tfrac{1}{6}x^3 + \tfrac{1}{42}x$$

$$S.x^7 = \tfrac{1}{8}x^8 + \tfrac{1}{2}x^7 + \tfrac{7}{12}x^6 - \tfrac{7}{24}x^4 + \tfrac{1}{12}x^2$$

$$S.x^8 = \tfrac{1}{9}x^9 + \tfrac{1}{2}x^8 + \tfrac{2}{3}x^7 - \tfrac{7}{15}x^5 + \tfrac{2}{9}x^3 - \tfrac{1}{30}x$$

$$S.x^9 = \tfrac{1}{10}x^{10} + \tfrac{1}{2}x^9 + \tfrac{3}{4}x^8 - \tfrac{7}{10}x^6 + \tfrac{1}{2}x^4 - \tfrac{3}{20}x^2$$

$$S.x^{10} = \tfrac{1}{11}x^{11} + \tfrac{1}{2}x^{10} + \tfrac{5}{6}x^9 - x^7 + x^5 - \tfrac{1}{2}x^3 + \tfrac{5}{66}x$$

$$S.x^{11} = \tfrac{1}{12}x^{12} + \tfrac{1}{2}x^{11} + \tfrac{11}{12}x^{10} - \tfrac{11}{8}x^8 + \tfrac{11}{6}x^6 - \tfrac{11}{8}x^4 + \tfrac{5}{12}x^2$$

$$S.x^{12} = \tfrac{1}{13}x^{13} + \tfrac{1}{2}x^{12} + x^{11} - \tfrac{11}{6}x^9 + \tfrac{22}{7}x^7 - \tfrac{33}{10}x^5 + \tfrac{5}{3}x^3 - \tfrac{691}{2730}x$$

$$S.x^{13} = \tfrac{1}{14}x^{14} + \tfrac{1}{2}x^{13} + \tfrac{13}{12}x^{12} - \tfrac{143}{60}x^{10} + \tfrac{143}{28}x^8 - \tfrac{143}{20}x^6 + \tfrac{65}{12}x^4 - \tfrac{691}{420}x^2$$

$$S.x^{14} = \tfrac{1}{15}x^{15} + \tfrac{1}{2}x^{14} + \tfrac{7}{6}x^{13} - \tfrac{91}{30}x^{11} + \tfrac{143}{18}x^9 - \tfrac{143}{10}x^7 + \tfrac{91}{6}x^5 - \tfrac{691}{90}x^3 + \tfrac{7}{6}x$$

$$S.x^{15} = \tfrac{1}{16}x^{16} + \tfrac{1}{2}x^{15} + \tfrac{5}{4}x^{14} - \tfrac{91}{24}x^{12} + \tfrac{143}{12}x^{10} - \tfrac{429}{16}x^8 + \tfrac{455}{12}x^6 - \tfrac{691}{24}x^4 + \tfrac{35}{4}x^2$$

$$S.x^{16} = \tfrac{1}{17}x^{17} + \tfrac{1}{2}x^{16} + \tfrac{4}{3}x^{15} - \tfrac{14}{3}x^{13} + \tfrac{52}{3}x^{11} - \tfrac{143}{3}x^9 + \tfrac{260}{3}x^7 - \tfrac{1382}{15}x^5 + \tfrac{140}{3}x^3 - \tfrac{3617}{510}x$$

&c.

quae

quae ſummae ex forma generali vsque ad poteſtatem viceſimam nonam continuari poſſunt. Atque ad huc vlterius progredi liceret, ſi coefficientes illi numerici vlterius eſſent eruti.

63. Ceterum, in his formulis lex quaedam obſervatur, cuius ope quaelibet ex praecedente facile inueniri poteſt, excepto tantum termino vltimo, ſi in eo poteſtas ipſius x prima contineatur: tum enim in ſumma ſequente vnus terminus inſuper accedit. Hoc autem omiſſo, ſi fuerit

$$S.x^n = \alpha x^{n+1} + \beta x^n + \gamma x^{n-1} - \delta x^{n-3} + \varepsilon x^{n-5} - \zeta x^{n-7} + \eta x^{n-9} - \&c.$$

erit ſequens ſumma:

$$S.x^{n+1} = \frac{n+1}{n+2}\alpha x^{n+2} + \frac{n+1}{n+1}\beta x^{n+1} + \frac{n+1}{n}\gamma x^n - \frac{n+1}{n-2}\delta x^{n-2} + \frac{n+1}{n-4}\varepsilon x^{n-4} - \frac{n+1}{n-6}\zeta x^{n-6} + \frac{n+1}{n-8}\eta x^{n-8} - \&c.$$

vnde ſi n fuerit numerus par, ſequens ſumma vera prodit: at ſi n fuerit numerus impar, tum in ſequente ſumma praeterea deſiderabitur terminus vltimus, cuius forma erit $\pm \Phi x$. Interim tamen hic ſine aliis ſubſidiis ita inueniri poterit. Cum enim ſi ponatur $x = 1$, ſumma vnici tantum termini, (hoc eſt terminus primus, qui erit $= 1$,) oriri debeat: ponatur in omnibus terminis iam inventis $x = 1$, ipſaque ſumma ſtatuatur $= 1$, quo facto valor ipſius Φ elicietur, eoque inuento vlterius progredi licebit. Atque hoc pacto omnes iſtae ſummae inueniri potuiſſent. Sic, cum ſit

S. x^5

$$S.x^5 = \tfrac{1}{6}x^6 + \tfrac{1}{2}x^5 + \tfrac{5}{12}x^4 - \tfrac{1}{12}x^2$$

erit

$$S.x^6 = \tfrac{6}{7}\cdot\tfrac{1}{6}x^7 + \tfrac{6}{6}\cdot\tfrac{1}{2}x^6 + \tfrac{6}{5}\cdot\tfrac{5}{12}x^5 - \tfrac{6}{3}\cdot\tfrac{1}{12}x^3 + \Phi x$$

ſeu

$$S.x^6 = \tfrac{1}{7}x^7 + \tfrac{1}{2}x^6 + \tfrac{1}{2}x^5 - \tfrac{1}{6}x^3 + \Phi x.$$

Ponatur nunc $x = 1$, fiet $1 = \frac{1}{7} + \frac{1}{2} + \frac{1}{2} - \frac{1}{6} + \Phi$ ideoque $\Phi = \frac{1}{6} - \frac{1}{7} = \frac{1}{42}$, vti ex forma generali inuenimus.

64. Ope harum formularum ſummatoriarum nunc facile omnium ſerierum, quarum termini generales ſunt functiones ipſius x rationales integrae, termini ſummatorii inueniri poterunt, hocque multo expeditius, quam praecedente methodo per differentias.

EXEMPLUM I.

Inuenire terminum ſummatorium huius ſeriei

2, 7, 15, 26, 40, 57, 77, 100, 126, &c.

cuius terminus generalis eſt

$$\frac{3xx + x}{2}.$$

Cum terminus generalis conſtet duobus membris, quaeratur pro vtroque terminus ſummatorius ex formulis ſuperioribus

$$S.\tfrac{3}{2}xx = \tfrac{1}{2}x^3 + \tfrac{3}{4}xx + \tfrac{1}{4}x$$

&

$$S.\tfrac{1}{2}x = \quad . \quad . \quad . \quad \tfrac{1}{4}xx + \tfrac{1}{4}x$$

eritque

$$S.\frac{3xx+x}{2} = \tfrac{1}{2}x^3 + xx + \tfrac{1}{2}x = \tfrac{1}{2}x(x+1)^2$$

 qui

qui eſt terminus ſummatorius quaeſitus. Sic, ſi ponatur $x = 5$, erit $\frac{5}{2} \cdot 6^2 = 90$, ſumma quinque terminorum

$$2 + 7 + 15 + 26 + 40 = 90.$$

EXEMPLUM II.

Inuenire terminum ſummatorium ſeriei

$$1, \; 27, \; 125, \; 343, \; 729, \; 1331, \; \&c.$$

quae continet cubos numerorum imparium.

Terminus generalis huius ſeriei eſt

$$= (2x-1)^3 = 8x^3 - 12xx + 6x - 1,$$

vnde terminus ſummatorius ſequenti modo colligetur.

$$+ 8.\, S.x^3 = 2x^4 + 4x^3 + 2x^2$$

&

$$-12.\, S.x^2 = \; . \; -4x^3 - 6x^2 - 2x$$

atque

$$+6.\, S.x = \; . \; . \; . \; . \; . + 3x^2 + 3x$$

denique

$$-1.\, S.x^0 = \; . \; . \; . \; . \; . \; . \; . \; . - x.$$

Erit ſcilicet ſumma quaeſita $= 2x^4 - x^2 = xx(2xx-1)$.

Vti, ſi ponatur $x = 6$ erit $36.71 = 2556$ ſumma ſex terminorum ſeriei propoſitae $= 1 + 27 + 125 + 343 + 729 + 1331 = 2556$.

65. Quod ſi terminus generalis fuerit productum ex factoribus ſimplicibus, tum terminus ſummatorius facilius reperietur per ea, quae ſupra §. 32. & ſequentibus ſunt tradita. Cum enim, poſito $\omega = 1$, ſit

$\Sigma (x$

$$\Sigma\,(x+n) = \tfrac{1}{2}(x+n-1)\,(x+n)$$

&

$$\Sigma\,(x+n)\,(x+n+1) = \tfrac{1}{3}(x+n-1)\,(x+n)\,(x+n+1)$$

atque

$$\Sigma(x+n)(x+n+1)(x+n+2) = \tfrac{1}{4}(x+n-1)(x+n)(x+n+1)(x+n+2)$$

&c.

ſi ad has ſummas ipſos terminos generales addamus, ſimulque conſtantem adiiciamus, quae poſito $x = 0$, reddat terminum ſummatorium euaneſcentem, ſequentes obtinebimus terminos ſummatorios.

$$S.(x+n) = \tfrac{1}{2}(x+n)\,(x+n+1) - \tfrac{1}{2}\,n(n+1)$$

&

$$S.(x+n)(x+n+1) = \tfrac{1}{3}(x+n)(x+n+1)(x+n+2) - \tfrac{1}{3}n(n+1)(n+2)$$

atque

$$S.(x+n)(x+n+1)(x+n+2) = \tfrac{1}{4}(x+n)(x+n+1)(x+n+2)(x+n+3)$$
$$- \tfrac{1}{4}n(n+1)(n+2)(n+3)$$

ſicque porro.

Si ergo fuerit vel $n = 0$ vel $n = -1$, quantitas conſtans in his ſummis euaneſcit.

66. Seriei ergo 1, 2, 3, 4, 5, &c. cuius terminus generalis eſt $= x$; terminus ſummatorius erit $= \tfrac{1}{2}x(x+1)$ ſeriesque ſummatrix haec: 1, 3, 6, 10, 15, &c. cuius porro terminus ſummatorius erit $= \frac{x(x+1)(x+2)}{1.\ 2.\ 3}$, & ſeries ſummatrix haec: 1, 4, 10, 20, 35, &c. Haec vero denuo terminum ſummatorium habebit $= \frac{x(x+1)(x+2)(x+3)}{1.\ 2.\ 3.\ 4}$, qui

erit

erit terminis generalis ſeriei 1, 5, 15, 35, 70, &c. huiusque terminus ſummatorius erit $= \frac{x(x+1)(x+2)(x+3)(x+4)}{1.\ 2.\ 3.\ 4.\ 5}$. Hae autem ſeries prae reliquis probe ſunt notandae, quoniam earum vbique ampliſſimus eſt vſus. Ex his enim deſumuntur coefficientes binomii ad dignitates eleuati, qui quam late pateant, cuique in his rebus parum verſato abunde conſtat.

67. Ex his etiam illi termini ſummatorii, quos ante ex differentiis elicuimus, facile inueniuntur. Cum enim ibi terminum generalem ſequenti forma inuenerimus expreſſum

$$a + \frac{(x-1)}{1} b + \frac{(x-1)(x-2)}{1.\ 2} c + \frac{(x-1)(x-2)(x-3)}{1.\ 2.\ 3} d + \&c.$$

ſi cuiusque membri terminum ſummatorium quaeramus eosque omnes addamus, habebimus terminum ſummatorium huic termino generali conuenientem. Sic cum ſit

$$S\ 1 = x$$

&

$$S\ (x-1) = \tfrac{1}{2}\, x(x-1)$$

atque

$$S\ (x-1)(x-2) = \tfrac{1}{3}\, x(x-1)(x-2)$$

&

$$S\ (x-1)(x-2)(x-3) = \tfrac{1}{4}\, x(x-1)(x-2)(x-3)$$

&c.

erit terminus ſummatorius quaeſitus:

$$xa + \frac{x(x-1)}{1.\ 2} b + \frac{x(x-1)(x-2)}{1.\ 2.\ 3} c + \frac{x(x-1)(x-2)(x-3)}{1.\ 2.\ 3.\ 4} d + \&c.$$

quae

quae forma non diſcrepat ab ea, quam ante ex differentiis obtinuimus.

68. Deinde etiam haec terminorum ſummatoriorum inuentio ad fractiones accommodari poteſt: quia enim ſupra §. 34. inuenimus eſſe, ponendo $\omega = 1$

$$\Sigma \frac{1}{(x+n)(x+n+1)} = -1 \cdot \frac{1}{x+n}$$

erit

$$S. \frac{1}{(x+n)(x+n+1)} = -1 \cdot \frac{1}{x+n+1} + \frac{1}{n+1}$$

Simili modo, ſi ad ſummas ſupra inuentas ipſos terminos generales addamus, ſeu quod idem eſt, ſi in illis expreſſionibus loco x ponamus $x+1$ habebimus

$$S. \frac{1}{(x+n)(x+n+1)(x+n+2)} = -\frac{1}{2} \cdot \frac{1}{(x+n+1)(x+n+2)} + \frac{1}{2} \cdot \frac{1}{(n+1)(n+2)}$$

&

$$S. \frac{1}{(x+n)(x+n+1)(x+n+2)(x+n+3)} = -\frac{1}{3} \cdot \frac{1}{(x+n+1)(x+n+2)(x+n+3)} + \frac{1}{3} \cdot \frac{1}{(n+1)(n+2)(n+3)}$$

quae formae facile pro lubitu vlterius continuantur.

69. Quia erit $S. \frac{1}{(x+n)(x+n+1)} = \frac{1}{n+1} - \frac{1}{x+n+1}$ erit quoque

$$S. \frac{1}{x+n} - S. \frac{1}{x+n+1} = \frac{1}{n+1} - \frac{1}{x+n+1}.$$

 Etſi

Etſi ergo neuter horum duorum terminorum ſummatoriorum ſeorſim exhiberi poteſt, tamen eorum differentia cognoſcitur; hincque in pluribus caſibus ſummae ſerierum ſatis expedite aſſignantur: id quod vſu venit, ſi terminus generalis fuerit fractio, cuius denominator in factores ſimplices reſolui poteſt. Tum enim tota fractio in fractiones partiales reſoluatur; quo facto, ope huius lemmatis mox patebit, vtrum terminus ſummatorius exhiberi queat nec ne?

EXEMPLUM I.

Inuenire terminum ſummatorium ſeriei huius:

$$1 + \tfrac{1}{3} + \tfrac{1}{6} + \tfrac{1}{10} + \tfrac{1}{15} + \tfrac{1}{21} + \&c.$$

cuius terminus generalis eſt $= \frac{2}{xx+x}$.

Terminus iſte generalis per reſolutionem reducitur ad hanc formam $\frac{2}{x} - \frac{2}{x+1}$. Hinc terminus ſummatorius erit $= 2\,S.\,\frac{1}{x} - 2\,S.\,\frac{1}{x+1}$, qui ergo per praecedens lemma erit $= 2 - \frac{2}{x+1} = \frac{2x}{x+1}$. Sic, ſi ſit $x = 4$, erit $\frac{8}{5} = 1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10}$.

EXEMPLUM II.

Quaeratur terminus ſummatorius ſeriei huius:

$$\tfrac{1}{5},\ \tfrac{1}{21},\ \tfrac{1}{45},\ \tfrac{1}{77},\ \tfrac{1}{117},\ \&c.$$

cuius terminus generalis eſt $= \frac{1}{4xx+4x-3}$.

Quia

Quia termini generalis denominator habet factores $2x - 1$ & $2x + 3$, is reſoluetur in has partes:

$$\frac{1}{4}\cdot\frac{1}{2x-1} - \frac{1}{4}\cdot\frac{1}{2x+3} = \frac{1}{8}\cdot\frac{1}{x-\frac{1}{2}} - \frac{1}{8}\cdot\frac{1}{x+\frac{3}{2}}.$$

At eſt

$$S.\frac{1}{x-\frac{1}{2}} = S.\frac{1}{x+\frac{1}{2}} + 2 - \frac{1}{x+\frac{1}{2}}$$

&

$$S.\frac{1}{x+\frac{1}{2}} = S.\frac{1}{x+\frac{3}{2}} + \frac{2}{3} - \frac{1}{x+\frac{3}{2}}$$

ergo

$$S.\frac{1}{x-\frac{1}{2}} - S.\frac{1}{x+\frac{3}{2}} = 2 + \frac{2}{3} - \frac{1}{x+\frac{1}{2}} - \frac{1}{x+\frac{3}{2}}$$

cuius pars octaua dabit terminum ſummatorium quaeſitum nempe

$$\frac{1}{4} + \frac{1}{12} - \frac{1}{8x+4} - \frac{1}{8x+12} = \frac{x}{4x+2} + \frac{x}{3(4x+6)} =$$

$$= \frac{x(4x+5)}{3(2x+1)(2x+3)}.$$

70. Quoniam numeri figurati, quos coefficientes binomii ad dignitates euecti praebent, prae ceteris notari merentur, ſummas ſerierum exhibeamus, quarum numeratores ſint $= 1$, denominatores vero numeri figurati; id quod ex §. 68. facile fiet. Seriei ergo cuius

Terminus generalis eſt	Terminus ſummatorius erit
$\frac{1.\,2}{x(x+1)}$	$\frac{2}{1}-\frac{2}{x+1}$
$\frac{1.\,2.\,3}{x(x+1)(x+2)}$	$\frac{3}{2}-\frac{1.\,3}{(x+1)(x+2)}$
$\frac{1.\,2.\,3.\,4}{x(x+1)(x+2)(x+3)}$	$\frac{4}{3}-\frac{1.\,2.\,4}{(x+1)(x+2)(x+3)}$
$\frac{1.\,2.\,3.\,4.\,5}{x(x+1)(x+2)(x+3)(x+4)}$	$\frac{5}{4}-\frac{1.\,2.\,3.\,5}{(x+1)(x+2)(x+3)(x+4)}$
&c.	&c.

vnde lex, qua iſtae expreſſiones progrediuntur, ſponte apparet. Neque vero hinc terminus ſummatorius, qui conueniat termino generali $\frac{1}{x}$, colligi poteſt, quippe qui per formulam definitam exprimi nequit.

71. Quoniam terminus ſummatorius praebet ſummam tot terminorum, quot vnitates continentur in indice x; manifeſtum eſt harum ſerierum in infinitum continuatarum ſummas obtineri, ſi ponatur index x infinitus: quo caſu expreſſionum modo inuentarum termini poſteriores, ob denominatores in infinitum abeuntes, euaneſcent.

Hinc

Hinc iſtae ſeries infinitae finitas habebunt ſummas, quae erunt

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \text{\&c.} = \frac{2}{1}$$

$$1 + \frac{1}{4} + \frac{1}{10} + \frac{1}{20} + \frac{1}{35} + \text{\&c.} = \frac{3}{2}$$

$$1 + \frac{1}{5} + \frac{1}{15} + \frac{1}{35} + \frac{1}{70} + \text{\&c.} = \frac{4}{3}$$

$$1 + \frac{1}{6} + \frac{1}{21} + \frac{1}{56} + \frac{1}{126} + \text{\&c.} = \frac{5}{4}$$

$$1 + \frac{1}{7} + \frac{1}{28} + \frac{1}{84} + \frac{1}{210} + \text{\&c.} = \frac{6}{5}$$

&c.

Omnium ergo ſerierum, quarum termini ſummatorii habentur, in infinitum continuatarum ſummae exhiberi poterunt poſito $x = \infty$, dummodo hoc caſu ſummae fiant finitae: quod quidem euenit, ſi in termino ſummatorio x tot habeat dimenſiones in denominatore, quot habet in numeratore.

CAPUT III.

DE INFINITIS ATQUE INFINITE PARUIS.

72.

Cum omnis Quantitas, quantumuis ſit magna, vlterius augeri poſſit, neque quicquam obſtet, quominus ad datam quantitatem quamcunque alia quantitas eiusdem generis addi queat; omnis quoque quantitas ſine fine augeri poterit: neque enim vnquam tam magna fiet, vt ipſi nihil amplius adiici poſſet. Nulla igitur datur quantitas tam magna, qua maior concipi nequeat: hincque extra dubium erit poſitum, *omnem quantitatem in infinitum augeri poſſe.* Qui enim hoc negauerit, is affirmare cogitur, dari limitem, quem quantitas, cum attigerit, ſuperare nequeat, atque ideo ſtatuere debebit quantitatem, cui nihil amplius adiici poſſet; quod cum ſit abſurdum atque quantitatis notioni aduerſetur, neceſſario concedendum eſt, omnem quantitatem ſine fine continuo magis, hoc eſt, in infinitum augeri poſſe.

73. In ſingulis quantitatum ſpeciebus hoc etiam clarius perſpicietur. Sic, nemo facile reperietur, qui ſtatuerit ſeriem numerorum naturalium 1, 2, 3, 4, 5, 6, &c. ita vsquam eſſe determinatam, vt vlterius continuari non poſſit. Nullus enim datur numerus, ad quem non inſuper vnitas addi, ſicque numerus ſequens maior exhiberi

que-

queat; hinc ſeries numerorum naturalium ſine fine progreditur, neque vnquam peruenitur ad numerum maximum, quo maior prorſus non detur. Simili modo linea recta nunquam eousque produci poteſt, vt inſuper vlterius prolongari non poſſet. Quibus euincitur, tam numeros in infinitum augeri, quam lineas in infinitum produci poſſe. Quae cum ſint ſpecies quantitatum, ſimul intelligitur, omni quantitate, quantumuis ſit magna, adhuc dari maiorem, hacque denuo maiorem, ſicque augendo continuo vlterius ſine fine, hoc eſt in infinitum, procedi poſſe.

74. Quanquam autem haec ſunt adeo perſpicua, vt qui ea negare vellet, ſibi ipſe contradicere deberet; tamen iſta infiniti doctrina a pluribus, qui eam explicare ſunt conati, tantopere eſt offuſcata, tantisque difficultatibus atque etiam contradictionibus obuoluta, vt, qua ſe extricarent, nulla via pateret. Ex eo, quod quantitas in infinitum augeri poſſit, quidam concluſerunt, dari reuera quantitatem infinitam, eamque ita deſcripſerunt, vt nullum amplius augmentum ſuſcipere poſſit. Hoc autem ipſo ideam quantitatis euertunt, dum eiusmodi quantitatem ſtatuunt, quae vlterius augeri nequeat. Praeterea vero ſecum ipſi infinitum admittentes pugnant; dum enim incrementi, quo quantitas ſit capax, finem faciunt, ſimul negant quantitatem ſine fine augeri poſſe; negant ergo quoque quantitatem in infinitum augeri poſſe, quoniam vtraque locutio congruit: ſicque, dum quantitatem infinitam ſtatuunt, eam ſimul tollunt. Si enim quantitas

titas ſine fine, hoc eſt in infinitum, augeri nequeat, certe nulla quantitas infinita exiſtere poterit.

75. Hinc igitur ex eo ipſo, quod omnis quantitas in infinitum augeri poſſit, ſequi videatur nullam dari quantitatem infinitam. Quantitas enim continuis incrementis aucta, infinita non euadet, niſi iam ſine fine increuerit: quod autem ſine fine fieri debet, id non tanquam iam factum concipi poteſt. Interim tamen non ſolum huiusmodi quantitatem, ad quam incrementis ſine fine congeſtis peruenitur, certo charactere indicare, ſicque debito modo in calculum inducere licet, vti mox fuſius oſtendemus; ſed etiam in mundo eiusmodi caſus exiſtere, vel ſaltem concipi poſſunt, quibus numerus infinitus actu exiſtere videatur. Sic ſi materia in infinitum ſit diuiſibilis, vti plures Philoſophi ſtatuerunt, numerus partium, quibus datum quodque materiae fruſtum conſtat, reuera erit infinitus; ſi enim ſtatueretur finitus, materia certe non in infinitum foret diuiſibilis. Simili modo ſi vniuerſus mundus eſſet infinitus, vti pluribus placuit, numerus corporum mundum componentium finitus certe eſſe non poſſet, foretque ideo quoque infinitus.

76. Haec etiamſi inter ſe pugnare videantur, tamen ſi attentius perpendantur, a cunctis incommodis liberari poterunt. Qui enim ſtatuit materiam in infinitum eſſe diuiſibilem, is negat in diuiſione materiae continua unquam ad partes tam paruas perueniri, quae vlterius diuidi nequeant: nullas ergo materia habebit partes,

ctes, vlterius indiuiduas; cum ſingulae particulae, ad quas per continuam diuiſionem iam ſit peruentum, vlterius ſe ſubdiuidi patiantur. Qui igitur dicit hoc caſu numerum partium fore infinitum, is partes vltimas, quae vlterius ſint indiuiduae, intelligit; ad quas cum nunquam perueniatur, & quae propterea nullae ſint, is has ipſas partes, quae nullae ſunt, numerare conatur. Si enim materia ſine fine continuo vlterius ſubdiuidi poteſt, partibus indiuiduis ſeu ſimplicibus prorſus caret: neque adeo quicquam ſupereſt, quod numerari queat. Hanc obrem qui materiam in infinitum diuiſibilem ſtatuit, is ſimul negat, materiam ex partibus ſimplicibus eſſe compoſitam.

77. Quod ſi autem, dum de partibus alicuius corporis ſeu materiae loquimur, non vltimas ſeu ſimplices, quippe quae nullae ſunt, intelligamus, ſed eas, quas diuiſio reuera produxit; tum, admiſſa hac hypotheſi de diuiſibilitate materiae in infinitum, vnumquodque vel minimum materiae fruſtum non ſolum in plurimas partes diſſecari, ſed etiam nullus numerus tum magnus aſſignari poterit, quo non maior partium ex illo fruſto ſectarum numerus exhiberi queat. Numerus ergo partium non quidem vltimarum, ſed quae ipſae adhuc ſint vlterius diuiſibiles, quae vnumquodque corpus componunt, omni numero aſſignabili erit maior. Simili modo, ſi vniuerſus mundus ſit infinitus, numerus corporum mundum conſtituentium pariter omni aſſignabili erit maior; qui cum finitus eſſe nequeat, ſequitur numerum

infinitum & numerum omni aſſignabili maiorem eſſe nomina ſynonyma.

78. Qui ergo hoc modo diuiſibilitatem materiae in infinitum intuetur, nullis incommodis, quae vulgo huic opinioni imputantur, ſe implicat, nihilque affirmare cogitur, quod ſanae rationi aduerſetur. Qui autem contra materiam in infinitum diuiſibilem eſſe negant, ii in maximas difficultates prolabuntur, ex quibus ſe nullo prorſus modo extrahere poſſunt. Statuere enim coguntur vnumquodque corpus nonniſi in certum partium numerum diſſecari poſſe, ad quas ſi fuerit peruentum, nulla diuiſio vlterior locum inueniat; quas vltimas particulas alii *atomos*, alii *monades* atque *entia ſimplicia* vocant. Cur autem iſtae vltimae particulae nullam amplius diuiſionem admittant, duplex eſſe poteſt cauſa: altera, quod omni extenſione careant; altera quod quidem ſint extenſae, ſed tamen tam durae atque ita comparatae, vt nulla vis ad eas diſſecandas ſufficiat. Vtrumuis patroni huius opinionis dicant, ſeſe aeque difficultatibus implicant.

79. Sint enim vltimae particulae omnis extenſionis expertes, ita vt partibus prorſus careant; qua explicatione quidem ideam entium ſimplicium optime tuentur. At, quemadmodum corpus ex finito huiusmodi particularum numero conſtare queat, concipi nullo modo poteſt. Ponamus pedem cubicum materiae ex mille huiusmodi entibus ſimplicibus eſſe compoſitum, huncque actu in mille partes ſecari; quae ſi ſint aequales, erunt digiti cubici: ſin autem ſint inaequales, aliae erunt ma-

maiores aliae minores. Vnus igitur digitus cubicus foret ens ſimplex, ſicque maxima reſultaret contradictio; niſi forte in digito cubico ineſſe tantum vnum ens ſimplex, reliquumque ſpatium vacuum eſſe dicere velint: at vero hoc modo continuitatem corporum tollerent, praeterquam quod iſti Philoſophi vacuum plane ex mundo profligant. Quodſi obiiciant numerum entium ſimplicium, quae pedem cubicum materiae conſtituunt, millenario longe eſſe maiorem, nihil omnino lucrantur: incommodum enim, quod ex numero millenario ſequitur, ex quouis alio numero quantumuis magno aeque manat. Hanc difficultatem Acutiſſimus LEIBNIZIVS, primus monadum inuentor, probe perſpexit, dum materiam abſolute in infinitum diuiſibilem eſſe ſtatuit. Neque ergo ante ad monades peruenire licet, quam corpus actu in infinitum ſit diuiſum. Hoc ipſo autem exiſtentiam entium ſimplicium, ex quibus corpora conſtent, penitus tollit: nam qui negat corpora ex entibus ſimplicibus eſſe compoſita, & ille qui ſtatuit corpora in infinitum eſſe diuiſibilia, in eadem prorſus ſunt ſententia.

80. Neque magis autem ſibi conſtant, ſi dicunt vltimas corporum particulas extenſas quidem eſſe, ſed ob ſummam duritiem in partes diuelli non poſſe. Cum primum enim in vltimis particulis extenſionem admittunt, eas ex partibus compoſitas eſſe ſtatuunt, quae, vtrum reuera a ſe inuicem ſeparari queant nec ne? parum refert; etiamſi nullam cauſam aſſignare poſſint, vnde tanta durities ſit orta. Nunc autem plerique, qui

diuiſibilitatem materiae in infinitum negant, hoc poſterius incommodum ſatis ſenſiſſe videntur, quia priori ideae partium vltimarum potiſſimum inhaerent; hasque difficultates aliter diluere non poſſunt, niſi aliquot leuiusculis metaphyſicis diſtinctionibus, quae maximam partem eo tendunt, vt ne conſequentiis, quae ſecundum mathematica principia formantur, fidamus: neque dimenſiones in partibus ſimplicibus adhiberi oportere regerunt. At primum demonſtrare debuiſſent, iſtas ſuas partes vltimas, quarum determinatus numerus corpus conſtituat, extenſas prorſus non eſſe.

81. Cum igitur ex hoc labyrintho exitum nullum inuenire, neque obiectionibus debito modo occurrere queant, ad diſtinctiones confugiunt, reſpondentes has obiectiones a ſenſibus atque imaginatione ſuppeditari, in hoc autem negotio ſolum intellectum purum adhiberi oportere; ſenſus autem ac ratiocinia inde pendentia ſaepiſſime fallere. Intellectus ſcilicet purus agnoſcet fieri poſſe, vt pars milleſima pedis cubici materiae omni extenſione careat, quod imaginationi abſurdum videatur. Tum vero, quod ſenſus ſaepenumero fallant, res vera quidem eſt, at nemini minus quam mathematicis opponi poteſt. Matheſis enim nos imprimis a fallacia ſenſuum defendit, atque docet obiecta, quae ſenſibus percipiuntur, aliter reuera eſſe comparata, aliter vero apparere: haecque ſcientia tutiſſima tradit praecepta, quae qui ſequuntur, ab illuſione ſenſuum immunes ſunt. Huiusmodi ergo reſponſionibus, tantum abeſt, vt Metaphyſici ſuam doctrinam tueantur, vt eam potius magis ſuſpectam efficiant.

82.

82. Verum vt ad propoſitum reuertamur, etiamſi quis neget in mundo numerum infinitum reuera exiſtere; tamen in ſpeculationibus mathematicis ſaepiſſime occurrunt quaeſtiones, ad quas, niſi numerus infinitus admittatur, reſponderi non poſſet. Sic, ſi quaeratur ſumma omnium numerorum, qui hanc ſeriem $1+2+3+4+5+$&c. conſtituunt; quia iſti numeri ſine fine progrediuntur, atque creſcunt, eorum omnium ſumma certe finita eſſe non poterit: quo ipſo efficitur, eam eſſe infinitam. Hinc, quae quantitas tanta eſt, vt omni quantitate finita ſit maior, ea non infinita eſſe nequit. Ad huiusmodi quantitatem deſignandam Mathematici vtuntur hoc ſigno ∞, quo denotatur quantitas omni quantitate finita, ſeu aſſignabili, maior. Sic cum Parabola ita definiri queat, vt dicatur eſſe Ellipſis infinite longa, recte affirmare poterimus axem Parabolae eſſe Lineam rectam infinitam.

83. Haec autem Infiniti doctrina magis illuſtrabitur, ſi, quid ſit infinite paruum Mathematicorum, expoſuerimus. Nullum autem eſt dubium, quin omnis quantitas eousque diminui queat, quoad penitus euaneſcat, atque in nihilum abeat. Sed quantitas infinite parua nil aliud eſt niſi quantitas euaneſcens, ideoque reuera erit $=0$. Conſentit quoque ea infinite paruorum definitio, qua dicuntur omni quantitate aſſignabili minora: ſi enim quantitas tam fuerit parua, vt omni quantitate aſſignabili ſit minor, ea certe non poterit non eſſe nulla; namque niſi eſſet $=0$, quantitas aſſignari poſſet ipſi aequalis, quod eſt contra hypotheſin. Quaerenti ergo, quid ſit quantitas

infinite parua in Matheſi, reſpondemus eam eſſe reuera $=0$: neque ergo in hac idea tanta Myſteria latent, quanta vulgo putantur, & quae pluribus calculum infinite paruorum admodum ſuſpectum reddiderunt. Interim tamen dubia, ſi quae ſupererunt, in ſequentibus, vbi hunc calculum ſumus tradituri, funditus tollentur.

84. Cum igitur oſtenderimus, quantitatem infinite paruam reuera eſſe cyphram, primum occurrendum eſt obiectioni, cur quantitates infinite paruas non perpetuo eodem charactere o deſignemus, ſed peculiares notas ad eas deſignandas adhibeamus. Quia enim omnia nihila ſunt inter ſe aequalia, ſuperfluum videtur variis ſignis ea denotare. Verum quamquam duae quaeuis cyphrae ita inter ſe ſunt aequales, vt earum differentia ſit nihil: tamen, cum duo ſint modi comparationis, alter arithmeticus, alter geometricus; quorum illo differentiam, hoc vero quotum ex quantitatibus comparandis ortum ſpectamus; ratio quidem arithmetica inter binas quasque cyphras eſt aequalitatis, non vero ratio geometrica. Facillime hoc perſpicietur ex hac proportione geometrica $2:1=0:0$, in qua terminus quartus eſt $=0$, vti tertius. Ex natura autem proportionis, cum terminus primus duplo ſit maior quam ſecundus, neceſſe eſt, vt & tertius duplo maior ſit quam quartus.

85. Haec autem etiam in vulgari Arithmetica ſunt planiſſima: cuilibet enim notum eſt, cyphram per quemvis numerum multiplicatam dare cyphram, eſſeque $n.0=0$, ſicque

ſicque fore $n : 1 = 0 : 0$. Vnde patet fieri poſſe, vt duae cyphrae quamcunque inter ſe rationem geometricam teneant, etiamſi, rem arithmetice ſpectando, earum ratio ſemper ſit aequalitatis. Cum igitur inter cyphras ratio quaecunque intercedere poſſit, ad hanc diuerſitatem indicandam conſulto varii characteres vſurpantur; praeſertim tum, cum ratio geometrica, quam cyphrae variae inter ſe tenent, eſt inueſtiganda. In calculo autem infinite paruorum nil aliud agitur, niſi vt ratio geometrica inter varia infinite parua indagetur, quod negotium propterea, niſi diuerſis ſignis ad ea indicanda vteremur, in maximam confuſionem illaberetur, neque vllo modo expediri poſſet.

86. Si ergo, prouti in Analyſi infinitorum modus ſignandi eſt receptus, denotet dx quantitatem infinite paruam, erit vtique tam $dx = 0$, quam $adx = 0$, denotante a quantitatem quamcunque finitam. Hoc tamen non obſtante erit ratio geometrica $adx : dx$ finita, nempe vt $a : 1$; & hanc obrem haec duo infinite parua dx & adx, etiamſi vtrumque ſit $= 0$, inter ſe confundi non poſſunt, ſi quidem eorum ratio inueſtigetur. Simili modo, ſi diuerſa occurrunt infinite parua dx & dy, etiamſi vtrumque ſit $= 0$, tamen eorum ratio non conſtat. Atque in inueſtigatione rationis inter duo quaeque huiusmodi infinite parua omnis vis calculi differentialis verſatur. Vſus autem huius comparationis, etiamſi primo intuitu admodum exiguus videatur, tamen ampliſſimus deprehenditur, atque adhuc indies magis elucet.

87. Cum

87. Cum igitur infinite paruum ſit reuera nihil, patet quantitatem finitam neque augeri neque diminui, ſi ad eam infinite paruum vel addamus vel ab ea ſubtrahamus. Sit a quantitas finita atque dx infinite parua, erit tam $a + dx$, quam $a - dx$, & generaliter $a \pm n dx = a$. Siue enim relationem inter $a \pm n dx$ & a arithmetice intueamur ſiue geometrice, vtroque caſu ratio aequalitatis deprehendetur. Arithmetica quidem ratio aequalitatis manifeſta eſt; cum enim ſit $n dx = 0$, erit $a \pm n dx - a = 0$: geometrica vero ratio aequalitatis inde patet, quod ſit $\frac{a \pm n dx}{a} = 1$. Hinc ſequitur canon ille maxime receptus, quod *infinite parua prae finitis euaneſcant, atque adeo horum reſpectu reiici queant.* Quare illa obiectio, qua Analyſis infinitorum rigorem geometricum negligere arguitur, ſponte cadit, cum nil aliud reiiciatur, niſi quod reuera ſit nihil. Ac propterea iure affirmare licet, in hac ſublimiori ſcientia rigorem geometricum ſummum, qui in Veterum libris deprehenditur, aeque diligenter obſeruari.

88. Quoniam quantitas infinite parua dx reuera eſt $= 0$, eius quoque quadratum dx^2, cubus dx^3, & quaeuis alia poteſtas affirmatiuum habens exponentem erit $= 0$, ideoque aeque prae quantitatibus finitis euaneſcent. At vero etiam quantitas infinite parua dx^2 prae ipſa dx euaneſcit; erit enim $dx \pm dx^2$ ad dx in ratione aequalitatis, ſiue comparatio arithmetice ſiue geometrice inſtituatur. De priori quidem dubium eſt nullum, at geometrice comparando erit

dx

$$dx + dx^2 : dx = \frac{dx + dx^2}{dx} = 1 + dx = 1.$$

Pari modo erit $dx + dx^3 = dx$, & generaliter $dx + dx^{n+1} = dx$, dummodo ſit n numerus nihilo maior: erit enim ratio geometrica $dx + dx^{n+1} : dx = 1 + dx^n$; ideoque, ob $dx^n = 0$, ratio aequalitatis. Si igitur vti in poteſtatibus ſit, vocetur dx infinite paruum primi ordinis, dx^2 ſecundi ordinis, dx^3 tertii ordinis & ita porro, manifeſtum eſt prae infinite paruis primi ordinis, euaneſcere infinite parua altiorum ordinum.

89. Simili modo oſtendetur infinite parua tertii ac ſuperiorum ordinum euaneſcere prae infinite paruis ordinis ſecundi; atque in genere infinite parua cuiusque ordinis ſuperioris euaneſcere prae infinite paruis ordinis inferioris. Ita ſi m fuerit numerus minor quam n, erit $adx^m + bdx^n = adx^m$, quia dx^n euaneſcit prae dx^m, vti oſtendimus. Hocque etiam in exponentibus fractis habet locum; ita dx euaneſcet prae $\sqrt{dx}$ ſeu $dx^{\frac{1}{2}}$, eritque $a\sqrt{dx} + bdx = a\sqrt{dx}$. Quodſi autem exponens ipſius dx ſit $= 0$, erit $dx^0 = 1$, quamuis ſit $dx = 0$; hinc poteſtas dx^n, cum fiat $= 1$, ſi ſit $n = 0$, ex finita ſtatim fit quantitas infinite parua, atque exponens n nihilo ſit maior. Hinc ergo infiniti ordines infinite paruorum exiſtunt, quae etſi omnia ſunt $= 0$, tamen inter ſe probe diſtingui debent, ſi ad earum relationem mutuam, quae per rationem geometricam explicatur, attendamus.

90. Stabilita notione infinite paruorum facilius indolem infinitorum ſeu infinite magnorum exponere poterimus. Notum eſt valorem fractionis $\frac{1}{z}$ eo maiorem euadere, quo magis diminuatur denominator z; quare ſi z fiat quantitas omni aſſignabili quantitate minor, ſeu infinite parua, neceſſe eſt vt valor fractionis $\frac{1}{z}$ fiat omni aſſignabili quantitate maior, ideoque infinitus. Quamobrem ſi vnitas ſeu quaeuis alia quantitas finita diuidatur per infinite paruum ſeu o, quotus erit infinite magnus, ideoque quantitas infinita. Cum igitur hoc ſignum ∞ denotet quantitatem infinite magnam, iſta habebitur aequatio $\frac{a}{dx} = \infty$; cuius veritas quoque hinc patet, quod ſit inuertendo $\frac{a}{\infty} = dx = 0$. Namque quo maior ſtatuitur fractionis $\frac{a}{z}$ denominator z, eo minor fit fractionis valor, atque ſi z fiat quantitas infinite magna ſeu $z = \infty$, neceſſe eſt, vt fractionis valor $\frac{a}{\infty}$ fiat infinite paruus.

91. Qui vtrumuis horum ratiociniorum negauerit, eum in maxima incommoda prolabi, atque adeo certiſſima Analyſeos fundamenta euertere neceſſe eſt. Qui enim ſtatuit valorem fractionis $\frac{a}{0}$ eſſe finitum vti b, vtrinque per denominatorem multiplicando prodiret $a = 0 . b$, atque ideo quantitas finita b per nihil o multiplicata praeberet

beret quantitatem finitam a, quod eſſet abſurdum. Multo minus valor ille b fractionis $\frac{a}{0}$ poterit eſſe $=0$: nam 0 per 0 multiplicata quantitatem a producere nullo modo poterit. In idem abſurdum incidit, qui negat eſſe $\frac{a}{\infty}=0$, ei enim dicendum erit eſſe $\frac{a}{\infty}=$ quantitati finitae b: quare cum ex aequatione $\frac{a}{\infty}=b$ legitime ſequatur haec $\infty=\frac{a}{b}$, foret valor fractionis $\frac{a}{b}$, cuius numerator ac denominator ſunt quantitates finitae, infinite magnus, quod perinde foret abſurdum. Neque vero etiam valores fractionum $\frac{a}{0}$ & $\frac{a}{\infty}$ imaginarii ſtatui poſſunt; propterea quod valor fractionis, cuius numerator eſt finitus denominator vero imaginarius, neque infinite magnus neque infinite paruus eſſe poteſt.

92. Quantitas ergo infinite magna, ad quam nos haec conſideratio perduxit, & quae ſola in Analyſi infinitorum locum habet, commodiſſime definitur dicendo, quantitatem infinite magnam eſſe quotum, qui ex diuiſione quantitatis finitae per infinite paruam oritur. Viciſſim ergo erit quantitas infinite parua quotus, qui oritur ex diuiſione quantitatis finitae per infinite magnam. Quare, cum eiusmodi proportio geometrica ſubſiſtat, vt ſit quantitas infinite parua ad finitam, ita finita ad infinite magnam; vti quantitas infinita infinities maior eſt quam finita, ita quantitas finita infinities maior erit quam infinite

nite parua. Huiusmodi igitur locutiones, quibus plures offenduntur, non ſunt improbandae, cum certiſſimis innitantur principiis. Deinde etiam ex aequatione $\frac{a}{0} = \infty$ ſequitur fieri poſſe, vt nihil per quantitatem infinite magnam multiplicatum producat quantitatem finitam, quod alienum videri poſſet, niſi planiſſime per legitimam conſequentiam eſſet deductum.

93. Quoniam inter infinite parua, ſi ſecundum rationem geometricam inter ſe comparantur, maximum deprehenditur diſcrimen, ita quoque inter quantitates infinite magnas multo maior differentia intercedit, cum non ſolum geometrice ſed etiam arithmetice comparatae diſcrepent. Ponatur quantitas illa infinita, quae ex diviſione quantitatis finitae a per infinite paruam dx oritur, $= A$, ita vt ſit $\frac{a}{dx} = A$: erit vtique $\frac{2a}{dx} = 2A$ & $\frac{na}{dx} = nA$; cum igitur & nA ſit quantitas infinita, ſequitur inter quantitates infinite magnas rationem quamcunque locum habere poſſe. Hincque, ſi quantitas infinita per numerum finitum ſiue multiplicetur, ſiue diuidatur, prodibit quantitas infinita. Neque ergo de quantitatibus infinitis negari poteſt, eas vlterius augeri poſſe. Facile autem perſpicitur, ſi ratio geometrica, quam duae quantitates infinitae inter ſe tenent, non fuerit aequalitatis, multo minus earum rationem arithmeticam aequalitatis eſſe poſſe, cum potius earum differentia ſemper ſit infinite magna.

94. Quan-

94. Quantumuis autem nonnullis idea infiniti, qua in Matheſi vtimur, ſuſpecta videatur, qui hanc ob cauſam Analyſin infinitorum profligandam arbitrantur; tamen hac idea ne in partibus quidem Matheſeos triuialibus carere poſſumus. In Arithmetica enim, vbi doctrina logarithmorum tradi ſolet, logarithmus cyphrae & negatiuus & infinite magnus ſtatuitur, neque quisquam eſt tam mente captus, vt hunc logarithmum vel finitum vel adeo nihilo aequalem dicere audeat. In Geometria autem & Trigonometria hoc clarius apparet; quis enim vnquam negabit tangentem ſecantemue auguli recti non eſſe infinite magnam? & cum rectangulum ex tangente in cotangentem ſit radii quadrato aequale, cotangens autem anguli recti ſit $= 0$; in Geometria adeo concedi debet, productum ex nihilo & infinito eſſe poſſe finitum.

95. Cum ſit $\frac{a}{dx}$ quantitas infinita A, patet hanc quantitatem $\frac{A}{dx}$ fore quantitatem infinities maiorem, quam A: eſt enim $\frac{a}{dx} : \frac{A}{dx} = a : A$, hoc eſt vt numerus finitus ad infinite magnum. Dantur ergo inter quantitates infinite magnas eiusmodi relationes, vt aliae aliis infinities maiores eſſe queant. Sic $\frac{a}{dx^2}$ erit quantitas infinita infinities maior quam $\frac{a}{dx}$; poſito enim $\frac{a}{dx} = A$ erit $\frac{a}{dx^2} = \frac{A}{dx}$. Simili modo erit $\frac{a}{dx^3}$ quantitas infinita infi-

nities maior quam $\frac{a}{dx^2}$, ideoque infinities infinities maior quam $\frac{a}{dx}$. Dantur ergo infiniti gradus infinitorum, quorum quisque infinities maior eſt quam praecedentes: atque adeo ſi numerus m vel tantillum maior ſit quam n, erit $\frac{a}{dx^m}$ quantitas infinita infinities maior quam quantitas infinita $\frac{a}{dx^n}$.

96. Quemadmodum in quantitatibus infinite paruis dantur rationes geometricae inaequales, cum tamen rationes arithmeticae omnes ſint aequales: ita in quantitatibus infinite magnis dantur rationes geometricae aequales, cum tamen arithmeticae ſint quantumuis inaequales. Si enim a & b denotent quantitates finitas, hae duae quantitates infinitae $\frac{a}{dx} + b$ & $\frac{a}{dx}$ rationem geometricam habent aequalitatis; erit enim quotus ex earum diuiſione ortus $= 1 + \frac{b\,dx}{a} = 1$ ob $dx = 0$: interim tamen, ſi arithmetice comparentur, ob differentiam $= b$, ratio erit inaequalitatis. Simili modo $\frac{a}{dx^2} + \frac{a}{dx}$ ad $\frac{a}{dx^2}$ rationem geometricam habet aequalitatis, exponens enim rationis eſt $= 1 + dx = 1$; verum tamen differentia eſt $\frac{a}{dx}$ ideoque infinita. Hinc ſi ad rationem geometricam ſpectemus, infinite magna inferiorum gra-

duum

duum prae infinite magnis ſuperiorum graduum euaneſcunt.

97. His de gradibus infinitorum praemonitis, mox apparebit fieri poſſe, vt productum ex quantitate infinite magna in infinite paruam non ſolum quantitatem finitam producat, quod ſupra eueniſſe vidimus; ſed etiam huiusmodi productum eſſe poterit ſiue infinite magnum ſiue infinite paruum. Sic quantitas infinita $\frac{a}{dx}$, ſi per infinite paruam dx multiplicetur, dat productum finitum $= a$; ſin autem $\frac{a}{dx}$ multiplicetur per infinite parvum dx^2, vel dx^3, vel alius ſuperioris ordinis, productum erit vel adx, vel adx^2, vel adx^3 &c. ideoque infinite paruum. Eodem modo intelligetur, ſi quantitas infinita $\frac{a}{dx^2}$ multiplicetur per infinite paruam dx, productum fore infinite magnum: atque generatim ſi $\frac{a}{dx^n}$ multiplicetur per bdx^m, productum $abdx^{m-n}$ erit infinite paruum ſi m ſuperat n; finitum ſi m aequat n; & infinite magnum ſi m ſuperatur ab n.

98. Quantitates tam infinite paruae, quam infinite magnae in ſeriebus numerorum ſaepiſſime occurrunt, in quibus cum ſint numeris finitis permixtae, ex iis luculenter patebit, quemadmodum ſecundum leges continuitatis a quantitatibus finitis ad infinite magnas atque infinite paruas tranſitio fiat. Conſideremus primum ſeriem nume-

numerorum naturalium, quae ſimul retro continuata erit

&c. $-4-3-2-1+0+1+2+3+4+$ &c.

Numeri ergo continuo decreſcendo praebent tandem o ſeu infinite paruum, vnde vlterius continuati negatiui euadunt. Quamobrem hinc intelligitur a numeris finitis affirmatiuis decreſcentibus tranſiri per o ad negatiuos creſcentes. Sin autem eorum numerorum quadrata ſpectentur, quia omnia ſunt affirmatiua

&c. $16+9+4+1+0+1+4+9+16+$ &c.

erit o quoque tranſitus numerorum affirmatiuorum decreſcentium ad affirmatiuos creſcentes; atque ſi ſigna mutentur, erit quoque o tranſitus numerorum negatiuorum decreſcentium ad negatiuos creſcentes.

99. Si ſeries conſideretur, cuius terminus generalis eſt $\sqrt{x}$, quae etiam retro continuata erit huiusmodi

&c. $+\sqrt{-3}+\sqrt{-2}+\sqrt{-1}+0+\sqrt{1}+\sqrt{2}+\sqrt{3}+\sqrt{4}+$ &c.

ex qua patet o, quoque tanquam limitem conſiderari poſſe, per quem a quantitatibus realibus ad imaginaria tranſeatur. Si iſti termini tanquam applicatae curuarum conſiderentur, perſpicitur, ſi eae fuerint affirmatiuae atque eousque decreuerint vt tandem euaneſcant, tum eas vlterius continuatas vel fieri negatiuas, vel iterum affirmatiuas, vel adeo imaginarias. Idem eueniet, ſi applicatae primum fuerint negatiuae; tum enim aeque poſtquam euanuerint, ſi vlterius continuentur, vel affirmatiuae

tiuae fient, vel negatiuae vel imaginariae; quorum phaenomenorum plurima exempla praebet doctrina de lineis curuis in libro praecedente tractata.

100. Eodem modo in ſeriebus occurrunt ſaepe termini infiniti: ſic in ſerie harmonica, cuius terminus generalis eſt $\frac{1}{x}$, indici $x = 0$ reſpondebit terminus infinite magnus $\frac{1}{0}$; totaque ſeries ita ſe habebit:

$$\&c. - \tfrac{1}{4} - \tfrac{1}{3} - \tfrac{1}{2} - \tfrac{1}{1} + \tfrac{1}{0} + \tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{3} + \&c.$$

A dextra ergo ad ſiniſtram progrediendo termini creſcunt, ita vt $\frac{1}{0}$ iam ſit infinite magnus, quem cum tranſierint, fient negatiui decreſcentes. Hinc quantitas infinite magna ſpectari poteſt tanquam limes, per quem numeri affirmatiui progreſſi fiunt negatiui, & viciſſim: vnde pluribus viſum eſt, numeros negatiuos conſiderari poſſe, tanquam infinito maiores, propterea quod in hac ſerie termini continuo creſcentes, poſtquam infinitum attigerint, abeant in negatiuos. At vero ſi ad ſeriem, cuius terminus generalis eſt $\frac{1}{xx}$, attendamus, poſt tranſitum per infinitum rurſus prodeunt termini affirmatiui.

$$\&c. \tfrac{1}{9} + \tfrac{1}{4} + \tfrac{1}{1} + \tfrac{1}{0} + \tfrac{1}{1} + \tfrac{1}{4} + \tfrac{1}{9} + \&c.$$

quos tamen nemo infinito maiores dixerit.

101. Saepenumero quoque in ſeriebus terminus infinitus conſtituit limitem, terminos reales ab imagina-

riis ſegregantem, vti fit in ſerie hac, cuius terminus generalis eſt $\frac{1}{\sqrt{x}}$

$$\&c. + \frac{1}{\sqrt{-3}} + \frac{1}{\sqrt{-2}} + \frac{1}{\sqrt{-1}} + \frac{1}{0} + \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \&c.$$

neque tamen hinc ſequitur, imaginaria eſſe infinito maiora: quoniam ex ſerie ante allata

$$\&c. + \sqrt{-3} + \sqrt{-2} + \sqrt{-1} + 0 + \sqrt{1} + \sqrt{2} + \sqrt{3} + \&c.$$

aeque ſequeretur, imaginaria eſſe nihilo minora. Deinde vero etiam a terminis realibus tranſitus ad imaginarios exhiberi poteſt, quorum limes neque ſit 0 neque ∞, vti fit, ſi terminus generalis fuerit $1 + \sqrt{x}$. His autem caſibus, cum ob irrationalitatem quilibet terminus geminum habeat valorem, in limite inter realia & imaginaria ſemper bini illi valores fiunt inter ſe aequales. At quoties termini, qui ante erant affirmatiui, abeunt in negatiuos, tranſitus ſemper fit per limitem vel infinite paruum, vel infinite magnum, quae omnia ex lege continuitatis, quam in lineis curuis deprehendimus, clarius elucent.

102. Ex ſummatione quoque ſerierum in infinitum excurrentium plura hic afferri poſſunt, quae cum ad hanc infiniti doctrinam magis illuſtrandam, tum vero ad plura dubia, quae in hoc negotio ſuboriri ſolent, delenda inſeruiunt. Ac primo quidem, ſi ſeries conſtet ex terminis aequalibus, vt

$$1 + 1 + 1 + 1 + 1 + 1 + \&c.$$

ea-

eaque ſine fine, hoc eſt in infinitum continuetur, nullum certe eſt dubium, quin omnium horum terminorum ſumma maior ſit omni numero aſſignabili; eaque propterea infinita ſit neceſſe eſt. Hoc quoque confirmat eius origo, dum oritur ex euolutione fractionis

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \&c.$$

ponendo $x = 1$; erit ergo

$$\frac{1}{1-1} = 1 + 1 + 1 + 1 + \&c.$$

ideoque ſumma $= \frac{1}{1-1} = \frac{1}{0} =$ infinito.

103. Quamuis autem hic nullum dubium naſci queat, cum idem numerus finitus infinities ſumtus in infinitum abire debeat; tamen ipſa origo ex ſerie generali

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \&c.$$

grauiſſima incommoda afferre videtur: ſi enim pro x ſucceſſiue ponantur numeri 1, 2, 3, &c. ſequentes ſeries cum ſuis ſummis prodibunt.

A . . . $1 + 1 + 1 + 1 + 1 + \&c. = \frac{1}{1-1} =$ infinito

B . . . $1 + 2 + 4 + 8 + 16 + \&c. = \frac{1}{1-2} = -1$

C . . . $1 + 3 + 9 + 27 + 81 + \&c. = \frac{1}{1-3} = -\frac{1}{2}$

D . . . $1 + 4 + 16 + 64 + 256 + \&c. = \frac{1}{1-4} = -\frac{1}{3}$

&c.

Cum igitur ſeries B ſingulos terminos praeter primum habeat maiores, quam ſeries A, ſumma ſeriei B neceſſario multo maior eſſe deberet, quam ſumma ſeriei A: interim tamen iſte calculus oſtendit ſeriei A ſummam infinitam, ſeriei B vero ſummam negativam, hoc eſt nihilo minorem, quod concipi non poteſt. Multo minus cum ſolitis ideis conciliari poteſt, quemadmodum huius eſt ſequentium ſerierum C, D, &c. ſummae fiant negatiuae, cum tamen omnes termini ſint affirmatiui.

104. Ob hanc rationem opinio ſupra allata multis probabilis videri ſolet, quantitates ſcilicet negatiuas quandoque conſiderari poſſe tanquam infinito maiores ſeu plus quam infinitas; & cum etiam numeros vltra nihil diminuendo perueniatur ad negatiuos, diſcrimen ſtatuunt inter numeros negatiuos huiusmodi $-1, -2, -3$, &c. & huiusmodi $\frac{+1}{-1}, \frac{+2}{-1}, \frac{+3}{-1}$, &c. illos nihilo minores, hos vero infinito maiores dicendo. Verumtamen hoc pacto difficultatem non tollunt, quam ſuggerit haec ſeries

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + \&c. = \frac{1}{(1-x)^2}$$

vnde oriuntur ſequentes ſeries:

$$A \ldots 1 + 2 + 3 + 4 + 5 + \&c. = \frac{1}{(1-1)^2} = \frac{1}{0} = \text{infinito}$$

$$B \ldots 1 + 4 + 12 + 32 + 80 + \&c. = \frac{1}{(1-2)^2} = 1$$

vbi cum ſinguli termini ſeriei B ſint maiores, quam ſinguli termini ſeriei A, primis ſolis exceptis, quemadmodum

dum ſumma ſeriei A ſit infinita, ſeriei B vero ſumma aequalis 1, hoc eſt ſoli termino primo, ex illo principio explicari omnino nequit.

105. Quoniam autem ſi vellemus negare eſſe $-1 = \frac{+1}{-1}$, & $\frac{+a}{-b} = \frac{-a}{+b}$, firmiſſima Analyſeos fundamenta collaberentur, illa ante commemorata explicatio prorſus admitti non poteſt. Quin potius negare debebimus, illas, quas formulae generales ſuppeditauerant, ſummas eſſe veras. Cum enim hae ſeries ex continua diuiſione oriantur, dum reſiduum continuo vlterius diuiditur: reſiduum autem perpetuo fiat maius, quo longius progrediamur, id nunquam negligere poterimus; atque minime reſiduum vltimum, hoc eſt quod in diuiſione infiniteſima remanet, omitti poteſt, quippe quod ſit infinite magnum. Quia autem hoc in ſuperioribus ſeriebus non obſeruatur, dum nullius reſidui ratio habetur, mirum non eſt, eas ſummationes ad abſurdum deducere. Haecque reſponſio, vti eſt ex ipſa ſerierum geneſi petita, ita quoque eſt veriſſima, atque omnem dubitationem tollit.

106. Quo hoc clarius appareat, contemplemur euolutionem fractionis $\frac{1}{1-x}$, vti in terminis primum finitis tantum abſoluitur. Erit ergo

$$\frac{1}{1-x} = 1 + \frac{x}{1-x}$$

$$\frac{1}{1-x} = 1 + x + \frac{x^2}{1-x}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \frac{x^3}{1-x}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \frac{x^4}{1-x}.$$

&c.

qui ergo dicere vellet huius ſeriei finitae $1+x+x^2+x^3$ ſummam eſſe $\frac{1}{1-x}$, is erraret a vero quantitate $\frac{x^4}{1-x}$: & qui ſummam huius ſeriei

$$1+x+x^2+x^3+ \ldots \ldots \ldots +x^{1000}$$

ſtatuere vellet $= \frac{1}{1-x}$, is erraret quantitate $\frac{x^{1001}}{1-x}$ qui error ſi x ſit numerus unitate maior, foret maximus.

107. Ex his perſpicuum eſt eum, qui eiusdem ſeriei in infinitum continuatae ſeu huius:

$$1+x+x^2+x^3+ \ldots \ldots \ldots +x^{\infty}$$

ſummam ſtatuere velit $= \frac{1}{1-x}$, a veritate eſſe aberraturum quantitate $\frac{x^{\infty+1}}{1-x}$; quae ſi ſit $x > 1$ vtique erit infinite magna. Simul vero hinc ratio patet, cur ſeriei in infinitum continuatae

$$1+x+x^2+x^3+x^4+\&c.$$

ſum-

ſumma reuera ſit $= \frac{1}{1-x}$, ſi fuerit x fractio vnitate minor, tum enim error $\frac{x^{\infty+1}}{1-x}$ ſit infinite paruus, ideoque nullus; cuius propterea ratio tuto poteſt negligi. Sic poſito $x = \frac{1}{2}$, erit reuera

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \&c. = \frac{1}{1-\frac{1}{2}} = 2,$$

ſimiliterque reliquarum ſerierum, ſi x ſit fractio vnitate minor, ſumma vera hoc modo indicatur.

108. Haec eadem reſponſio valet de ſummis ſerierum diuergentium, in quibus ſigna $+$ & $-$ alternantur, quae vulgo ex eadem formula exhiberi ſolent, ponendo pro x numeros negatiuos. Cum enim ſit:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \&c.$$

niſi vltimi reſidui ratio habeatur, foret:

A . . . $1 - 1 + 1 - 1 + 1 - 1 + \&c. = \frac{1}{2}$
B . . . $1 - 2 + 4 - 8 + 16 - 32 + \&c. = \frac{1}{3}$
C . . . $1 - 3 + 9 - 27 + 81 - 243 - \&c. = \frac{1}{4}$
&c.

Patet autem ſeriei ſecundae B ſummam ideo non poſſe eſſe $= \frac{1}{3}$, cum quo plures termini actu ſummentur, aggregata eo magis ab $\frac{1}{3}$ recedant. Perpetuo autem cuiusque ſeriei ſumma debet eſſe limes, ad quem eo propius perueniatur, quo plures termini actu addantur.

109. Ex his quidam concluſerunt huiusmodi ſeries, quae vocantur diuergentes, prorſus nullas habere ſummas fixas;

fixas; propterea quod colligendis actu terminis ad nullum limitem fiat appropinquatio, qui pro ſumma ſeriei in infinitum continuatae haberi poſſet: quae ſententia, cum iſtae ſummae iam ob neglecta vltima reſidua erroneae ſint oſtenſae, veritati maxime eſt conſentanea. Interim tamen contra eam ſummo iure obiici poteſt, has memoratas ſummas, quantumuis a veritate abhorrere videantur, tamen nunquam in errores inducere; quin potius iis admiſſis plurima praeclara eſſe eruta, quibus ſi iſtas ſummationes prorſus reiicere vellemus, carendum eſſet. Neque vero hae ſummae, ſi eſſent falſae, perpetuo ad veritatem nos ducere poſſent; quin potius, cum non parum ſed infinite a veritate diſcrepent, nos quoque in infinitum a vero ſeducere deberent. Quod tamen cum non eueniat, difficillimus nobis reſtat nodus ſoluendus.

110. Dico igitur in voce *ſummae* latere totam difficultatem; ſi enim *ſumma* ſeriei, vt vulgo vſus fert, ſumatur pro aggregato omnium eius terminorum actu collectorum, tum dubium eſt nullum, quin earum tantum ſerierum in infinitum excurrentium ſummae exhiberi queant, quae ſint conuergentes, atque continuo propius ad certum ſtatumque valorem deducant, quo plures termini actu colligantur. Series autem diuergentes, quarum termini non decreſcunt, ſiue ſigna + & — alternentur ſiue ſecus, prorſus nullas habebunt ſummas fixas; ſi quidem vox ſummae hoc ſenſu pro aggregato omnium terminorum accipiatur. At vero in iis caſibus, quorum meminimus, quibus ex iſtiusmodi ſummis erroneis veritas tamen

men elicitur; id non fit, quatenus expreſſio finita, verbi gratia $\frac{1}{1-x}$, eſt ſumma ſeriei $1+x+x^2+x^3+$ &c. ſed quatenus ea expreſſio euoluta hanc ſeriem praebet; ſicque in hoc negotio nomen ſummae prorſus omitti poſſet.

111. Haec igitur incommoda, hasque apparentes contradictiones penitus euitabimus, ſi voci *ſummae* aliam notionem, atque vulgo fieri ſolet, tribuamus. Dicamus ergo ſeriei cuiusque infinitae *ſummam* eſſe expreſſionem finitam, ex cuius euolutione illa ſeries naſcatur. Hocque ſenſu ſeriei infinitae $1+x+x^2+x^3+$ &c. ſumma reuera erit $=\frac{1}{1-x}$, quia illa ſeries ex huius fractionis euolutione oritur: quicunque numerus loco x ſubſtituatur. Hoc pacto, ſi ſeries fuerit conuergens, iſta noua vocis ſummae definitio, cum conſueta congruet; & quia diuergentes nullas habent ſummas proprie ſic dictas, hinc nullum incommodum ex noua hac appellatione orietur. Denique ope huius definitionis vtilitatem ſerierum diuergentium tueri, atque ab omnibus iniuriis vindicare poterimus.

CAPUT IV.

DE DIFFERENTIALIUM CUIUSQUE ORDINIS NATURA.

112.

In capite primo vidimus, ſi quantitas variabilis x accipiat augmentum $= \omega$, tum cuiusuis functionis ipſius x augmentum inde oriundum tali forma exprimi $P\omega + Q\omega^2 + R\omega^3 + \&c.$ ſiue haec expreſſio ſit finita ſiue in infinitum excurrat. Functio ergo y, ſi in ea loco x ſcribatur $x+\omega$, valorem ſequentem induet:

$$y^{\mathrm{I}} = y + P\omega + Q\omega^2 + R\omega^3 + S\omega^4 + \&c.$$

a quo, ſi valor prior y ſubtrahatur, remanebit differentia functionis y, quae ita exprimetur

$$\Delta y = P\omega + Q\omega^2 + R\omega^3 + S\omega^4 + \&c.$$

atque cum valor ipſius x ſequens ſit $x^{\mathrm{I}} = x + \omega$, erit differentia ipſius x, nempe $\Delta x = \omega$. Litterae autem P, Q, R, &c. denotant functiones ipſius x pendentes ab y, quas capite primo inuenire docuimus.

113. Hinc ergo quocunque augmento ω augeatur quantitas variabilis x, ſimul definiri poterit augmentum, quod cuique ipſius x functioni y accedit; dummodo pro quouis ipſius y valore functiones P, Q, R, S, &c. definire valeamus. In hoc autem capite, atque in vniuerſa Analyſi infinitorum augmentum illud ω, quo quantitatem variabilem x creſcere ſumſimus, ſtatuemus infinite par-

paruum, atque adeo euaneſcens, ſeu $=0$. Vnde manifeſtum eſt, incrementum ſeu differentiam functionis y quoque fore infinite paruam. Cum autem in hac hypotheſi ſinguli termini expreſſionis

$$P\omega + Q\omega^2 + R\omega^3 + S\omega^4 + \&c.$$

prae antecedentibus euaneſcant, (88. & ſeqq.), ſolus primus $P\omega$ remanebit, eritque propterea hoc caſu, quo ω eſt infinite paruum, differentia ipſius y nempe $\Delta y = P\omega$.

114. Erit ergo Analyſis infinitorum, quam hic tractare caepimus, nil aliud, niſi caſus particularis methodi differentiarum in capite primo expoſitae, qui oritur, dum differentiae, quae ante finitae erant aſſumtae, ſtatuantur infinite paruae. Quo igitur iſte caſus, quo vniuerſa Analyſis infinitorum continetur, a methodo differentiarum diſtinguatur, cum peculiaribus nominibus, tum etiam ſignis ad differentias iſtas infinite paruas denotandas vti conueniet. Differentias igitur infinite paruas hic cum LEIBNIZIO *differentialia* vocabimus; atque cum differentiarum in primo capite diuerſos ordines conſtituiſſemus, ex iis nunc facile quoque intelligetur, quid differentialia prima, ſecunda, tertia, &c. cuiusque functionis ſignificent. Loco characteris autem Δ, quo ante differentias indicaueramus, nunc vtemur charactere d; ita vt dy ſignificet differentiale primum ipſius y; ddy differentiale ſecundum; d^3y tertium & ita porro.

115. Quoniam differentias infinite paruas, quas hic tractamus, *differentialia* vocamus, hinc totus calculus, quo differentialia inueſtigantur atque ad vſum accommodantur, appellari ſolet *Calculus differentialis*. Mathematici Angli, inter quos primum NEWTONUS aeque ac LEIBNIZIUS inter Germanos hanc nouam Analyſeos partem excolere coepit, aliis tam nominibus quam ſignis vtuntur. Differentias enim infinite paruas, quas nos differentialia vocamus, potiſſimum *fluxiones* nominare ſolent, interdum quoque *incrementa:* quae voces vti latino ſermoni magis conueniunt, ita quoque res, quas denotant, ſatis commode exprimunt. Quantitas enim variabilis creſcendo continuo alios atque alios valores recipiens tanquam fluens conſiderari poteſt, hincque vox fluxionis, quae primum a NEUTONO ad celeritatem creſcendi adhibebatur, ad incrementum infinite paruum, quod quantitas quaſi fluendo accipit, deſignandum analogice eſt translata.

116. Quamuis autem circa vocum vſum atque definitionem cum Anglis diſceptare abſonum foret, nosque coram iudice puritatem latinae linguae atque expreſſionum commoditatem ſpectante facile ſuperaremur; tamen nullum eſt dubium, quin Anglis ratione ſignorum palmam praeripiamus. Differentialia enim, quae ipſi fluxiones appellant, punctis, quae litteris ſuperſcribunt, denotare ſolent, ita vt $\dot{y}$, iis ſignificet fluxionem primam ipſius y; $\ddot{y}$ fluxionem ſecundam; $\dddot{y}$ fluxionem tertiam, atque ita porro. Qui notandi modus, vti ab arbitrio pen-

pendens, etſi improbari nequit, ſi punctorum numerus fuerit paruus, vt numerando facile percipi queat; tamen ſi plura puncta inſcribi debeant, maximam confuſionem plurimaque incommoda affert. Differentiale enim ſeu fluxio decima perquam incommode hoc modo $\overset{\vdots\vdots}{y}$ repraeſentatur, cum noſtro ſignandi modo $d^{10}y$ facillime comprehendatur. Oriuntur autem caſus, quibus multo adhuc ſuperiores differentialium ordines atque adeo indefiniti exprimi debent, ad quos Anglorum modus prorſus ſit ineptus.

117. Noſtris igitur tam nominibus quam ſignis vtemur, quippe quorum illa in noſtris regionibus iam ſunt vſu recepta atque plerisque familiaria, haec vero commodiora. Interim tamen non abs re erat, Anglorum denominationes & ſignationes hic commemorare, vt qui eorum libros euoluunt, eos quoque intelligere queant. Neque enim Angli ſuo mori tam pertinaciter adhaerent, vt quae noſtro more ſunt ſcripta, prorſus repudient, nec legere dignentur. Nos quidem ipſorum opera maxima cum auiditate perlegimus, ex iisque ſummum fructum percipimus; ſaepenumero vero etiam animaduertimus, ipſos noſtratium ſcripta non ſine vtilitate legiſſe. Quamobrem etſi idem vbique atque aequabilis modus cogitata ſua exprimendi maxime eſſet optandus, tamen non admodum eſt difficile, vt vtrique aſſueſcamus, quantum quidem intelligentia librorum alieno more ſcriptorum poſtulat.

118. Cum igitur littera ω nobis hactenus denotaverit differentiam ſeu incrementum, quo quantitas variabilis x creſcere concipitur, nunc autem ω ſtatuatur infinite paruum, erit ω differentiale ipſius x; & hancobrem recepto ſignandi modo erit $\omega = dx$; atque dx proinde erit differentia infinite parua, qua ipſa x creſcere concipitur. Simili modo differentiale ipſius y ita exprimetur dy; atque ſi y fuerit functio quaecunque ipſius x, differentiale dy denotabit incrementum, quod functio y capit, dum x abit in $x + dx$. Quare ſi in functione y vbique loco x ſubſtituatur $x + dx$, & quantitas reſultans ponatur $= y^{I}$, erit $dy = y^{I} - y$, hocque modo differentiale cuiusque functionis reperietur: quod quidem intelligendum eſt de differentiali primo ſeu primi ordinis; de reliquis enim poſtea videbimus.

119. Probe ergo tenendum eſt litteram d hic non quantitatem denotare, ſed tantum loco ſigni adhiberi, ad vocem *differentialis* exprimendam, eodem modo, quo in doctrina logarithmorum littera l pro ſigno logarithmi, & in Algebra charactere V pro ſigno radicis vti conſueuimus. Hinc dy non ſignificat, vti vulgo in Analyſi vſu eſt receptum, productum ex quantitate d in quantitatem y, ſed ita enunciari debet, vt dicatur differentiale ipſius y. Simili modo ſi ſcribatur d^2y, neque binarius exponentem, neque d^2 poteſtatem ipſius d ſignificat, ſed adhibetur tantum ad nomen *differentialis ſecundi* breviter & apte exprimendum. Cum igitur littera d in calculo differentiali non quantitatem, ſed ſignum tantum ex-

exhibeat, ad confuſionem vitandam in calculis, vbi plures quantitates conſtantes occurrunt, littera *d* ad earum deſignationem vſurpari nequit; perinde atque euitare ſolemus litteram *l* tanquam quantitatem in calculum inducere, vbi ſimul logarithmi occurrunt. Optandum autem eſſet, vt litterae iſtae *d* & *l* per characteres aliquantulum alteratos exprimerentur, ne cum litteris Alphabethi, quibus quantitates deſignari ſolent, confundantur: ſimili ſcilicet modo, quo loco litterae *r*, qua primum vox radicis indicabatur, nunc character iſte diſtortus *V* in vſum eſt receptus.

120. Quoniam igitur vidimus differentiale primum ipſius y, ſi y fuerit functio quaecunque ipſius x, habiturum eſſe huiusmodi formam $P\omega$; ob $\omega = dx$, erit $dy = Pdx$. Qualiscunque ſcilicet fuerit y functio ipſius x, eius differentiale dy exprimetur certa quadam functione ipſius x, pro qua hic ponimus P, per differentiale ipſius x, nempe per dx multiplicata. Etiamſi ergo differentialia ipſarum x & y reuera ſint infinite parua, ideoque nihilo aequalia; tamen inter ſe finitam habebunt rationem: erit ſcilicet $dy : dx = P : 1$. Inuenta ergo functione iſta P, innoteſcit ratio inter differentiale dx & differentiale dy. Cum igitur calculus differentialis in inuentione differentialium conſiſtat, in eo non tam ipſa differentialia, quae ſunt nihilo aequalia ac propterea nullo labore inuenirentur, quam eorum ratio mutua geometrica inueſtigatur.

121. Differentialia igitur multo facilius inueniuntur, quam differentiae finitae. Ad differentiam enim finitam

tam Δy, qua functio y creſcit, dum quantitas variabilis x incrementum ω accipit, non ſufficit functionem P noſſe, ſed indagari inſuper oportet functiones Q, R, S, &c. quae in differentiam finitam, quam poſuimus

$$= P\omega + Q\omega^2 + R\omega^3 + \&c.$$

ingrediuntur; ad differentiale ipſius y autem inueniendum ſatis eſt, ſi nouerimus ſolam functionem P. Quamobrem ex cognita differentia finita cuiusque functionis ipſius x, facillime eius differentiale definitur; verum contra ex differentiali eius functionis, nondum erui poteſt eius differentia finita. Interim tamen infra docebitur, quemadmodum ex differentialibus omnium ordinum ſimul cognitis differentia quaeuis finita cuiusque functionis propoſitae inueniri queat. Ceterum ex his manifeſtum eſt differentiale primum $dy = P\,dx$, praebere terminum primum differentiae finitae, quippe qui eſt $= P\omega$.

122. Si igitur incrementum ω, quod quantitas variabilis x accipere concipitur, fuerit vehementer paruum, ita vt in expreſſione $P\omega + Q\omega^2 + R\omega^3 + \&c.$ termini $Q\omega^2$ & $R\omega^3$, multoque magis reliqui, fiant tam parui, vt in computo, quo ſummus rigor non obſeruatur, prae primo $P\omega$ negligi queant; tum cognito differentiali $P\,dx$, ex eo differentia finita vero proxime cognoſcetur, quippe quae erit $= P\omega$: vnde in pluribus occaſionibus, quibus calculus ad praxin adhibetur, non parum fructus hauritur. Atque hinc nonnulli arbitrantur, differentialia tanquam incrementa vehementer parua conſiderari poſſe, eaque nihilo reuera aequalia eſſe negant, atque tantum inde-

indefinite parua ſtatuunt. Haecque idea aliis occaſionem praebuit Analyſin infinitorum accuſandi, quod non veras rerum quantitates eliciat, ſed tantum vero proximas; quae obiectio ſemper aliquam vim retineret, niſi infinite parua prorſus nihilo aequalia ſtatueremus.

123. Qui autem nolunt infinite parua plane in nihilum abire, ii vt vim obiectionis deſtruere videantur, differentialia comparant minimis puluisculis ratione totius terrae, cuius quantitatem nemo non veram tradidiſſe cenſeretur, qui vnico puluiſculo a veritate aberrauerit. Talem igitur rationem inter quantitatem finitam & infinite paruam eſſe volunt, qualis eſt inter totam terram minimumque puluisculum: atque ſi cui hoc diſcrimen adhuc non ſatis magnum videatur, eam rationem millies magisque adaugent, vt paruitas amplius omnino percipi nequeat. Interim tamen agnoſcere coguntur, ſummum rigorem geometricum aliquantulum infringi; quare quo huic obiectioni occurrant, ad eiusmodi exempla confugiunt, quorum tam per Geometriam quam per Analyſin infinitorum ſolutiones inueniri poſſunt, ex earumque congruentia bonitatem poſterioris methodi concludunt. Quanquam autem hoc argumentum negotium non conficit, cum ſaepe numero per erroneas methodos verum elici queat; tamen quia hoc vitio non laborat, potius euincit, eas quantitates, quae in calculo ſint neglectae, non ſolum non incomprehenſibiliter paruas, ſed plane nullas eſſe, vti nos aſſumimus. Ex quo rigori geometrico nullam omnino vim inferimus.

124. Progrediamur ad differentialium ſecundi ordinis naturam explicandam, quae oriuntur ex differentiis ſecundis in capite primo expoſitis, ponendo quantitatem ω infinite paruam $= dx$. Cum igitur ſi ponamus quantitatem variabilem x aequalibus incrementis creſcere, ita vt ſi valor ſecundus x^{I} fuerit $= x + dx$, ſequentes futuri ſint $x^{II} = x + 2dx$; $x^{III} = x + 3dx$ &c. ob differentias primas conſtantes $= dx$, differentiae ſecundae euaneſcent: erit ergo quoque differentiale ſecundum ipſius x nempe $ddx = 0$, atque ob hanc rationem quoque differentialia vlteriora erunt $= 0$, ſcilicet $d^3x = 0$, $d^4x = 0$; $d^5x = 0$; &c. Obiici quidem poteſt, haec differentialia, cum ſint infinite parua, per ſe eſſe $= 0$, neque hoc proprium eſſe eius quantitatis variabilis x, cuius incrementa aequalia concipiantur: at vero hanc euaneſcentiam ita interpretari oportet, vt differentialia ddx, d^3x &c. non ſolum in ſe ſpectata ſint nulla, ſed etiam ratione poteſtatum ipſius dx, cum quibus alias comparari poſſent, euaneſcere.

125. Quae quo clarius intelligantur, recordandum eſt differentiam ſecundam cuiusque functionis ipſius x, quae ſit y, huiusmodi forma exprimi $P\omega^2 + Q\omega^3 + R\omega^3 +$ &c. Quodſi ergo ω ſit infinite paruum, termini $Q\omega^3$, $R\omega^4$ &c. prae primo $P\omega^2$ euaneſcent, vnde poſito $\omega = dx$ differentiale ſecundum ipſius y erit $= Pdx^2$, denotante dx^2 quadratum differentialis dx. Quare etſi differentiale ſecundum ipſius y, nempe ddy per ſe ſit $= 0$, tamen cum ſit $ddy = Pdx^2$, ad dx^2 habebit rationem finitam,

vti

vti P ad 1: ſin autem ſit $y = x$, tum fit $P = o$, $Q = o$, $R = o$, &c. ideoque hoc caſu differentiale ſecundum ipſius x etiam reſpectu dx^2 altiorumque ipſius dx poteſtatum euaneſcit. Hocque modo intelligenda ſunt ea, quae ante diximus, eſſe ſcilicet $ddx = o$, $d^3x = o$, &c.

126. Cum differentia ſecunda nil aliud ſit, niſi differentia differentiae primae; differentiale quoque ſecundum ſeu vti ſaepe vocari ſolet, differentio-differentiale nil aliud erit praeter differentiale differentialis primi. Quia deinde quantitas conſtans nulla neque augmenta neque decrementa accipit, nullasque admittit differentias, quippe quae ſolis quantitatibus variabilibus ſunt propriae, dicimus eodem ſenſu quantitatum conſtantium differentialia omnia cuiusque ordinis eſſe $= o$, hoc eſt prae omnibus adeo poteſtatibus ipſius dx euaneſcere. Cum igitur differentiale ipſius dx hoc eſt ddx ſit $= o$; differentiale dx tanquam quantitas conſtans conſiderari poteſt, & quoties differentiale cuiuspiam quantitatis dicitur conſtans, toties ea quantitas intelligenda eſt continuo aequalia incrementa accipere. Sumimus hic autem x pro ea quantitate, cuius differentiale ſit conſtans, hicque ſingularum eius functionum variabilitatem, cui earum differentialia ſunt obnoxia, aeſtimabimus.

127. Ponamus differentiale primum ipſius y eſſe $= p dx$; atque ad eius differentiale ſecundum inueniendum, ipſius $p dx$ denuo differentiale quaeri debet. Cum autem dx ſit conſtans, neque varietur eiamſi loco x ſcri-

batur $x+dx$, tantum opus eſt, vt quantitatis finitae p differentiale quaeratur: ſit igitur $dp=qdx$, quoniam vidimus omnium functionum ipſius x differentialia ad huiusmodi formam reuocari: & cum ſit, vti de differentiis finitis oſtendimus, differentiale ipſius $np=nqdx$, ſi n ſit quantitas conſtans, ponatur dx loco n, eritque differentiale ipſius $pdx=qdx^2$. Hancobrem ſi ſit $dy=pdx$ & $dp=qdx$, erit differentiale ſecundum $ddy=qdx^2$, ſicque conſtat, quod iam ante innuimus, differentiale ſecundum ipſius y ad dx^2 habere rationem finitam.

128. In Capite primo iam notauimus differentias ſecundas atque ſequentes conſtitui non poſſe, niſi valores ſucceſſiui ipſius x certa quadam lege progredi aſſumantur, quae lex cum ſit arbitraria, his valoribus progreſſionem arithmeticam tanquam facillimam ſimulque aptiſſimam tribuimus. Ob eandem ergo rationem de differentialibus ſecundis nihil certi ſtatui poterit, niſi differentialia prima, quibus quantitas variabilis x continuo creſcere concipitur, ſecundum datam legem progrediantur; ponimus itaque differentialia prima ipſius x, nempe dx, dx^{I}, dx^{II}, &c. omnia inter ſe aequalia, vnde fiunt differentialia ſecunda

$$ddx=dx^{\mathrm{I}}-dx=0;\ ddx^{\mathrm{I}}=dx^{\mathrm{II}}-dx^{\mathrm{I}}=0, \&c.$$

Quoniam ergo differentialia ſecunda & vlteriora ab ordine, quem differentialia quantitatis variabilis x inter ſe tenent, pendent, hicque ordo ſit arbitrarius, quae conditio differentialia prima non afficit; hinc ingens diſcrimen inter differentialia prima ac ſequentia ratione inuentionis intercedit.

129. Quodſi autem ſucceſſiui ipſius x valores x, x^{I}, x^{II}, x^{III}, x^{IV}, &c. non ſecundum arithmeticam progreſſionem ſtatuantur, ſed alia quacunque lege progredi ponantur, tum eorum quoque differentialia prima dx, dx^{I}, dx^{II}, &c. non erunt inter ſe aequalia, neque propterea erit $ddx = 0$. Hancobrem differentialia ſecunda quarumvis functionum ipſius x aliam formam induent; ſi enim huiusmodi functionis y differentiale primum fuerit $= pdx$ ad eius differentiale ſecundum inueniendum non ſufficit differentiale ipſius p per dx multiplicaſſe, ſed inſuper ratio differentialis ipſius dx, quod eſt ddx haberi debet. Quoniam enim differentiale ſecundum oritur, ſi pdx a valore eius ſequente, qui oritur dum $x + dx$ loco x & $dx + ddx$ loco dx ponitur, ſubtrahatur, ponamus valorem ipſius p ſequentem eſſe $= p + qdx$, eritque ipſius pdx valor ſequens

$$= (p + qdx)(dx + ddx) = pdx + pddx + qdx^2 + qdxddx;$$

a quo ſubtrahatur pdx, eritque differentiale ſecundum

$$ddy = pddx + qdx^2 + qdxddx = pddx + qdx^2,$$

quia $qdxddx$ prae $pddx$ euaneſcit.

130. Quanquam autem ratio aequalitatis eſt ſimpliciſſima atque aptiſſima, quae continuo ipſius x incrementis tribuatur, tamen frequenter euenire ſolet, vt non eius quantitatis variabilis x, cuius y eſt functio, incrementa aequalia aſſumantur, ſed alius cuiuspiam quantitatis, cuius ipſa x ſit functio quaedam. Quin etiam ſaepe eiusmodi alius quantitatis differentialia prima ſtatuuntur ae-

qualia, cuius nequidem relatio ad x conſtet. Priori caſu pendebunt differentialia ſecunda & ſequentia ipſius x a ratione, quam x tenet ad illam quantitatem, quae aequabiliter creſcere ponitur, ex eaque pari modo definiri debent, quo hic differentialia ſecunda ipſius y ex differentialibus ipſius x definire docuimus. Poſteriori autem caſu differentialia ſecunda & ſequentia ipſius x tanquam incognita ſpectari, eorumque loco ſigna ddx, d^3x, d^4x, &c. vſurpari debebunt.

131. Cum autem, quemadmodum his caſibus differentiationes ſingulas abſolui oporteat, infra fuſius ſimus oſtenſuri, hic pergamus quantitatem variabilem x tanquam vniformiter creſcentem aſſumere, ita vt eius differentialia prima dx, dx^{I}, dx^{II}, &c. inter ſe omnia aequalia, ac propterea differentialia ſecunda ac ſequentia nihilo aequalia ſtatuantur: quae conditio ita enunciari ſolet vt differentiale ipſius x nempe dx conſtans aſſumi dicatur. Sit deinde y functio quaecunque ipſius x, quae cum per x & conſtantes definiatur, ſingula quoque eius differentialia prima, ſecunda, tertia, quarta, &c. quae his ſignis indicantur dy, ddy, d^3y, d^4y, &c. per x & dx exprimi poterunt. Scilicet ſi in y loco x ſcribatur $x+dx$, ab hocque valore prior ſubtrahatur, remanebit differentiale primum dy: in quo ſi porro loco x ponatur $x+dx$, prodibit dy^{I}, eritque $ddy = dy^{\text{I}} - dy$, ſimili modo ponendo $x+dx$ loco x, ex ddy naſcetur ddy^{I}, atque $ddy^{\text{I}} - ddy$ dabit d^3y & ita porro: in quibus

bus operationibus differentiale dx perpetuo tanquam quantitas conſtans ſpectatur, quae nullum differentiale recipiat.

132. Ex ratione, qua functio y per x determinatur, tam ope methodi differentiarum finitarum, quam multo expeditius ex iis, quae poſtea ſumus tradituri, definietur valor functionis p, quae per dx multiplicata praebeat differentiale primum dy. Poſito ergo $dy = pdx$, differentiale ipſius pdx dabit differentiale ſecundum ddy; vnde ſi fuerit $dp = qdx$, ob dx conſtans, orietur $ddy = qdx^2$, vti iam ante oſtendimus. Vlterius igitur progrediendo, cum differentialis ſecundi differentiale praebeat differentiale tertium, ponamus eſſe $dq = rdx$, eritque $d^3y = rdx^3$: ſimili modo ſi huius functionis r differentiale quaeratur, fueritque $dr = sdx$, habebitur differentiale quartum $d^4y = sdx^4$; ſicque porro, dummodo nouerimus differentiale primum cuiusque functionis inuenire, differentiale cuiusque ordinis aſſignare poterimus.

133. Quo igitur formae ſingulorum horum differentialium, ſimulque ratio ea inueniendi clarius menti repraeſentetur, ea ſequenti tabella complecti viſum eſt.

Si y

Si y fuerit functio quaecunque ipſius x,

erit	atque poſito
$dy = p\,dx$	$dp = q\,dx$
$ddy = q\,dx^2$	$dq = r\,dx$
$d^3y = r\,dx^3$	$dr = s\,dx$
$d^4y = s\,dx^4$	$ds = t\,dx$
$d^5y = t\,dx^5$	&c.

Cum igitur functio p ex functione y per differentiationem cognoſcatur, ſimilique modo ex p inueniatur q, hincque porro r, & ex eo vlterius s, &c. differentialia cuiusuis ordinis ipſius y facile reperientur, dummodo differentiale dx aſſumatur conſtans.

134. Cum p, q, r, s, t, &c. ſint quantitates finitae, functiones nimirum ipſius x, differentiale primum ipſius y, rationem finitam habebit ad differentiale primum ipſius x, ſcilicet vt p ad 1; hancque ob cauſam differentialia dx & dy vocantur homogenea. Deinde cum ddy ad dx^2 habeat rationem finitam vt q ad 1, erunt ddy & dx^2 homogenea; ſimili modo homogenea erunt d^3q & dx^3, itemque d^4y & dx^4, & ita porro. Vnde vti differentialia prima ſunt inter ſe homogenea, ſeu rationem finitam tenentia; ſic differentialia ſecunda cum quadratis differentialium primorum, differentialia autem tertia cum cubis differentialium primorum atque ita porro erunt homogenea. Atque generatim differentiale ipſius

ſius y ordinis n, quod ita exprimitur $d^n y$, homogeneum erit cum dx^n, hoc eſt cum poteſtate differentialis dx, cuius exponens eſt n.

135. Cum igitur prae dx euaneſcant omnes eius poteſtates, quarum exponentes ſunt vnitate maiores, prae dy quoque euaneſcent dx^2, dx^3, dx^4, &c. & quae ad has poteſtates rationem finitam tenent differentialia altiorum ordinum ddy, d^3y, d^4y, &c. Simili modo prae ddy quia eſt homogeneum cum dx^2, omnes ipſius dx poteſtates quadrato ſuperiores dx^3, dx^4, &c. euaneſcent, euaneſcent ergo quoque d^3y, d^4y, &c. Atque prae d^3y, euaneſcent dx^4, d^4y; dx^5, d^5y; &c. Hincque facile, ſi propoſitae fuerint quaecunque expreſſiones huiusmodi differentialia inuoluentes, dignoſci poterunt, vtrum ſint homogeneae nec ne? Reſpici enim debebunt tantum differentialia, omiſſis quantitatibus finitis, quippe quae homogeneitatem non turbant; atque pro differentialibus ſecundi altiorumque ordinum ſcribantur poteſtates ipſius dx ipſis homoneae, quae ſi praebeant vbique eundem dimenſionum numerum, expreſſiones erunt homogeneae.

136. Ita patebit has expreſſiones $Pddy^2$ & $Qdyd^3y$ eſſe inter ſe homogeneas. Nam ddy^2 denotat quadratum ipſius ddy, & quia ddy homogeneum eſt cum dx^2, erit ddy^2 homogeneum cum dx^4. Deinde quia dy cum dx & d^3y cum dx^3 homogeneum eſt, erit productum dyd^3y cum dx^4 homogeneum: ex quo ſequitur expreſſiones $Pddy^2$ & $Qdyd^3y$ inter ſe eſſe homogeneas, ideo-

que rationem inter ſe finitam habere. Simili modo colligetur has expreſſiones $\frac{P d^3 y^2}{dx ddy}$ & $\frac{Q d^5 y}{dy^2}$ eſſe homogeneas; ſubſtitutis enim pro dy, ddy, $d^3 y$ & $d^5 y$ his ipſius dx poteſtatibus ipſis homogeneis dx, dx^2, dx^3 & dx^5, orientur hae expreſſiones $P dx^3$ & $Q dx^3$, quae vtique erunt inter ſe homogeneae.

137. Quod ſi facta hac reductione expreſſiones propoſitae non contineant aequales ipſius dx poteſtates, tum non erunt homogeneae, neque propterea inter ſe rationem finitam tenebunt. Erit ergo altera infinities ſiue maior ſiue minor altera, hincque vna reſpectu alterius euaneſcet. Sic $\frac{P d^3 y}{dx^2}$ ad $\frac{Q ddy^2}{dy}$ rationem habebit infinite magnam: prior enim expreſſio reducitur ad $P dx$ & altera ad $Q dx^3$, vnde haec prae illa euaneſcet. Quamobrem ſi in quopiam calculo aggregatum huiusmodi binarum formularum occurrat, $\frac{P d^3 y}{dx^2} + \frac{Q ddy^2}{dy}$, poſterior terminus prae priori tuto reiici, ſolusque primus $\frac{P d^3 y}{dx^2}$ in calculo retineri poterit: ſubſiſtet enim perfecta ratio aequalitatis inter expreſſiones

$$\frac{P d^3 y}{dx^2} + \frac{Q ddy^2}{dy} \;\&\; \frac{P d^3 y}{dx^2},$$

quia exponens rationis eſt

$$= 1 + \frac{Q dx^2 ddy^2}{P dy d^3 y} = 1 \text{ ob } \frac{Q dx^2 ddy^2}{P dy d^3 y} = 0.$$

Hoc-

Hocque pacto expreſſiones differentiales quandoque mirifice contrahi poſſunt.

138. In calculo differentiali praecepta traduntur, quorum ope cuiusuis quantitatis propoſitae differentiale primum inueniri poteſt: & quoniam differentialia ſecunda ex differentiatione primorum, tertia per eandem operationem ex ſecundis & ita porro ſequentia ex praecedentibus reperiuntur, calculus differentialis continet methodum omnia cuiusque ordinis differentialia inueniendi. Ex voce autem *differentialis*, qua differentia infinite parva denotatur, alia nomina deriuantur, quae vſu ſunt recepta. Sic verbum habetur *differentiare*, quod ſignificat *differentiale inuenire*, quantitasque *differentiari* dicitur, quando eius differentiale elicitur. *Differentiatio* autem denotat operationem, qua differentialia inueniuntur. Hinc calculus differentialis quoque vocatur methodus *differentiandi*, cum modum differentialia inueniendi contineat.

139. Quemadmodum in calculo differentiali cuiusvis quantitatis differentiale inueſtigatur, ita viciſſim calculi ſpecies conſtituitur quoque in inuentione eius quantitatis, cuius differentiale proponitur, qui calculus integralis vocatur. Si enim propoſitum fuerit differentiale quodcunque, eius reſpectu ea quantitas, cuius eſt differentiale, vocari ſolet integrale. Cuius denominationis ratio eſt, quod, cum differentiale conſiderari poſſit, tanquam pars infinite parua, qua quantitas quaepiam creſcit, ipſa illa quantitas reſpectu huius partis tanquam totum ſeu integrum ſpectari poteſt, hancque ob cauſam eius vo-

catur integrale. Sic cum dy ſit differentiale ipſius y, viciſſim y erit integrale ipſius dy, & cum ddy ſit differentiale ipſius dy, erit dy integrale ipſius ddy. Similique modo erit ddy integrale ipſius d^3y, & d^3y ipſius d^4y & ita porro: vnde quaelibet differentiatio, ſi inuerſe ſpectatur, integrationis exemplum exhibet.

140. Origo & natura integralium pariter ac differentialium clariſſime ex differentiarum finitarum doctrina in capite primo expoſita explicari poteſt. Poſtquam enim eſſet oſtenſum, quomodo cuiusque quantitatis differentiam inueniri oporteat, retrogrediendo quoque monſtravimus, quomodo, ſi propoſita fuerit differentia, ea quantitas inueniri queat, cuius illa ſit differentia; quam quantitatem reſpectu ſuae differentiae vocauimus eius ſummam. Vti igitur ad infinita parua procedendo differentiae in differentialia abierunt, ita ſummae quae ibi erant vocatae, integralium nomen ſortiuntur: & hanc ob cauſam integralia quoque non raro ſummae appellari ſolent. Angli qui differentialia fluxiones nominant, integralia vocant quantitates fluentes; eorumque loquendi more datae fluxionis fluentem inuenire, idem eſt, quod noſtro more dati differentialis integrale inuenire dicimus.

141. Vti differentialia charactere d deſignamus, ita ad integralia indicanda hac littera $\int$ vtimur, quae ergo quantitatibus differentialibus praefixa eas denotabit quantitates, quarum illa ſunt differentialia. Sic ſi differentiale ipſius y fuerit pdx, ſeu $dy = pdx$, erit y integrale ipſius

pdx,

$p\,dx$, quod hoc modo ſcribitur $y = \int p\,dx$, cum ſit $y = \int dy$. Integrale ergo ipſius $p\,dx$, quod per $\int p\,dx$ indicatur, denotat quantitatem, cuius differentiale eſt $p\,dx$. Simili modo cum ſit $ddy = q\,dx^2$ exiſtente $dp = q\,dx$; erit integrale ipſius ddy hoc eſt $dy = p\,dx$, atque ob $p = \int q\,dx$, erit $dy = dx \int q\,dx$, ac propterea $y = \int dx \int q\,dx$. Si vlterius ſit $dq = r\,dx$, erit $q = \int r\,dx$ & $dp = dx \int r\,dx$; vnde ſi character $\int$ denuo praefigatur, fiet $p = \int dx \int r\,dx$, porroque $dy = dx \int dx \int r\,dx$, atque $y = \int dx \int dx \int r\,dx$.

142. Quia differentiale dy eſt quantitas infinite parva, eius integrale autem y quantitas finita, parique modo differentiale ſecundum ddy infinities minus eſt, quam eius integrale dy, manifeſtum eſt differentialia prae ſuis integralibus euaneſcere. Quae affectio quo melius percipiatur, infinite parua in ordines diuidi ſolent, diciturque infinite paruum primi ordinis, ad quod referuntur differentialia prima dx, dy. Infinite paruum ſecundi ordinis complectitur differentialia ſecundi ordinis, quae homogenea ſunt cum dx^2; ſimilique modo infinite parva, quae cum dx^3 ſunt homogenea, vocantur ordinis tertii, ad quem ergo pertinent differentialia tertia omnia; ſicque porro. Vnde vti infinite parua primi ordinis prae quantitatibus finitis euaneſcunt, ſic infinite parua ſecundi ordinis prae infinite paruis primi ordinis, atque generatim infinite parua cuiusque ordinis altioris prae infinite paruis ordinis inferioris euaneſcent.

143. His igitur infinite paruorum ordinibus conſtitutis, vti differentiale quantitatis finitae eſt infinite par-

vum primi ordinis, atque differentiale infinite parui primi ordinis eſt infinite paruum ſecundi ordinis, & ita porro; ita viciſſim manifeſtum eſt integrale infinite parui primi ordinis eſſe quantitatem finitam, integrale autem infinite parui ſecundi ordinis eſſe infinite paruum primi ordinis ſicque deinceps. Quare ſi differentiale propoſitum fuerit infinite paruum ordinis n, eius integrale erit infinite paruum ordinis $n-1$; hincque vti differentiando ordo infinite paruorum augetur, ita integratione ad ordines inferiores progredimur, donec ad ipſas quantitates finitas perueniamus. Sin autem quantitates finitas denuo integrare velimus, tum ſecundum hanc legem perueniemus ad quantitates infinite magnas, ab harumque integratione inſtituta ad quantitates adhuc infinities maiores, ſicque progrediendo obtinebimus ſimiles infinitorum ordines, quorum quisque praecedentem infinities ſuperat.

144. Supereſt vt in hoc Capite quaedam de vſu ſignorum recepto moneamus, ne ambiguitati vllus locus relinquatur. Ac primo quidem ſignum differentiationis d tantum afficit literam immediate ſequentem ſolam: ſic dxy non denotat differentiale producti xy, ſed differentiale ipſius x per ipſam quantitatem y multiplicatum. Solet autem, quominus confuſio naſcatur, quantitas y ante ſignum d hoc modo ſcribi ydx, quo productum ex y in dx indicatur. Attamen ſi y ſit quantitas vel ſignum radicale $\surd$ vel logarithmicum habens praefixum, tum poſt differentiale poni ſolet: nimirum $dx\surd(aa-xx)$ ſignificat productum ex quantitate finita $\surd(aa-xx)$ in dif-

differentiale dx, ſimilique modo $dxl(1+x)$ eſt productum ex logarithmo quantitatis $1+x$, per dx multiplicato. Ob eandem rationem $ddy\sqrt{x}$ exprimit productum differentialis ſecundi ddy & quantitatis finitae $\sqrt{x}$.

145. Neque vero ſignum d litteram immediate ſequentem ſolam afficit, ſed etiam nequidem exponentem ſi quem habet, ſpectat. Ita dx^2 non exprimit differentiale ipſius x^2, ſed quadratum differentialis ipſius x, ita vt exponens 2 non ad x, ſed ad dx referri debeat. Poſſet etiam ſcribi $dxdx$, quemadmodum productum duorum differentialium dx & dy hoc modo $dxdy$ exponitur, verum prior modus dx^2, vti eſt breuior, ita vſitatior. Praeſertim ſi altiores poteſtates ipſius dx eſſent indicandae, nimis prolixum foret dx toties repeti: ſic dx^3 denotat cubum ipſius dx, & in differentialibus altiorum ordinum ſimilis ratio obſeruatur. Scilicet ddy^4 denotat poteſtatem quartam differentialis ſecundi ordinis ddy; atque $d^3y^2\sqrt{x}$ ſignificat quadratum differentialis tertii ordinis ipſius y multiplicatum eſſe per $\sqrt{x}$; ſin autem per quantitatem rationalem x multiplicari deberet, ea praefigitur hoc modo xd^3y^2.

146. Sin autem velimus, vt ſignum d plus quam ſolam litteram ſubſequentem afficiat, id peculiari modo indicari debet. Vtimur hoc caſu praecipue vncinulis, quibus ea quantitas includitur, cuius differentiale debet indicare. Vti $d(xx+yy)$ denotat differentiale quantitatis $xx+yy$; verum ſi velimus differentiale poteſtatis hu-

huiusmodi quantitatis deſignare, ambiguitatem vix euitare poſſumus: ſi enim ſcribamus $d(xx+yy)^2$, intelligi poſſet quadratum ipſius $d(xx+yy)$. Poterimus autem hoc caſu punctum in auxilium vocare, ita vt $d.(xx+yy)^2$ denotet differentiale ipſius $(xx+yy)^2$, omiſſo autem puncto $d(xx+yy)^2$ quadratum ipſius $d(xx+yy)$. Puncto ſcilicet commode indicari poteſt ſignum d ad totam quantitatem poſt punctum ſequentem pertinere: ſic $d.xdy$ exprimet differentiale ipſius xdy; & $d.^3 xdyV(aa+xx)$ differentiale tertii ordinis expreſſionis $xdyV(aa+xx)$, quae eſt productum ex quantitatibus finitis x & $V(aa+xx)$ atque ex differentiali dy.

147. Quemadmodum autem ſignum differentiationis d ſolam quantitatem immediate ſequentem afficit, niſi puncto interpoſito eius vis ad totam expreſſionem ſequentem extendatur; ita contra ſignum integrationis $\int$ ſemper totam expreſſionem, cui eſt praefixum, complectitur. Ita $\int ydx(aa-xx)^n$ denotat integrale ſeu eam quantitatem, cuius differentiale eſt $ydx(aa-xx)^n$, atque haec expreſſio $\int xdx\int dxlx$ denotat quantitatem, cuius differentiale eſt $xdx\int dxlx$. Hinc ſi velimus productum duorum integralium ſcilicet $\int ydx$ & $\int zdx$ exprimere, id hoc modo $\int ydx\int zdx$ perperam fiet, intelligeretur enim integrale quantitatis $ydx\int zdx$. Hanc ob cauſam iterum puncto ſolet haec ambiguitas tolli, ita vt $\int ydx . \int zdx$ ſignificet productum integralium $\int ydx$ & $\int zdx$.

148.

148. Analyſis infinitorum igitur cum in differentialibus tum in integralibus inueniendis verſatur, & hanc obrem in duas praecipuas partes diuiditur, quarum altera vocatur Calculus differentialis, altera Calculus integralis. In priori praecepta traduntur quantitatum quarumvis differentialia inueniendi; in poſteriori vero via monſtratur differentialium propoſitorum integralia inueſtigandi: in vtroque autem ſimul ſummus vſus, quem iſti calculi tam ad ipſam Analyſin quam ad Geometriam ſublimiorem afferunt, indicatur. Quam ob cauſam iſta Analyſeos pars iam tanta accepit incrementa, vt modico volumine prorſus comprehendi nequeat. Imprimis vero in calculo integrali indies tam nova artificia integrandi, quam adiumenta eius in ſoluendis variis generis problematibus, deteguntur, vt ob haec noua inuenta, quae continuo accedunt, nunquam exhauriri, multo minus perfecte deſcribi atque explicari poſſit. Dabo autem operam, vt quae adhuc ſunt reperta, vel cuncta in his libris exponam, vel ſaltem methodos explicem, vnde ea facile deduci queant.

149. Solent vulgo plures Analyſeos infinitorum partes numerari; praeter calculos enim differentialem & integralem inueniuntur paſſim calculi differentio-differentialis atque exponentialis. In calculo differentio-differentiali tradi ſolet methodus differentialia ſecundi atque altiorum ordinum inueniendi: quoniam autem modum cuiusque ordinis differentialia inueniendi in ipſo calculo differentiali ſum expoſiturus, hac ſubdiuiſione, quae potius

Q ex

ex merito inuentionis, quam ex re ipſa facta eſſe videtur, ſuperſedebimus. Quod deinde ad calculum exponentialem attinet, quo Celeb. IOH. BERNOULLI, cui ob innumera eaque maxima incrementa Analyſeos infinitorum aeternas debemus gratias, methodos differentiandi atque integrandi ad quantitates exponentiales transtulit, quia vtrumque calculum ad omnis generis quantitates tam algebraicas quam transcendentes accommodare conſtitui, hinc partem peculiarem facere ſuperfluum atque inſtituto contrarium foret.

150. Primum igitur calculum differentialem in hoc libro pertractare ſtatui, modumque ſum expoſiturus, cuius ope omnium quantitatum variabilium differentialia non ſolum prima, ſed etiam ſecunda & altiorum ordinum expedite inueniri queant. Primum ergo quantitates algebraicas contemplabor, ſiue ſint functiones vnius variabilis, ſiue plurium, ſiue demum explicite dentur, ſiue per aequationes. Deinde inuentionem differentialium quoque accommodabo ad quantitates non algebraicas, ad quarum notitiam quidem ſine calculi integralis ſubſidio peruenire licet: cuiusmodi ſunt logarithmi, atque quantitates exponentiales; deinde etiam arcus circuli, viciſſimque arcuum circularium ſinus, & tangentes. Denique etiam quantitates vtcunque ex his compoſitas & permixtas differentiare docebo; ſicque calculi differentialis pars prior, methodus ſcilicet differentiandi abſoluetur.

151. Altera pars vſui, quem methodus differentiandi tam ad Analyſin quam Geometriam ſublimiorem affert, explicando eſt deſtinata. In Algebram autem communem inde plurima redundant commoda, partim ad radices aequationum inueniendas, partim ad ſeries tractandas atque ſummandas, partim ad maxima minimaque eruenda, partim ad valores expreſſionum, quae certis caſibus indeterminatae videantur, definiendos, & quae ſunt alia. Geometria autem ſublimior ex calculo differentiali maxima accepit incrementa, dum eius ope tangentes linearum curuarum, eorumque curuatura ipſa mira facilitate definiri, multaque alia problemata circa radios a lineis curuis vel reflexos vel refractos reſolui poſſunt. Quibus etſi ampliſſimus tractatus impleri poſſet, tamen conabor, quantum fieri licet, omnia breuiter ac perſpicue explicare.

CAPUT V.

DE DIFFERENTIATIONE FUNCTIONUM ALGEBRAICARUM VNICAM VARIABILEM INUOLUENTIUM.

152.

Quia quantitatis variabilis x differentiale eſt $= dx$ erit x in proximum promouendo $x^{I} = x + dx$. Quare ſi fuerit y quaecunque functio ipſius x, ſi in ea loco x ponatur $x + dx$, ea abibit in y^{I}, atque differentia $y^{I} - y$ dabit differentiale ipſius y. Si igitur ponamus $y = x^n$ fiet

$$y^{I} = (x + dx)^n = x^n + nx^{n-1}dx + \frac{n(n-1)}{1.\ 2}x^{n-2}dx^2 + \&c.$$

eritque ergo

$$dy = y^{I} - y = nx^{n-1}dx + \frac{n(n-1)}{1.\ 2}x^{n-2}dx^2 + \&c.$$

At in hac expreſſione terminus ſecundus cum reliquis ſequentibus prae primo euaneſcit, eritque idcirco $nx^{n-1}dx$ differentiale ipſius x^n, ſeu $d.x^n = nx^{n-1}dx$. Vnde ſi a ſit numerus ſeu quantitas conſtans, erit quoque $d.ax^n = nax^{n-1}dx$. Cuiuscunque ergo ipſius x poteſtatis differentiale inuenitur, multiplicando eam per exponentem, diuidendo per x, & reliquum per dx multiplicando, quae regula facile memoria retinetur.

153. Cognito differentiali primo ipſius x^n, ex eo facile differentiale ſecundum reperitur, dummodo, vt

hic

hic conſtanter aſſumemus, differentiale dx conſtans ſtatuatur. Cum enim in differentiali $nx^{n-1}dx$ factor ndx ſit conſtans, alterius factoris x^{n-1} differentiale ſumi debet, quod proinde erit $(n-1)x^{n-2}dx$. Hoc ego per ndx multiplicatum dabit differentiale ſecundum: $dd.x^n = n(n-1)x^{n-2}dx^2$. Simili modo ſi differentiale ipſius x^{n-2} quod eſt $=(n-2)x^{n-3}dx$ multiplicetur per $n(n-1)dx^2$ prodibit differentiale tertium

$$d.^3x^n = n(n-1)(n-2)x^{n-3}dx^3.$$

Porro itaque erit differentiale quartum

$$d.^4x^n = n(n-1)(n-2)(n-3)x^{n-4}dx^4,$$

& differentiale quintum

$$d.^5x^n = n(n-1)(n-2)(n-3)(n-4)x^{n-5}dx^5;$$

vnde ſimul forma ſequentium differentialium facillime colligitur.

154. Quoties ergo n eſt numerus integer affirmatiuus, toties ad differentialia tandem peruenitur euaneſcentia; quae ſcilicet ita ſunt $=0$, vt prae omnibus ipſius dx poteſtatibus euaneſcant. Horum autem notandi ſunt caſus ſimpliciores.

$d.x = dx$; $dd.x = 0$; $d.^3x = 0$; &c.

$d.x^2 = 2xdx$; $dd.x^2 = 2dx^2$; $d.^3x^2 = 0$; $d.^4x^2 = 0$ &c.

$d.x^3 = 3x^2dx$; $dd.x^3 = 6xdx^2$; $d.^3x^3 = 6dx^3$; $d.^4x^3 = 0$

$d.x^4 = 4x^3dx$; $dd.x^4 = 12x^2dx^2$; $d.^3x^4 = 24xdx^3$; $d.^4x^4 = 24dx^4$

$d.x^5 = 5x^4dx$; $dd.x^5 = 20x^3dx^2$; $d.^3x^5 = 60x^2dx^3$; $d.^4x^5 = 120xdx^4$; $d.^5x^5 = 120dx^5$; $d.^6x^5 = 0$.

Patet ergo ſi n fuerit numerus integer affirmatiuus, poteſtatis x^n differentiale ordinis n eſſe conſtans, nempe $= 1.2.3. \ldots n dx^n$, adeoque differentialia ſuperiorum ordinum omnium eſſe $= 0$.

155. Si n ſit numerus integer negatiuus, huiusmodi ipſius x poteſtatum negatiuarum $\frac{1}{x}$, $\frac{1}{x^2}$, $\frac{1}{x^3}$, &c. differentialia ſumi poterunt, cum ſit $\frac{1}{x} = x^{-1}$; $\frac{1}{xx} = x^{-2}$, & generaliter $\frac{1}{x^m} = x^{-m}$. Si ergo in formula antecedente ponatur $n = -m$, erit ipſius $\frac{1}{x^m}$ differentiale primum $= \frac{-m\,dx}{x^{m+1}}$; differentiale ſecundum $= \frac{m(m+1)\,dx^2}{x^{m+2}}$; differentiale tertium $= \frac{-m(m+1)(m+2)\,dx^3}{x^{m+3}}$ &c. vnde ſequentes caſus ſimpliciores imprimis notari merentur.

$$d.\frac{1}{x} = \frac{-dx}{x^2}; \quad dd.\frac{1}{x} = \frac{2dx^2}{x^3}; \quad d.^3\frac{1}{x} = \frac{-6dx^3}{x^4}$$

$$d.\frac{1}{x^2} = \frac{-2dx}{x^3}; \quad dd.\frac{1}{x^2} = \frac{6dx^2}{x^4}; \quad d.^3\frac{1}{x^2} = \frac{-24dx^3}{x^5}$$

$$d.\frac{1}{x^3} = \frac{-3dx}{x^4}; \quad dd.\frac{1}{x^3} = \frac{12dx^2}{x^5}; \quad d.^3\frac{1}{x^3} = \frac{-60dx^3}{x^6}$$

$$d.\frac{1}{x^4} = \frac{-4dx}{x^5}; \quad dd.\frac{1}{x^4} = \frac{20dx^2}{x^6}; \quad d.^3\frac{1}{x^4} = \frac{-120dx^3}{x^7}$$

$$d.\frac{1}{x^5} = \frac{-5dx}{x^6}; \quad dd.\frac{1}{x^5} = \frac{30dx^2}{x^7}; \quad d.^3\frac{1}{x^5} = \frac{-210dx^3}{x^8}$$

&c. 156.

156. Ponendis deinde pro n numeris fractis differentialia formularum irrationalium obtinebimus. Sit enim $n = \frac{\mu}{\nu}$, erit formulae $x^{\frac{\mu}{\nu}}$ ſeu $\sqrt[\nu]{x^{\mu}}$ differentiale primum

$$= \frac{\mu}{\nu} x^{\frac{\mu-\nu}{\nu}} dx = \frac{\mu}{\nu} dx \sqrt[\nu]{x^{\mu-\nu}} \quad \text{ſecundum}$$

$$= \frac{\mu(\mu-\nu)}{\nu^2} x^{\frac{\mu-2\nu}{\nu}} dx^2 = \frac{\mu(\mu-\nu)}{\nu\nu} dx^2 \sqrt[\nu]{x^{\mu-2\nu}} \quad \&c.$$

Hinc erit :

$$d.\sqrt{x} = \frac{dx}{2\sqrt{x}}; \quad dd.\sqrt{x} = \frac{-dx^2}{4x\sqrt{x}}; \quad d.^3\sqrt{x} = \frac{1.3\,dx^3}{8x^2\sqrt{x}}$$

$$d.\sqrt[3]{x} = \frac{dx}{3\sqrt[3]{x^2}}; \quad dd.\sqrt[3]{x} = \frac{-2\,dx^2}{9x\sqrt[3]{x^2}}; \quad d.^3\sqrt[3]{x} = \frac{2.5\,dx^3}{27x^2\sqrt[3]{x^2}}$$

$$d.\sqrt[4]{x} = \frac{dx}{4\sqrt[4]{x^3}}; \quad dd.\sqrt[4]{x} = \frac{-3\,dx^2}{16x\sqrt[4]{x^3}}; \quad d.^3\sqrt[4]{x} = \frac{3.7\,dx^3}{64x^2\sqrt[4]{x^3}}$$

quae expreſſiones ſi pauliſper inſpiciantur, facile habitus acquiretur huiusmodi differentialia, etiam ſine praeuia reductione ad formam poteſtatis, inueniendi.

157. Si μ non fuerit 1, ſed numerus alius ſiue affirmatiuus ſiue negatiuus integer, differentialia aeque facile definientur. Cum autem differentialia ſecunda & altiorum ordinum eadem lege ex primis, qua haec ex ipſis poteſtatibus, deriuentur, exempla ſimpliciora primorum tantum differentialium apponamus.

$d.x\sqrt{x} = \frac{3}{2}dx\sqrt{x}$; $d.x^2\sqrt{x} = \frac{5}{2}xdx\sqrt{x}$; $d.x^3\sqrt{x} = \frac{7}{2}x^2dx\sqrt{x}$;

$d.\frac{1}{\sqrt{x}} = \frac{-dx}{2x\sqrt{x}}$; $d.\frac{1}{x\sqrt{x}} = \frac{-3dx}{2xx\sqrt{x}}$; $d.\frac{1}{xx\sqrt{x}} = \frac{-5dx}{2x^3\sqrt{x}}$;

$d.\sqrt[3]{x^2} = \frac{2}{3}\frac{dx}{\sqrt[3]{x}}$; $d.x\sqrt[3]{x} = \frac{4}{3}dx\sqrt[3]{x}$; $d.x\sqrt[3]{x^2} = \frac{5}{3}dx\sqrt[3]{x^2}$;

$d.xx\sqrt[3]{x} = \frac{7}{3}xdx\sqrt[3]{x}$; $d.xx\sqrt[3]{x^2} = \frac{8}{3}xdx\sqrt[3]{x^2}$; &c.

$d.\frac{1}{\sqrt[3]{x}} = \frac{-dx}{3x\sqrt[3]{x}}$; $d.\frac{1}{\sqrt[3]{x^2}} = \frac{-2dx}{3x\sqrt[3]{x^2}}$; $d.\frac{1}{x\sqrt[3]{x}} = \frac{-4dx}{3x^2\sqrt[3]{x}}$;

$d.\frac{1}{x\sqrt[3]{x^2}} = \frac{-5dx}{3x^2\sqrt[3]{x^2}}$; $d.\frac{1}{x^2\sqrt[3]{x}} = \frac{-7dx}{3x^3\sqrt[3]{x}}$; &c.

158. Ex his iam functionum omnium algebraicarum rationalium integrarum differentialia poterunt inueniri, propterea quod earum ſinguli termini ſunt poteſtates ipſius x, quas differentiare nouimus. Cum enim quantitas huiusmodi $p + q + r + s +$ &c. poſito $x + dx$ loco x abeat in $p + dp + q + dq + r + dr + s + ds +$ &c. erit eius differentiale $= dp + dq + dr + ds +$ &c. Quare ſi ſingularum quantitatum p, q, r, s, differentialia aſſignare queamus, ſimul quoque aggregati earum differentiale innoteſcet. Atque cum multipli ipſius p differentiale ſit aeque multiplum ipſius dp, hoc eſt $d.ap = adp$; erit quantitatis $ap + bq + cr$ differentiale $= adp + bdq + cdr$. Cum denique quantitatum conſtantium differentialia ſint nulla, erit quoque quantitatis huius $ap + bq + cr + f$ differentiale $= adp + bdq + cdr$.

159. In functionibus ergo rationalibus integris cum ſinguli termini ſint vel conſtantes vel poteſtates ipſius x, diffe-

differentiatio ſecundum praecepta data facile abſoluetur. Sic erit:

$d(a+x) = dx$; $d(a+bx) = bdx$;

$d(a+xx) = 2xdx$; $d(aa-xx) = -2xdx$;

$d(a+bx+cxx) = bdx+2cxdx$;

$d(a+bx+cxx+ex^3) = bdx+2cxdx+3ex^2dx$;

$d(a+bx+cxx+ex^3+fx^4) = bdx+2cxdx+3ex^2ax$
$+4fx^3dx.$

Atque ſi exponentes fuerint indefiniti erit:

$d(1-x^n) = -nx^{n-1}dx$; $d(1+x^m) = mx^{m-1}dx$;

$d(a+bx^m+cx^n) = mbx^{m-1}dx+ncx^{n-1}dx$.

160. Cum igitur functiones rationales integrae ſecundum maximam ipſius x dignitatem in gradus diſtinguantur, manifeſtum eſt, ſi huiusmodi functionum continuo differentialia capiantur, ea tandem fieri conſtantia, poſteaque in nihilum abire, ſi quidem differentiale dx aſſumatur conſtans. Sic functionis primi gradus $a+bx$ differentiale primum bdx eſt conſtans, ſecundum cum ſequentibus nullum. Sit functio ſecundi gradus

$$a+bx+cxx = y; \text{ erit } dy = bdx+2cxdx;$$
$$ddy = 2cdx^2 \;;\; d^3y = 0.$$

Simili modo ſi ponatur functio tertii gradus

$a+bx+cxx+ex^3 = y$; erit $dy = bdx+2cxdx+3ex^2dx$;
$ddy = 2cdx^2+6exdx^2$ & $d^3y = 6edx^3$ atque $d^4y = 0$.

Quare generaliter ſi huiusmodi functio ſit gradus n, eius differentiale ordinis n erit conſtans, & ſequentia omnia nulla.

161. Neque etiam differentiatio turbabitur, ſi inter poteſtates ipſius x, quae huiusmodi functionem componunt, occurrant tales, quarum exponentes ſint numeri negatiui ſeu fracti. Ita

I. Si ſit $y = a + b\sqrt{x} - \frac{c}{x}$

erit $dy = \frac{bdx}{2\sqrt{x}} + \frac{cdx}{xx}$.

II. Si ſit $y = \frac{a}{\sqrt{x}} + b + c\sqrt{x} - ex$

erit $dy = \frac{-adx}{2x\sqrt{x}} + \frac{cdx}{2\sqrt{x}} - edx$,

& $ddy = \frac{3adx^2}{4xx\sqrt{x}} - \frac{cdx^2}{4x\sqrt{x}}$.

III. Si ſit $y = a + \frac{b}{\sqrt[3]{xx}} - \frac{c}{x\sqrt[3]{x}} + \frac{f}{xx}$

erit $dy = \frac{-2bdx}{3x\sqrt[3]{xx}} + \frac{4cdx}{3xx\sqrt[3]{x}} - \frac{2fdx}{x^3}$,

& $ddy = \frac{10bdx^2}{9x^2\sqrt[3]{xx}} - \frac{28cdx^2}{9x^3\sqrt[3]{x}} + \frac{6fdx^2}{x^4}$.

cuiusmodi exempla ſecundum praecepta data facillime abſoluuntur.

162. Si quantitas differentianda propoſita fuerit poteſtas eiusmodi functionis, cuius differentiale exhibere valemus, praecedentia praecepta ſufficiunt ad eius differentiale primum definiendum. Sit enim p functio quaecunque ipſius x, cuius differentiale dp in poteſtate eſt, erit ipſi-

ipſius poteſtatis p^n differentiale primum $= np^{n-1}dp$. Hinc ſequentia exempla ſoluuntur:

I. Si ſit $y = (a+x)^n$; erit $dy = n(a+x)^{n-1}dx$

II. Si ſit $y = (aa-xx)^2$; erit $dy = -4xdx(aa-xx)$

III. Si ſit $y = \frac{1}{aa+xx}$ ſeu $y = (aa+xx)^{-1}$

erit $dy = \frac{-2xdx}{(aa+xx)^2}$.

IV. Si ſit $y = \sqrt{(a+bx+cxx)}$ erit

$$dy = \frac{bdx+2cxdx}{2\sqrt{(a+bx+cxx)}}.$$

V. Si ſit $y = \sqrt[3]{(a^4-x^4)^2}$ ſeu $y = (a^4-x^4)^{\frac{2}{3}}$

erit $dy = -\frac{8}{3}x^3dx(a^4-x^4)^{-\frac{1}{3}} = \frac{-8x^3dx}{3\sqrt[3]{(a^4-x^4)}}$.

VI. Si ſit $y = \frac{1}{\sqrt{(1-xx)}}$ ſeu $y = (1-xx)^{-\frac{1}{2}}$

erit $dy = xdx(1-xx)^{-\frac{3}{2}} = \frac{xdx}{(1-xx)\sqrt{(1-xx)}}$.

VII. Si ſit $y = \sqrt[3]{(a+\sqrt{bx}+x)}$

erit $dy = \frac{dx\sqrt{b}:2\sqrt{x}+dx}{3\sqrt[3]{(a+\sqrt{bx}+x)^2}} = \frac{dx\sqrt{b}+2dx\sqrt{x}}{6\sqrt{x}.\sqrt[3]{(a+\sqrt{bx}+x)^2}}$.

VIII. Si ſit $y = \frac{1}{x+\sqrt{(aa-xx)}}$,

ob $d.\sqrt{(aa-xx)} = \frac{xdx}{\sqrt{(aa-xx)}}$, erit

$$dy = \frac{-dx+xdx:\sqrt{(aa-xx)}}{(x+\sqrt{(aa-xx)})^2} = \frac{xdx-dx\sqrt{(aa-xx)}}{(x+\sqrt{(aa-xx)})^2\sqrt{(aa-xx)}}$$

ſeu $dy = \frac{dx(x-\sqrt{(aa-xx)})^3}{(2xx-aa)^2\sqrt{(aa-xx)}}$.

IX.

IX. Si ſit $y = \sqrt[4]{\left(1 - \frac{1}{\sqrt{x}} + \sqrt[3]{(1-xx)^2}\right)^3}$.

Ponatur $\frac{1}{\sqrt{x}} = p$ & $\sqrt[3]{(1-xx)^2} = q$;

ob $y = \sqrt[4]{(1-p+q)^3}$, erit $dy = \frac{-3dp+3dq}{4\sqrt[4]{(1-p+q)}}$.

Iam per antecedentia eſt

$$dp = \frac{-dx}{2x\sqrt{x}} \quad \& \quad dq = \left(\frac{3\sqrt[3]{(1-xx)}}{-4xdx}\right)\frac{-4xdx}{3\sqrt[3]{(1-xx)}},$$

quibus valoribus ſubſtitutis fiet:

$$dy = \frac{3dx : 2x\sqrt{x} - 4xdx : \sqrt[3]{(1-xx)}}{4\sqrt[4]{\left(1 - \frac{1}{\sqrt{x}} + \sqrt[3]{(1-xx)^2}\right)}}.$$

Simili autem modo ſingulares litteras loco terminorum aliquantum compoſitorum ſubſtituendo omnium huiusmodi functionum differentialia facile eruuntur.

163. Si quantitas differentianda fuerit productum ex duabus pluribusue functionibus ipſius x, quarum differentialia conſtant, eius differentiale ſequente modo commodiſſime inuenietur. Sint p & q functiones ipſius x, quarum differentialia dp & dq iam ſunt cognita, quia poſito $x+dx$ loco x; p abit in $p+dp$ & q in $q+dq$; productum pq transmutabitur in

$$(p+dp)(q+dq) = pq + pdq + qdp + dpdq,$$

vnde producti pq differentiale erit $= pdq + qdp + dpdq$; vbi cum pdq & qdp ſint infinite parua primi ordinis, at $dpdq$ ſecundi ordinis, vltimus terminus euaneſcet, eritque igitur $d.pq = pdq + qdp$. Differentiale ergo producti pq conſtat ex duobus membris, quae obtinentur, ſi

ſi vterque factor per differentiale alterius factoris multiplicetur. Hinc facile deducitur differentiatio producti pqr ex tribus factoribus conſtantis: ponatur enim $qr = z$, fiet $pqr = pz$, & $d.pqr = pdz + zdp$, verum ob $z = qr$ erit $dz = qdr + rdq$, quibus valoribus loco z & dz ſubſtitutis erit

$$d.pqr = pqdr + prdq + qrdp,$$

Simili modo ſi quantitas differentianda quatuor habeat factores erit:

$$d.pqrs = pqrds + pqsdr + prsdq + qrsdp,$$

vnde quilibet differentiationem plurium factorum facile perſpiciet.

I. Si ergo fuerit $y = (a + x)(b - x)$, erit

$$dy = -dx(a + x) + dx(b - x) = -adx + bdx - 2xdx$$

quod idem differentiale quoque inuenitur, ſi quantitas propoſita euoluatur: fit enim $y = ab - ax + bx - xx$, ideoque per ſuperiora praecepta $dy = -adx + bdx - 2xdx$.

II. Si fuerit $y = \frac{1}{x}V(aa - xx)$.

Ponatur $\frac{1}{x} = p$ & $V(aa - xx) = q$, quia eſt $dp = \frac{-dx}{xx}$ & $dq = \frac{-xdx}{V(aa - xx)}$, erit

$$dy = pdq + qdp = \frac{-dx}{V(aa - xx)} - \frac{dx}{xx}V(aa - xx);$$

quae ad eundem denominatorem reductae dabunt $\frac{-xxdx - aadx + xxdx}{xxV(aa - xx)} = \frac{-aadx}{xxV(aa - xx)}$. Hinc erit differentiale quaeſitum, $dy = \frac{-aadx}{xxV(aa - xx)}$.

III.

III. Si fuerit $y = \frac{xx}{\sqrt{(a^4 + x^4)}}$.

Ponatur $xx = p$, & $\frac{1}{\sqrt{(a^4 + x^4)}} = q$; quia inuenimus $dp = 2xdx$ & $dq = \frac{-2x^3dx}{(a^4+x^4)^{\frac{3}{2}}}$, erit
$$pdq + qdp = \frac{-2x^5dx}{(a^4+x^4)^{\frac{3}{2}}} + \frac{2xdx}{\sqrt{(a^4+x^4)}} = \frac{2a^4xdx}{(a^4+x^4)^{\frac{3}{2}}}.$$
Hinc ergo erit differentiale quaeſitum
$$dy = \frac{2a^4xdx}{(a^4+x^4)\sqrt{(a^4+x^4)}}.$$

IV. Si fuerit $y = \frac{x}{x + \sqrt{(1+xx)}}$.

Ponendo $x = p$ & $\frac{1}{x+\sqrt{(1+xx)}} = q$, ob $dp = dx$ & $dq = \frac{-dx - xdx : \sqrt{(1+xx)}}{(x+\sqrt{(1+xx)})^2} = \frac{-dx(x+\sqrt{(1+xx)})}{(x+\sqrt{(1+xx)})^2\sqrt{(1+xx)}}$ $= \frac{-dx}{(x+\sqrt{(1+xx)})\sqrt{(1+xx)}}$, erit $pdq + qdp =$ $= \frac{-xdx}{(x+\sqrt{(1+xx)})\sqrt{(1+xx)}} + \frac{dx}{x+\sqrt{(1+xx)}} =$ $= \frac{dx(\sqrt{(1+xx)} - x)}{(x+\sqrt{(1+xx)})\sqrt{(1+xx)}}$. Fiet ergo differentiale quaeſitum $dy = \frac{dx(\sqrt{(1+xx)}-x)}{(x+\sqrt{(1+xx)})\sqrt{(1+xx)}}$; cuius fractionis ſi numerator ac denominator multiplicetur per $\sqrt{(1-}$

$V(1+xx)-x$, fiet $dy = \frac{dx(1+2xx-2xV(1+xx))}{V(1+xx)}$
$= \frac{dx+2xxdx}{V(1+xx)} - 2xdx$.

Idem differentiale alio modo commodius inueniri poteſt; cum enim ſit $y = \frac{x}{x+V(1+xx)}$, multiplicetur numerator ac denominator per $V(1+xx)-x$, fietque

$$y = xV(1+xx) - xx = V(x^2+x^4) - xx,$$

cuius differentiale per priorem regulam eſt

$$dy = \frac{xdx+2x^3dx}{V(xx+x^4)} - 2xdx = \frac{dx+2xxdx}{V(1+xx)} - 2xdx.$$

V. Si fuerit $y = (a+x)(b-x)(x-c)$, erit

$$dy = (a+x)(b-x)dx - (a+x)(x-c)dx + (b-x)(x-c)dx.$$

VI. Si fuerit $y = x(aa+xx)V(aa-xx)$.

Ob tres factores ergo reperietur

$$dy = dx(aa+xx)V(aa-xx) + 2xxdxV(aa-xx)$$
$$- \frac{xxdx(aa+xx)}{V(aa-xx)} = \frac{dx(a^4+aaxx-4x^4)}{V(aa-xx)}.$$

164. Quanquam etiam fractiones in factoribus comprehendi poſſunt, tamen commodius vtemur regula fractionibus differentiandi inſeruiente. Sit ergo propoſita haec fractio $\frac{p}{q}$, cuius differentiale inueniri oporteat. Quoniam poſito $x+dx$ loco x fractio illa abit in

$p+$

$$\frac{p+dp}{q+dq}=(p+dp)\left(\frac{1}{q}-\frac{dq}{qq}\right)=\frac{p}{q}-\frac{pdq}{qq}+\frac{dp}{q}-\frac{dpdq}{qq},$$

vnde ſi fractio ipſa $\frac{p}{q}$ ſubtrahatur, remanet eius differentiale $d.\frac{p}{q}=\frac{dp}{q}-\frac{pdq}{qq}$, ob euaneſcentem terminum $\frac{dpdq}{qq}$. Hinc ergo erit $d.\frac{p}{q}=\frac{qdp-pdq}{qq}$, vnde haec regula pro differentiatione cuiusque fractionis enaſcitur. *A differentiali numeratoris per denominatorem multiplicato ſubtrahatur differentiale denominatoris per numeratorem multiplicatum, reſiduum dividatur per quadratum denominatoris, quotusque erit differentiale fractionis quaeſitum.* Cuius regulae vſus per ſequentia exempla illuſtrabitur.

I. Si fuerit $y=\frac{x}{aa+xx}$, erit per hanc regulam

$$dy=\frac{(aa+xx)\,dx-2xxdx}{(aa+xx)^2}=\frac{(aa-xx)\,dx}{(aa+xx)^2}.$$

II. Si fuerit $y=\frac{\sqrt{(aa+xx)}}{aa-xx}$; reperitur

$$dy=\frac{(aa-xx)xdx:\sqrt{(aa+xx)}+2xdx\sqrt{(aa+xx)}}{(aa-xx^2)},$$

& facta reductione $dy=\frac{(3aa+xx)\,xdx}{(aa-xx)^2\sqrt{(aa+xx)}}$.

Saepenumero expedit ea regula vti, quae ſequitur ex formula priori $d.\frac{p}{q}=\frac{dp}{q}-\frac{pdq}{qq}$, qua differentiale fractionis aequale reperitur differentiali numeratoris per de-

no-

nominatorem diuiſo, demto differentiali denominatoris per numeratorem multiplicato at per quadratum denominatoris diuiſo. Ita

III. Si fuerit $y = \frac{aa - xx}{a^4 + aaxx + x^4}$, erit

$$dy = \frac{-2xdx}{a^4 + aaxx + x^4} - \frac{(aa - xx)(2aaxdx + 4x^3dx)}{(a^4 + aaxx + x^4)^2}$$

quae ad eundem determinatorem reuocata praebet.

$$dy = \frac{-2xdx(2a^4 + 2aaxx - x^4)}{(a^4 + aaxx + x^4)^2}.$$

165. Haec iam ſufficiunt ad cuiusque functionis rationalis ipſius x propoſitae differentiale inueſtigandum; ſi enim fuerit integra modus differentiandi iam ſupra eſt expoſitus. Sit igitur functio propoſita fracta, quae ſemper ad huiusmodi formam reducetur:

$$y = \frac{A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c.}{\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \zeta x^5 + \&c.}$$

Ponatur numerator $= p$ & denominator $= q$, vt fiat

$y = \frac{p}{q}$; eritque $dy = \frac{qdp - pdq}{qq}$. At cum ſit:

$$p = A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.$$

& $q = \alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \&c.$

erit $dp = Bdx + 2Cxdx + 3Dx^2dx + 4Ex^3dx + \&c.$

& $dq = \beta dx + 2\gamma xdx + 3\delta x^2dx + 4\varepsilon x^3dx + \&c.$

vnde per multiplicationem obtinebitur:

$$\begin{aligned}
qdp = {} & \alpha B\,dx + 2\alpha C x\,dx + 3\alpha D x^2 dx + 4\alpha E x^3 dx + \&c. \\
& \qquad + \beta B x\,dx + 2\beta C x^2 dx + 3\beta D x^3 dx + \&c. \\
& \qquad\qquad + \gamma B x^2 dx + 2\gamma C x^3 dx + \&c. \\
& \qquad\qquad\qquad + \delta B x^3 dx + \&c. \\
pdq = {} & \beta A\,dx + \beta B x\,dx + \beta C x^2 dx + \beta D x^3 dx + \&c. \\
& \qquad + 2\gamma A x\,dx + 2\gamma B x^2 dx + 2\gamma C x^3 dx + \&c. \\
& \qquad\qquad + 3\delta A x^2 dx + 3\delta B x^3 dx + \&c. \\
& \qquad\qquad\qquad + 4\varepsilon A x^3 dx + \&c.
\end{aligned}$$

Ex his ita que obtinebitur differentiale quaeſitum:

$$dy = \frac{\begin{matrix} +\alpha B \\ -\beta A \end{matrix}\,dx \; \begin{matrix} +2\alpha C \\ -2\gamma A \end{matrix}\,x\,dx \; \begin{matrix} +3\alpha D \\ +\beta C \\ -\gamma B \\ -3\delta A \end{matrix}\,x^2 dx \; \begin{matrix} +4\alpha E \\ +2\beta D \\ -2\delta B \\ -4\varepsilon A \end{matrix}\,x^3 dx \; \begin{matrix} +5\alpha F \\ +3\beta E \\ +\gamma D \\ -\delta C \\ -3\varepsilon B \\ -5\zeta A \end{matrix}\,x^4 dx \; \&c.}{(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \zeta x^5 + \&c.)^2}$$

Quae expreſſio ad cuiusuis functionis rationalis differentiale expedite inueniendum maxime eſt accommodata. Quamadmodum enim numerator differentialis ex coefficientibus numeratoris ac denominatoris functionis propoſitae combinatur, ex inſpectione mox intelligitur. Denominator vero differentialis eſt quadratum denominatoris functionis propoſitae.

166. Si fractionis propoſitae vel numerator vel denominator vel vterque ex factoribus conſtet, multiplicatione actu inſtituta orietur quidem forma, qualem modo differentiauimus; attamen facilius pro his caſibus regula peculiaris formabitur. Sit igitur propoſita huiusmodi fractio $y = \frac{pr}{q}$. Ponatur numerator $pr = P$, vt ſit $dP = pdr + rdp$. Atque ob $y = \frac{P}{q}$, erit $dy = \frac{qdP - Pdq}{qq}$, ſubſtitutis autem loco P & dP valoribus, habebitur:

I. Si fuerit $y = \frac{pr}{q}$;

eius diff. $dy = \frac{pqdr + qrdp - prdq}{qq}$.

Si ſit $y = \frac{p}{qs}$, poſito denominatore $qs = Q$, erit $dQ = qds + sdq$, & $dy = \frac{Qdp - pdQ}{qqss}$. Quare

II. Si fuerit $y = \frac{p}{qs}$,

erit $dy = \frac{qsdp - pqds - psdq}{qqss}$.

Si fuerit $y = \frac{pr}{qs}$, ponatur $pr = P$ & $qs = Q$, vt habeatur $y = \frac{P}{Q}$, & $dy = \frac{QdP - PdQ}{QQ}$. Cum autem ſit $dP = pdr + rdp$ & $dQ = qds + sdq$, prodibit ſequens differentiatio:

III. Si fuerit $y = \frac{pr}{qs}$,

erit $dy = \frac{pqsdr + qrsdp - pqrds - prsdq}{qqss}$,

ſeu $dy = \frac{rdp}{qs} + \frac{pdr}{qs} - \frac{prdq}{qqs} - \frac{prds}{qss}$.

Simili modo, ſi numerator ac denominator fractionis propoſitae plures habeant factores, differentialia eadem ratione inueſtigabuntur; neque ad hoc ampliori manuductione erit opus. Quamobrem quoque exempla huc pertinentia praetermitto, cum mox modus generalis has omnes differentiandi methodos particulares complectens afferetur.

167. Dantur autem caſus tam productorum quam fractionum, quibus differentiale commodius exprimi poteſt, quam per regulas generaliores hic expoſitas. Euenit hoc ſi factores, qui vel functionem ipſam, vel functionis numeratorem aut denominatorem conſtituunt, fuerint poteſtates.

Ponamus functionem differentiandam eſſe $y = p^m q^n$, ad cuius differentiale inueniendum ſit $p^m = P$ & $q^n = Q$, vt fiat $y = PQ$ & $dy = PdQ + QdP$. Cum autem ſit $dP = mp^{m-1}dp$ & $dQ = nq^{n-1}dq$, fiet his valoribus ſubſtitutis:

$dy = np^m q^{n-1}dq + mp^{m-1}q^n dp = p^{m-1}q^{n-1}(npdq + mqdp)$; vnde ſequens oritur regula:

I. Si

I. Si fuerit $y = p^m q^n$;

erit $dy = p^{m-1} q^{n-1} (npdq + mqdp)$.

Simili modo ſi tres fuerint factores, differentiale inuenietur, ac reperietur hoc modo expreſſum.

II. Si fuerit $y = p^m q^n r^k$;

erit $dy = p^{m-1} q^{n-1} r^{k-1} (mqrdp + nprdq + kpqdr)$.

168. Sin autem fuerit propoſita fractio, cuius vel numerator vel denominator habeat factorem, qui eſt poteſtas, regulae quoque particulares tradi poterunt. Sit primum propoſita huiusmodi fractio $y = \frac{p^m}{q}$, erit per regulam fractionibus inſeruientem $dy = \frac{mp^{m-1}qdp - p^m dq}{qq}$, quod differentiale commodius ſic exprimetur.

I. Si fuerit $y = \frac{p^m}{q}$,

erit $dy = \frac{p^{m-1}(mqdp - pdq)}{qq}$.

Sit iam $y = \frac{p}{q^n}$, fiet per eandem ſuperiorem regulam $dy = \frac{q^n dp - npq^{n-1}dq}{q^{2n}}$, cuius expreſſionis ſi numerator ac denominator per q^{n-1} diuidatur, erit $dy = \frac{qdp - npdq}{q^{n+1}}$. Quamobrem.

II. Si fuerit $y = \frac{p}{q^n}$, erit $dy = \frac{qdp - npdq}{q^{n+1}}$.

Quod ſi vero proponatur $y = \frac{p^m}{q^n}$; inuenietur

$$dy = \frac{mp^{m-1}q^n dp - np^m q^{n-1}dq}{q^{2n}},$$ quae reducitur ad

$$dy = \frac{mp^{m-1}qdp - np^m dq}{q^{n+1}}.$$ Quocirca

III. Si fuerit $y = \frac{p^m}{q^n}$;

erit $dy = \frac{p^{m-1}(mqdp - npdq)}{q^{n+1}}$.

Denique ſi propoſita fuerit huiusmodi fractio $y = \frac{r}{p^m q^n}$, habebitur per regulam fractionum generalem

$$dy = \frac{p^m q^n dr - mp^{m-1}q^n rdp - np^m q^{n-1}rdq}{p^{2m}q^{2n}},$$

cuius expreſſionis cum numerator & denominator ſit diviſibilis per $p^{m-1}q^{n-1}$:

IV. Si fuerit $y = \frac{r}{p^m q^n}$;

erit $dy = \frac{pqdr - mqrdp - nprdq}{p^{m+1}q^{n+1}}$.

Si plures occurrant factores, huiusmodi regulae ſpeciales, quas verbis exprimere ſuperfluum foret, facili negotio pro quouis caſus erui poterunt.

169. Regulae differentiandi quas hactenus expoſuimus tam late patent, vt nulla excogitari poſſit functio ipſius x algebraica, quae non earum ope differentiari que-

queat. Si enim functio ipſius x fuerit rationalis, vel erit integra vel fracta, priori caſu §. 159. modum dedimus eiusmodi fructiones differentiandi, poſteriori vero caſu in §. 165. negotium abſoluimus. Simul vero etiam compendia, ſi factores inuoluantur, differentiationis exhibuimus. Deinde vero etiam quantitates irrationales cuiusuis generis differentiare docuimus, quae quomodocunque functionem propoſitam afficiant, ſiue ei per additionem, ſiue per ſubtractionem ſiue multiplicationem, ſiue diuiſionem ſint implicatae, perpetuo ad caſus iam tractatos reuocari poterunt. Intelligenda autem haec ſunt de functionibus explicitis; nam de implicitis, quarum natura per aequationem datur, infra, poſtquam functiones duarum pluriumue variabilium differentiare docuerimus, tractandi locus erit.

170. Si regulas hic traditas ſingulas perpendamus atque inter ſe conferamus, eas omnes ad vnam maxime vniuerſalem reducere poterimus; quam autem infra demum rigida demonſtratione munire licebit; interim tamen & hoc loco non adeo difficile erit eius veritatem attendenti intueri. Functio quaecunque algebraica compoſita eſt ex partibus, quae vel additione vel ſubtractione vel multiplicatione vel diuiſione inter ſe erunt complicatae; haeque partes erunt vel rationales vel irrationales. Vocemus ergo iſtas quantitates functionem quamvis conſtituentes eius partes. *Tum pro qualibet parte functio propoſita ſeorſim ita differentietur, quaſi ea pars ſola eſſet variabilis, reliquae vero partes omnes conſtantes.*

Quo

Quo facto ſingula iſta differentialia, quae ex ſingulis partibus modo deſcripto eliciuntur, in vnam ſummam colligantur, ſicque obtinebitur differentiale functionis propoſitae. Huiusque regulae ope omnes omnino functiones differentiari poterunt, nequidem transcendentibus exceptis, vti infra oſtendetur.

171. Ad regulam hanc illuſtrandam ponamus functionem y duabus conſtare partibus, ſiue per additionem ſiue ſubtractionem connexis, ita vt ſit $y = p \pm q$. Ponatur primo ſola pars p variabilis, altera q conſtans erit differentiale $= dp$, deinde ponatur altera pars $\pm q$ ſola variabilis, altera vero p conſtans, eritque differentiale $= \pm dq$. Atque ex his differentialibus differentiale quaeſitum ita componetur, vt ſit $dy = dp \pm dq$, omnino vti idem iam ſupra inuenimus. Hinc vero ſimul liquet, ſi functio pluribus conſtet partibus, ſiue inuicem additis ſiue ſubtractis, nempe $q = p \pm q \pm r \pm s$, ope huius regulae inuentum iri $dy = dp \pm dq \pm dr \pm ds$, plane vti & ſuperior regula docebat.

172. Si partes ſint in ſe inuicem multiplicatae, ita vt ſit $y = pq$; manifeſtum eſt poſita ſola parte p variabili, fore differentiale $= q\,dp$; at ſi altera pars q ſola variabilis ſtatuatur, erit differentiale $= p\,dq$. Addantur ergo haec duo differentialia inuicem, atque prodibit differentiale quaeſitum $dy = q\,dp + p\,dq$, quemadmodum ex iam allatis conſtat. Si plures fuerint partes per multiplicationem connexae, ſcilicet $y = pqrs$, ſi ſucceſſiue vna-

vnaquaeque ſola variabilis ſtatuatur, orientur iſta differentialia $qrsdp$, $prsdq$, $pqsdr$, & $pqrds$, quorum ſumma dabit differentiale quaeſitum, nempe

$$dy = qrsdp + prsdq + pqsdr + pqrds,$$

prorſus vti iam ante inuenimus. Differentiale ergo ex totidem partibus componitur, ſiue partes functionem conſtituentes ſint inuicem additae ſubtractaeve, ſiue in ſe inuicem multiplicatae.

173. Si partes functionem formantes per diuiſionem ſint connexae, nempe $y = \frac{p}{q}$, ponatur ſecundum regulam primum ſola pars p variabilis, eritque ob q conſtans differentiale $= \frac{dp}{q}$; deinde ponatur ſola pars q variabilis ob $y = pq^{-1}$, erit differentiale $= -\frac{pdq}{qq}$, quae duo differentialia collecta dabunt differentiale functionis propoſitae $dy = \frac{dp}{q} - \frac{pdq}{qq} = \frac{qdp - pdq}{qq}$, ſicut iam ſupra inuenimus. Simili modo ſi functio propoſita ſit $y = \frac{pq}{rs}$, ponendo ſucceſſiue ſingulas partes ſolas p, q, r & s variabiles, prodibunt ſequentia differentialia: $\frac{qdp}{rs}$; $\frac{pdq}{rs}$; $\frac{-pqdr}{rrs}$; & $\frac{pqds}{rss}$, vnde fit

$$dy = \frac{qrsdp + prsdq - pqsdr - pqrds}{rrss}.$$

174. Dummodo ergo ſingulae partes, ex quibus functio componitur, ita fuerint comparatae, vt earum differentialia exhiberi queant, ſimul quoque totius functionis differentiale inueniri poterit. Quodſi igitur partes fuerint functiones rationales, tum earum differentialia non ſolum ope praeceptorum ante iam datorum inueniuntur, ſed ea quoque ex hac ipſa regula generali erui poterunt: ſin autem partes fuerint irrationales, quia irrationalitas ad poteſtates, quarum exponentes ſunt numeri fracti, reducitur, eae per differentiationem poteſtatum, qua eſt $d.x^n = nx^{n-1}dx$ differentiabuntur. Atque ex eodem fonte haurietur quoque differentiatio eiusmodi formularum irrationalium, quae alias inſuper expreſſiones ſurdas inuoluunt. Vnde patet ſi cum regula generali hic data, infra vero demonſtranda, coniungatur regula differentiandi poteſtates, tum omnium omnino functionum algebraicarum differentialia exhiberi poſſe.

175. Ex his omnibus iam dilucide ſequitur, ſi y fuerit functio quaecunque ipſius x, differentiale eius dy huiusmodi habiturum eſſe formam $dy = pdx$, in qua valor ipſius p per praecepta hic expoſita ſemper aſſignari queat. Erit autem p functio ipſius x quoque algebraica, cum in eius determinationem nullae aliae operationes ingrediantur, niſi conſuetae, quibus functiones algebraicae conſtitui ſolent. Hancobrem ſi y fuerit functio algebraica ipſius x, erit quoque $\frac{dy}{dx}$ functio algebraica ipſius x. Atque ſi z fuerit etiam functio algebraica ipſius x, ita vt

ſi dz

ſi $dz = q\,dx$, ob q functionem algebraicam ipſius x, erit quoque $\frac{dz}{dy}$ functio algebraica ipſius x, quippe quae eſt $= \frac{p}{q}$. Quare ſi huiusmodi formulae $\frac{dz}{dy}$ in expreſſionem cetera algebraicam ingrediantur, eae non impedient, quominus ea expreſſio ſit algebraica, dummodo y & z fuerint functiones algebraicae.

176. Poterimus autem hoc ratiocinium extendere ad differentialia ſecunda & ſuperiorem ordinum. Si enim manente y functione algebraica ipſius x, fuerit $dy = p\,dx$, atque $dp = q\,dx$; erit ſumto differentiali dx conſtante, $ddy = q\,dx^2$ vti ſupra vidimus. Cum igitur ob rationes ante allegatas ſit quoque q functio algebraica ipſius x, erit quoque $\frac{ddy}{dx^2}$ non ſolum quantitas finita, ſed etiam functio algebraica ipſius x, dummodo y fuerit eiusmodi functio. Simili modo perſpicietur, fore $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$, &c. functiones algebraicas ipſius x, modo y fuerit talis; atque ſi z ſit quoque functio algebraica ipſius x, omnes expreſſiones finitae; quae ex differentialibus cuiusuis ordinis ipſarum y, z, & ex dx componuntur cuiusmodi ſunt $\frac{ddy}{ddz}$; $\frac{d^3y}{dz\,ddy}$; $\frac{dx\,d^4y}{dy^3\,ddz}$; &c. ſimul erunt functiones algebraicae ipſius x.

177. Cum igitur nunc methodus ſit tradita cuiusque functionis ipſius x algebraicae differentiale primum inueniendi, eadem methodo poterimus quoque differentialia ſecunda altiorumque ordinum inueſtigare. Si enim y fuerit functio quaecunque algebraica ipſius x, ex eius differentiatione $dy = pdx$ innoteſcet valor ipſius p. Qui ſi denuo differentietur atque reperiatur $dp = qdx$, erit $ddy = qdx^2$, poſito dx conſtante, ſicque definietur differentiale ſecundum. Differentiando porro q, vt ſit $dq = rdx$, habebitur differentiale tertium $d^3y = rdx^3$; ſicque vlterius differentialia altiorum ordinum indagabuntur; quoniam quantitates p, q, r, &c. omnes ſunt functiones ipſius x algebraicae, ad quas differentiandas praecepta data ſufficiunt. Hoc ergo efficietur continua differentiatione; omiſſis enim dx, in differentiatione ipſius y, prodibit valor ipſius $\frac{dy}{dx} = p$, qui denuo differentiatus ac diuiſus per dx, quod fit dum vbique differentiale dx omittatur, dabit valorem ipſius $q = \frac{ddy}{dx^2}$. Simili modo porro inuenitur $r = \frac{d^3y}{dx^3}$ &c.

I. Sit $y = \frac{aa}{aa + xx}$ cuius differentialia tam prima quam ſequentium ordinum requiruntur.

Primum ergo differentiando ſimulque per dx diuidendo erit $\frac{dy}{dx} = \frac{-2aax}{(aa+xx)^2}$, hincque porro

ddy

$$\frac{ddy}{dx^2} = \frac{-2a^4 + 6aaxx}{(aa+xx)^3}$$

$$\frac{d^3y}{dx^3} = \frac{24a^4x - 24aax^3}{(aa+xx)^4}$$

$$\frac{d^4y}{dx^4} = \frac{24a^6 - 240a^4xx + 120aax^4}{(aa+xx)^5}$$

$$\frac{d^5y}{dx^5} = \frac{-720a^6x + 2400a^4x^3 - 720aax^5}{(aa+xx)^6}$$

&c.

II. Sit $y = \frac{1}{\sqrt{(1-xx)}}$, eruntque differentialia primum & ſequentia:

$$\frac{dy}{dx} = \frac{x}{(1-xx)^{\frac{3}{2}}}$$

$$\frac{ddy}{dx^2} = \frac{1+2xx}{(1-xx)^{\frac{5}{2}}}$$

$$\frac{d^3y}{dx^3} = \frac{9x+6x^3}{(1-xx)^{\frac{7}{2}}}$$

$$\frac{d^4y}{dx^4} = \frac{9+72x^2+24x^4}{(1-xx)^{\frac{9}{2}}}$$

$$\frac{d^5y}{dx^5} = \frac{225x+600x^3+120x^5}{(1-xx)^{\frac{11}{2}}}$$

$$\frac{d^6y}{dx^6} = \frac{225+4050x^2+5400x^4+720x^6}{(1-xx)^{\frac{13}{2}}}$$

&c.

Haec

Haec Differentialia facile vlterius continuantur; interim tamen lex, qua termini eorum progrediuntur, non cito patet. Coefficiens quidem ſupremarum ipſius x poteſtatum ſemper eſt productum numerorum naturalium ab 1 vsque ad ordinem differentialis, quod quaeritur. Interim ſi has formas vlterius continuemus atque perpendamus, deprehendemus fore generaliter, ſi $y = \frac{1}{\sqrt{(1-xx)}}$,

$$\frac{d^n y}{dx^n} = \frac{1.2.3 \ldots\ldots n}{(1-xx)^{n+\frac{1}{2}}} \Big(x^n + \frac{1}{2} \cdot \frac{n(n-1)}{1.\ 2.} x^{n-2} +$$

$$\frac{1.3}{2.4} \cdot \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4} x^{n-4} + \frac{1.3.5}{2.4.6} \cdot \frac{n(n-1)\ldots(n-5)}{1.\ 2 \ldots 6} x^{n-6}$$

$$+ \frac{1.3.5.7}{2.4.6.8} \cdot \frac{n(n-1) \ldots\ldots (n-7)}{1.\ 2 \ldots\ldots 8} x^{n-8} + \&c. \Big)$$

Huiusmodi ergo exempla non ſolum inſeruiunt, ad habitum in differentiationis negotio acquirendum, ſed etiam leges, quae in differentialibus omnium ordinum obſeruantur, per ſe ſunt notatu digniſſimae, atque ad alias inuentiones deducere poſſunt.

CAPUT

CAPUT VI.

DE DIFFERENTIATIONE FUNCTIONUM TRANSCENDENTIUM.

178.

Praeter infinita quantitatum transcendentium ſeu non algebraicarum genera, quae calculus integralis ſuppeditabit, in Introductione ad analyſin infinitorum ad cognitionem aliquot huiusmodi quantitatum magis vſitatarum nobis peruenire licuit, quas doctrina de logarithmis & arcubus circularibus ſuggeſſerat. Quoniam igitur harum quantitatum naturam tam dilucide expoſuimus, vt fere eadem facilitate atque quantitates algebraicae in calculo tractari queant, earum quoque differentialia in hoc capite inueſtigabimus, quo earum indoles ac proprietates clarius perſpiciantur; hocque pacto aditus ad calculum integralem, qui quantitatum transcendentium eſt fons proprius, patefiat.

179. Primum igitur occurrunt quantitates logarithmicae, ſeu eiusmodi functiones ipſius x, quae praeter expreſſiones algebraicas quoque logarithmum ipſius x, ſeu cuiusuis ipſius functionis inuoluunt. Ad quas differentiandas, cum quantitates algebraicae nullum negotium amplius faceſſant, omnis difficultas in inueniendo differentiali logarithmi cuiusque ipſius x functionis erit poſita. Quia vero logarithmorum plurima dantur genera diuerſa, quae

quae tamen inter ſe conſtantes tenent rationes, hic logarithmos hyperbolicos potiſſimum contemplabimur, cum ex iis omnes reliqui logarithmi facile formentur. Si enim functionis p logarithmus hyperbolicus fuerit $= lp$, tum eiusdem functionis p logarithmus ex alio canone deſumtus erit $= mlp$, denotante m numerum, quo relatio huius logarithmorum canonis ad hyperbolicos exprimitur. Hanc ob cauſam lp perpetuo hic deſignabit logarithmum hyperbolicum quantitatis p.

180. Quaeramus ergo differentiale logarithmi hyperbolici quantitatis x, ponaturque $y = lx$, ita vt differentialis dy valor definiri debeat. Ponatur $x + dx$ loco x, ſicque tranſibit y in $y^{I} = y + dy$; quare habebitur $y + dy = l(x + dx)$ & $dy = l(x + dx) - lx = l(1 + \frac{dx}{x})$. At iam ſupra logarithmum hyperbolicum huiusmodi expreſſionis $1 + z$ ita per ſeriem infinitam expreſſimus, vt eſſet $l(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} +$ &c. Poſito ergo $\frac{dx}{x}$ pro z, obtinebimus:

$$dy = \frac{dx}{x} - \frac{dx^2}{2x^2} + \frac{dx^3}{3x^3} - \text{\&c.}$$

Cum igitur huius ſeriei omnes termini prae primo euaneſcant, erit $d.\,lx = dy = \frac{dx}{x}$. Vnde alius cuiuscunque logarithmi, cuius ad hyperbolicum ratio eſt vt $n:1$, differentiale erit $= \frac{n\,dx}{x}$.

181.

181. Si igitur cuiusque ipſius x functionis p logarithmus lp proponatur, eodem ratiocinio reperietur eius differentiale eſſe $= \frac{dp}{p}$, vnde ad logarithmorum differentialia inuenienda haec habetur regula. *Quantitatis* p, *cuius logarithmus proponitur, ſumatur differentiale, hocque per ipſam quantitatem* p *diuiſum dabit differentiale logarithmi quaeſitum.* Sequitur haec eadem regula quoque ex forma $\frac{p^0 - 1^0}{0}$, ad quam ſuperiori libro logarithmum ipſius p reduximus. Sit $\omega = 0$, & cum ſit $lp = \frac{p^\omega - 1}{\omega}$; erit $d.lp = d.\frac{1}{\omega}p^\omega = p^{\omega-1}dp = \frac{dp}{p}$ ob $\omega = 0$. Notandum autem eſt $\frac{dp}{p}$ eſſe differentiale logarithmi hyperbolici ipſius p; ita vt, ſi logarithmus vulgaris ipſius p proponeretur, differentiale illud $\frac{dp}{p}$ multiplicari deberet per hunc numerum 0,43429448 &c.

182. Ope huius ergo regulae, cuiuscunque functionis ipſius x logarithmus proponatur, eius differentiale facillime inueniri poterit, quemadmodum ex ſequentibus exemplis perſpicietur:

I. Si ſit $y = lx$; erit $dy = \frac{dx}{x}$.

II. Si ſit $y = lx^n$; ponatur $x^n = p$, vt ſit $y = lp$, eritque $dy = \frac{dp}{p}$. At eſt $dp = nx^{n-1}dx$, vnde fit $dy = \frac{ndx}{x}$.

Idem quoque ex logarithmorum natura colligitur; cum enim ſit $lx^n = nlx$, erit $d.lx^n = nd.lx = \frac{ndx}{x}$.

III. Si ſit $y = l(1+xx)$, erit $dy = \frac{2xdx}{1+xx}$.

IV. Si ſit $y = l\frac{1}{\sqrt{(1-xx)}}$; quia erit $y = -l\sqrt{(1-xx)}$ $= -\frac{1}{2}l(1-xx)$, inuenitur $dy = \frac{xdx}{1-xx}$.

V. Si ſit $y = l\frac{x}{\sqrt{(1+xx)}}$, ob $y = lx - \frac{1}{2}l(1+xx)$, fiet $dy = \frac{dx}{x} - \frac{xdx}{1+xx} = \frac{dx}{x(1+xx)}$.

VI. Si ſit $y = l(x+\sqrt{(1+xx)})$, fiet

$$dy = \frac{dx + xdx : \sqrt{(1+xx)}}{x+\sqrt{(1+xx)}} = \frac{xdx + dx\sqrt{(1+xx)}}{(x+\sqrt{(1+xx)})\sqrt{(1+xx)}},$$

cuius fractionis cum numerator ac denominator per $x+\sqrt{(1+xx)}$ ſit diuiſibilis fiet $dy = \frac{dx}{\sqrt{(1+xx)}}$.

VII. Si ſit $y = \frac{1}{\sqrt{-1}}l(x\sqrt{-1}+\sqrt{(1-xx)})$, ponatur $x\sqrt{-1} = z$. Atque ob $y = \frac{1}{\sqrt{-1}}l(z+\sqrt{(1+zz)})$, erit per praecedens $dy = \frac{1}{\sqrt{-1}}dz : \sqrt{(1+zz)}$.

Quare, ob $dz = dx\sqrt{-1}$, fiet $dy = \frac{dx}{\sqrt{(1-xx)}}$. Quamuis ergo logarithmus propoſitus imaginaria inuoluat, tamen eius differentiale ſit reale.

183. Si quantitas, cuius logarithmus proponitur, habeat factores, tum ipſe logarithmus in plures alios reſoluetur hoc modo: Si proponatur $y = lpqrs$, quia erit $y = lp + lq + lr + ls$, erit $dy = \frac{dp}{p} + \frac{dq}{q} + \frac{dr}{r} + \frac{ds}{s}$. Haec reſolutio pariter locum habet, ſi illa quantitas, cuius logarithmus differentiari debet, fuerit fractio. Sit enim $y = l\frac{pq}{rs}$, ob $y = lp + lq - lr - ls$, erit $dy = \frac{dp}{p} + \frac{dq}{q} - \frac{dr}{r} - \frac{ds}{s}$. Neque etiam poteſtates difficultatem mouebunt, ſi enim fuerit $y = l\frac{p^m q^n}{r^\mu s^\nu}$, ob $y = mlp + nlq - \mu lr - \nu ls$, erit $dy = \frac{mdp}{p} + \frac{ndq}{q} - \frac{\mu dr}{r} - \frac{\nu ds}{s}$.

I. Si fuerit $y = l(a+x)(b+x)(c+x)$, quia erit $y = l(a+x) + l(b+x) + l(c+x)$, fiet differentiale quaeſitum $dy = \frac{dx}{a+x} + \frac{dx}{b+x} + \frac{dx}{c+x}$.

II. Si fuerit $y = \frac{1}{2} l \frac{1+x}{1-x}$, erit $y = \frac{1}{2} l(1+x) - \frac{1}{2} l(1-x)$, hincque $dy = \frac{\frac{1}{2}dx}{1+x} + \frac{\frac{1}{2}dx}{1-x} = \frac{dx}{1-xx}$.

III. Si ſit $y = \frac{1}{2} l \frac{\sqrt{(1+xx)}+x}{\sqrt{(1+xx)}-x}$, ob $y = \frac{1}{2} l(\sqrt{(1+xx)}+x) - \frac{1}{2} l(\sqrt{(1+xx)}-x)$, erit $dy = \frac{\frac{1}{2}dx}{\sqrt{(1+xx)}} + \frac{\frac{1}{2}dx}{\sqrt{(1+xx)}} = \frac{dx}{\sqrt{(1+xx)}}$. Hoc idem facilius inuenitur, ſi in fractione

$\frac{\sqrt{(1+xx)}+x}{\sqrt{(1+xx)}-x}$, irrationalitas in denominatore tollatur multiplicando numeratorem ac denominatorem per

$$\sqrt{(1+xx)}+x, \text{ prodibit enim}$$

$$y = \tfrac{1}{2}\, l(\sqrt{(1+xx)}+x)^2 = l(\sqrt{(1+xx)}+x),$$

cuius differentiale ante vidimus eſſe $dy = \frac{dx}{\sqrt{(1+xx)}}$.

IV. Si ſit $y = l\frac{\sqrt{(1+x)}+\sqrt{(1-x)}}{\sqrt{(1+x)}-\sqrt{(1-x)}}$. Ponatur huius fractionis numerator $\sqrt{(1+x)}+\sqrt{(1-x)} = p$ & denominator $\sqrt{(1+x)}-\sqrt{(1-x)} = q$, erit $y = l\frac{p}{q} = lp - lq$, & $dy = \frac{dp}{p} - \frac{dq}{q}$. Eſt vero $dp = \frac{dx}{2\sqrt{(1+x)}} - \frac{dx}{2\sqrt{(1-x)}} = \frac{-dx}{2\sqrt{(1-xx)}}(\sqrt{(1+x)}-\sqrt{(1-x)}) = \frac{-q\,dx}{2\sqrt{(1-xx)}}$; & $dq = \frac{dx}{2\sqrt{(1+x)}} + \frac{dx}{2\sqrt{(1-x)}} = \frac{p\,dx}{2\sqrt{(1-xx)}}$. Hinc fiet

$$\frac{dp}{p} - \frac{dq}{q} = \frac{-q\,dx}{2p\sqrt{(1-xx)}} - \frac{p\,dx}{2q\sqrt{(1-xx)}} = \frac{-(pp+qq)dx}{2pq\sqrt{(1-xx)}}.$$

At eſt $pp + qq = 4$ & $pq = 2x$, vnde erit $dy = -\frac{dx}{x\sqrt{(1-xx)}}$. Hoc autem differentiale facilius inuenietur, ſi logarithmus propoſitus ita transformetur,

$$y = l\frac{1+\sqrt{(1-xx)}}{x} = l\left(\frac{1}{x}+\sqrt{(\frac{1}{xx}-1)}\right).$$

Poſito

Poſito enim $\frac{1}{x} + \sqrt{\left(\frac{1}{xx} - 1\right)} = p$, erit

$$dp = \frac{-dx}{xx} - \frac{dx}{x^3\sqrt{(\frac{1}{xx} - 1)}} = \frac{-dx}{xx} - \frac{dx}{xx\sqrt{(1 - xx)}}$$

$= \frac{-dx(1 + \sqrt{(1 - xx)})}{xx\sqrt{(1 - xx)}}$, ideoque, ob $p = \frac{1 + \sqrt{(1 - xx)}}{x}$,

erit $dy = \frac{dp}{p} = \frac{-dx}{x\sqrt{(1 - xx)}}$ vt ante.

184. Cum igitur logarithmorum differentialia prima, ſi per dx diuidantur, ſint quantitates algebraicae, differentialia ſecunda ac ſequentium ordinum per praecepta praecedentis capitis facile inueniuntur, ſiquidem differentiale dx aſſumatur conſtans. Sic poſito

$$y = lx, \quad \text{erit}$$

$$dy = \frac{dx}{x}, \quad \& \quad \frac{dy}{dx} = \frac{1}{x}$$

$$ddy = \frac{-dx^2}{x^2}, \quad \& \quad \frac{ddy}{dx^2} = \frac{-1}{x^2}$$

$$d^3y = \frac{2dx^3}{x^3}, \quad \& \quad \frac{d^3y}{dx^3} = \frac{2}{x^3}$$

$$d^4y = \frac{-6dx^4}{x^4}, \quad \& \quad \frac{d^4y}{dx^4} = \frac{-6}{x^4}$$

&c.

Atque ſi p fuerit quantitas algebraica, ſitque $y = lp$, etiamſi y non ſit quantitas algebraica, tamen $\frac{dy}{dx}$; $\frac{ddy}{dx^2}$; $\frac{d^3y}{dx^3}$; &c. erunt functiones algebraicae ipſius x.

185. Expoſita logarithmorum differentiatione, functiones, quae ex algebraicis ac logarithmis ſunt permixtae, facile differentiabuntur, perinde atque eae, quae ex logarithmis ſolis componuntur; vti ex ſequentibus exemplis fiet perſpicuum.

I. Si ſit $y = (lx)^2$, ponatur $lx = p$, atque ob $y = p^2$ erit $dy = 2pdp$; verum $dp = \frac{dx}{x}$; ideoque erit $dy = \frac{2dx}{x} lx$.

II. Simili modo ſi ſit $y = (lx)^n$, erit $dy = \frac{ndx}{x}(lx)^{n-1}$, vnde, ſi ſit $y = V lx$, ob $n = \frac{1}{2}$, erit $dy = \frac{dx}{2xVlx}$.

III. Atque ſi p fuerit functio quaecunque ipſius x, ponaturque $y = (lp)^n$, erit $dy = \frac{ndp}{p}(lp)^{n-1}$. Quare cum differentiale dp per praecedentia aſſignari poſſit, erit quoque differentiale ipſius y cognitum.

IV. Si ſit $y = lp \,.\, lq$, fuerintque p & q functiones quaecunque ipſius x, per regulam factorum ſupra datam erit $dy = \frac{dp}{p} lq + \frac{dq}{q} lp$.

V. Si ſit $y = x\,lx$; erit per eandem regulam $dy = dx\,lx + \frac{xdx}{x} = dx\,lx + dx$.

VI.

VI. Si ſit $y = x^m lx - \frac{1}{m} x^m$, differentiatione ſecundum partes inſtituta, reperietur $d.x^m lx = mx^{m-1}dxlx + x^{m-1}dx$, & $d.\frac{1}{m}x^m = x^{m-1}dx$, vnde erit $dy = mx^{m-1}dxlx$.

VII. Si ſit $y = x^m(lx)^n$, fiet $dy = mx^{m-1}dx(lx)^n + nx^{m-1}dx(lx)^{n-1}$.

VIII. Si logarithmi logarithmorum occurrant, vti ſi fuerit $y = llx$, ponatur $lx = p$, erit $y = lp$, & $dy = \frac{dp}{p}$; at eſt $dp = \frac{dx}{x}$; vnde fiet $dy = \frac{dx}{xlx}$.

IX. Atque ſi fuerit $y = lllx$, ſi ſtatuatur $lx = p$, fiet $y = llp$, eritque per exemplum praecedens $dy = \frac{dp}{plp}$; at eſt $dp = \frac{dx}{x}$, quibus valoribus ſubſtitutis habebitur $dy = \frac{dx}{xlx.llx}$.

186. Expoſita logarithmorum differentiatione, progrediamur ad quantitates exponentiales, ſeu eiusmodi poteſtates, quarum exponentes ſint variabiles. Huiusmodi autem ipſius x functionum differentialia per logarithmorum differentiationem inueniri poſſunt hoc modo. Quaeratur differentiale ipſius a^x, ad quod inueſtigandum ponatur $y = a^x$, eritque logarithmis ſumendis $ly = xla$. Sumantur iam differentialia, atque obtinebitur $\frac{dy}{y} = dxla$; vnde ſit $dy = ydxla$, cum autem ſit $y = a^x$, erit $dy = a^x dxla$, quod eſt differentiale ipſius a^x. Simili mo-

modo, ſi ſit p functio quaecunque ipſius x, huius quantitatis exponentialis a^p differentiale erit $= a^p dpla$.

187. Hoc idem autem differentiale immediate ex natura quantitatum exponentialium in introductione expoſita deduci poteſt. Sit enim propoſita a^p, denotante p functionem quamcunque ipſius x, quae, poſito $x + dx$ loco x, abeat in $p + dp$. Vnde ſi ponatur $y = a^p$, ſi x abeat in $x + dx$, erit $y + dy = a^{p+dp}$, ideoque $dy = a^{p+dp} - a^p = a^p(a^{dp} - 1)$. Oſtendimus autem ſupra, quamuis quantitatem exponentialem a^z per huiusmodi ſeriem exprimi $1 + zla + \frac{z^2(la)^2}{2} + \frac{z^3(la)^3}{6} + \&c.$ vnde erit $a^{dp} = 1 + dpla + \frac{dp^2(la)^2}{2} + \&c.$, & $a^{dp} - 1 = dpla$, quia ſequentes termini prae $dpla$ omnes euaneſcunt. Conſequenter erit $dy = d.a^p = a^p dpla$. *Quare quantitatis exponentialis* a^p *differentiale erit productum ex ipſa quantitate exponentiali, exponentis differentiali* dp, *& logarithmo quantitatis conſtantis* a, *quae ad exponentem variabilem eſt euecta.*

188. Si igitur e ſit numerus, cuius logarithmus hyperbolicus eſt $= 1$, vt ſit $le = 1$, erit quantitatis e^x differentiale $= e^x dx$. Atque ſi dx ſumatur conſtans, erit huius differentiale $= e^x dx^2$, quod eſt differentiale ſecundum ipſius e^x. Simili modo differentiale tertium erit $= e^x dx^3$. Quare ſi ſit

$y =$

$y = e^{nx}$, erit $\frac{dy}{dx} = ne^{nx}$, & $\frac{ddy}{dx^2} = n^2 e^{nx}$

porroque $\frac{d^3 y}{dx^3} = n^3 e^{nx}$; $\frac{d^4 y}{dx^4} = n^4 e^{nx}$; &c.

Vnde patet ipſius e^{nx} differentialia primum, ſecundum & reliqua ſequentia conſtituere progreſſionem geometricam: eritque ergo differentiale ordinis m ipſius $e^{nx} = y$, nempe $\frac{d^m y}{dx^m} = n^m e^{nx}$; hincque igitur $\frac{d^m y}{y dx^m}$ quantitas conſtans n^m.

189. Si ipſa quantitas, quae eleuatur, fuerit variabilis, eius differentiale ſimili modo inueſtigabitur. Sint p & q functiones quaecunque ipſius x, ac proponatur quantitas exponentialis $y = p^q$. Sumtis logarithmis erit $ly = qlp$, quibus differentiatis erit $\frac{dy}{y} = dqlp + \frac{qdp}{p}$, vnde fit $dy = ydqlp + \frac{yqdp}{p} = p^q dqlp + qp^{q-1}dp$, ob $y = p^q$. Hoc ergo differentiale conſtat duobus membris, quorum prius $p^q dqlp$ oritur, ſi quantitas propoſita p^q ita differentietur, quaſi p eſſet quantitas conſtans, ſolusque exponens q variabilis: alterum vero membrum $qp^{q-1}dp$ oritur, ſi in quantitate propoſita p^q exponens q tanquam conſtans ſpectetur, ſolaque quantitas p, quaſi eſſet variabilis, tractetur. Hocque ergo differentiale per regulam generalem differentiandi ſupra traditam inueniri potuiſſet.

190. Eiusdem vero expreſſionis p^q differentiale quoque ex natura quantitatum exponentialium erui poteſt hoc modo: ſit $y = p^q$, eritque, loco x poſito $x + dx$, vtique $y + dy = (p + dp)^{q+dq}$, quae expreſſio, ſi more ſolito in ſeriem reſoluatur, fiet

$$y + dy = p^{q+dq} + (q + dq)\, p^{q+dq-1} dp$$
$$+ \frac{(q + dq)(q + dq - 1)}{1 \cdot 2} p^{q+dq-2} dp^2 + \&c.$$

ideoque

$$dy = p^{q+dq} - p^q + (q + dq)\, p^{q+dq-1} dp,$$

ſequentes enim termini, qui altiores ipſius dp poteſtates inuoluunt, prae $(q + dq)p^{q+dq-1}dp$ euaneſcunt. At eſt $p^{q+dq} - p^q = p^q(p^{dq} - 1) = p^q(1 + dqlp + \frac{dq^2(lp)^2}{2} + \&c. - 1)$ $= p^q dqlp$. In altero vero termino $(q + dq)p^{q+dq-1}dp$, ſi loco $q + dq$ ſcribamus q, orietur $qp^{q-1}dp$, ideoque differentiale erit vt ante $dy = p^q dqlp + qp^{q-1}dp$.

191. Facilius vero hoc idem differentiale ex natura quantitatum exponentialium inueſtigabitur, hoc modo: Cum, ſumto e pro numero, cuius logarithmus hyperbolicus eſt $= 1$, ſit $p^q = e^{qlp}$, vtriusque enim logarithmus eſt idem qlp; erit $y = e^{qlp}$. Quare, cum nunc quantitas eleuata e ſit conſtans, erit $dy = e^{qlp}\left(dqlp + \frac{qdp}{p}\right)$, vti ante oſtendimus in regula §. 187 data. Reſtituatur igitur p^q loco e^{qlp}, fietque

$$dy = p^q dqlp + p^q q dp : p = p^q dqlp + qp^{q-1}dp.$$

Si

Si igitur fuerit $y = x^x$, erit $dy = x^x dx lx + x^x dx$; atque hinc quoque eius vlteriora differentialia definientur: reperietur enim:

$$\frac{ddy}{dx^2} = x^x\left(\frac{1}{x} + (1 + lx)^2\right)$$

$$\frac{d^3y}{dx^3} = x^x\left((1 + lx)^3 + \frac{3(1 + lx)}{x} - \frac{1}{xx}\right)$$

&c.

192. Inter differentialia huiusmodi functionum, quae quantitates exponentiales complectuntur, imprimis ſunt notanda ſequentia exempla, quae ex differentiatione formulae $e^x p$ originem habent; eſt autem

$$d.\, e^x p = e^x dp + e^x p dx = e^x (dp + p dx).$$

I. Si ſit $y = e^x x^n$; erit $dy = e^x n x^{n-1} dx + e^x x^n dx$ ſeu $dy = e^x dx (n x^{n-1} + x^n)$

II. Si ſit $y = e^x (x - 1)$
Erit $dy = e^x x dx$.

III. Si ſit $y = e^x (x^2 - 2x + 2)$
Erit $dy = e^x xx dx$.

IV. Si ſit $y = e^x (x^3 - 3x^2 + 6x - 6)$
Erit $dy = e^x x^3 dx$.

193. Si ipſi exponentes fuerint denuo quantitates exponentiales, differentiatio ſecundum eadem praecepta in-

ſtituetur. Sic ſi haec quantitas e^{e^x} differentiari debeat, ſtatuatur $e^x = p$, vt ſit $y = e^{e^x} = e^p$, erit $dy = e^p dp$; at eſt $dp = e^x dx$, vnde ſi fuerit

$$y = e^{e^x}; \text{ erit } dy = e^{e^x} e^x dx,$$

$$\text{atque ſi ſit } y = e^{e^{e^x}}; \text{ erit } dy = e^{e^{e^x}} e^{e^x} e^x dx.$$

Quod ſi vero fuerit $y = p^{q^r}$, ſtatuatur $q^r = z$, erit $dy = p^z dz lp + z p^{z-1} dp$, at $dz = q^r dr lq + r q^{r-1} dq$, vnde $dy = p^z q^r dr lp.lq + p^z r q^{r-1} dq lp + p^z q^r dp : p$.

Quare ſi ſit:

$$y = p^{q^r}, \text{ erit } dy = p^{q^r} q^r \left(dr\, lp.lq + \frac{r\, dq\, lp}{q} + \frac{dp}{p} \right).$$

Hoc ergo modo, quaecunque occurrat quantitas exponentialis, eius differentiale inueniri poterit.

194. Pergamus ergo ad quantitates transcendentes, ad quarum cognitionem conſideratio arcuum circularium nos ſupra deduxit. Sit igitur in circulo, cuius radium conſtanter ponimus vnitati aequalem, propoſitus arcus, cuius ſinus ſit $= x$, quem arcum hoc modo exprimamus A ſin x; huiusque arcus differentiale inueſtigemus, ſeu incrementum quod accipit, ſi ſinus x differentiali ſuo dx au-

augeatur. Hoc autem ex differentiatione logarithmorum praeſtari poterit, quia in introductione oſtendimus hanc expreſſionem A ſin x reduci poſſe ad hanc logarithmicam: $\frac{1}{\sqrt{-1}} l(\sqrt{(1-xx)}+x\sqrt{-1})$. Poſito ergo $y = \text{A} \sin x$, erit quoque $y = \frac{1}{\sqrt{-1}} l(\sqrt{(1-xx)}+x\sqrt{-1})$; quae differentiata dat

$$dy = \frac{\frac{1}{\sqrt{-1}}\left(\frac{-xdx}{\sqrt{(1-xx)}}+dx\sqrt{-1}\right)}{\sqrt{(1-xx)}+x\sqrt{-1}} = \frac{dx(x\sqrt{-1}+\sqrt{(1-xx)})}{(\sqrt{(1-xx)}+x\sqrt{-1})\sqrt{(1-xx)}},$$

vnde fit $dy = \frac{dx}{\sqrt{(1-xx)}}$.

195. Iſtud arcus circularis differentiale etiam hoc modo facilius ſine logarithmorum ſubſidio inueniri poteſt. Si enim ſit $y = \text{A} \sin x$, erit x ſinus arcus y, ſeu $x = \sin y$. Cum igitur, poſito $x + dx$ loco x, abeat y in $y + dy$, fiet $x + dx = \sin(y + dy)$. At quia eſt

$$\sin(a+b) = \sin a.\cos b + \cos a.\sin b,\ \text{erit}$$
$$\sin(y+dy) = \sin y.\cos dy + \cos y.\sin dy:$$

arcus autem euaneſcentis dy ſinus ipſi illi arcui dy, eiusque coſinus ſinui toti aequatur, hanc ob rem fiet

$\sin(y+dy) = \sin y + dy \cos y$, ideoque $x + dx = \sin y + dy \cos y$.

Quia vero eſt

$\sin y = x$, erit coſinus ipſius y ſeu $\cos y = \sqrt{(1-xx)}$, quibus valoribus ſubſtitutis, erit $dx = dy\sqrt{(1-xx)}$, ex qua obtinebitur $dy = \frac{dx}{\sqrt{(1-xx)}}$.

Arcus ergo, cuius ſinus proponitur, differentiale aequatur differentiali ſinus per coſinum diuiſo.

196. Cum igitur, ſi p fuerit functio quaecunque ipſius x, atque y denotet arcum, cuius ſinus eſt $= p$, ſeu $y = \text{A}\,\text{ſin}\,p$, ſit huius arcus differentiale $dy = \frac{dp}{V(1-pp)}$, vbi $V(1-pp)$ exprimit coſinum eiusdem arcus, inueniri quoque poterit differentiale arcus, cuius coſinus proponitur. Sit enim $y = \text{A}\,\text{coſ}\,x$, erit eiusdem arcus ſinus $= V(1-xx)$, ideoque $y = \text{A}\,\text{ſin}\,V(1-xx)$. Facto ergo $p = V(1-xx)$, erit $dp = \frac{-x\,dx}{V(1-xx)}$, & $V(1-pp) = x$; vnde fiet $dy = \frac{-dx}{V(1-xx)}$. *Arcus ergo, cuius coſinus proponitur, differentiale aequatur differentiali coſinus negatiue ſumto, atque per ſinum eiusdem arcus diuiſo.* Quod etiam hoc modo oſtendi poteſt: ſi ſit $y = \text{A}\,\text{coſ}\,x$, ponatur $z = \text{A}\,\text{ſin}\,x$, erit $dz = \frac{dx}{V(1-xx)}$; at arcus $y + z$ ſimul ſumti dant arcum conſtantem $90°$, eritque $y + z =$ conſtans ideoque $dy + dz = 0$, ſeu $dy = -dz$; vnde fit $dy = \frac{-dx}{V(1-xx)}$, vt ante.

197. Si arcus proponatur differentiandus, cuius tangens detur, ita vt ſit $y = \text{A}\,\text{tang}\,x$. Arcus autem cuius tangens eſt x, ſinus erit $= \frac{x}{V(1+xx)}$, & coſinus $= \frac{1}{V(1+xx)}$. Poſito ergo $\frac{x}{V(1+xx)} = p$, vt ſit $V(1-pp) = \frac{1}{V(1+xx)}$, fiet

fiet $y = A\ \text{ſin}\ p$: vnde per regulam modo datam erit $dy = \frac{dp}{\surd(1-pp)}$. At, ob $p = \frac{x}{\surd(1+xx)}$, erit $dp = \frac{dx}{(1+xx)^{\frac{3}{2}}}$; quibus valoribus ſubſtitutis fiet $dy = \frac{dx}{1+xx}$. *Arcus ergo, cuius tangens proponitur, differentiale aequatur differentiali tangentis per quadratum ſecantis diuiſo.* Eſt enim $\surd(1+xx)$ ſecans, ſi x ſit tangens.

198. Simili modo ſi proponatur arcus, cuius cotangens datur, ita vt ſit $y = A\ \text{cot}\ x$; quia eiusdem arcus tangens eſt $= \frac{1}{x}$, poſito $\frac{1}{x} = p$, erit $y = A\ \text{tang}\ p$, ac propterea $dy = \frac{dp}{1+pp}$. Cum nunc ſit $dp = \frac{-dx}{xx}$, facta ſubſtitutione, erit $dy = \frac{-dx}{1+xx}$, quod eſt differentiale cotangentis negatiue ſumtum, atque per quadratum coſecantis diuiſum. Porro ſi proponatur $y = A\ \text{ſec.}\ x$, quia eſt $y = A\ \text{coſ}\ \frac{1}{x}$, fiet $dy = \frac{dx}{xx\surd\left(1-\frac{1}{xx}\right)} = \frac{dx}{x\surd(xx-1)}$. Atque, ſi ſit $y = A\ \text{coſec.}\ x$, erit $y = A\ \text{ſin}\ \frac{1}{x}$, ideoque $dy = \frac{-dx}{x\surd(xx-1)}$. Saepe etiam ſinus verſus occurrit, ita ſi proponatur $y = A\ \text{ſv.}\ x$, quia eſt $y = A\ \text{coſ}(1-x)$, huiusque arcus ſinus eſt $= \surd(2x-xx)$, fiet $dy = \frac{dx}{\surd(2x-xx)}$.

199. Quanquam ergo arcus, cuius ſinus, vel coſinus, vel tangens, vel cotangens, vel ſecans, vel coſecans, vel denique ſinus verſus datur, eſt quantitas transcendens, tamen eius differentiale, ſi per dx diuidatur, erit quantitas algebraica, ac propterea quoque eius differentialia ſecunda, tertia, quarta &c. ſi per poteſtates ipſius dx conuenientes diuidantur. Ceterum, quo haec differentiatio melius percipiatur, adiunximus ſequentia exempla.

I. Si ſit $y = A\,\text{ſin}\, 2x\sqrt{(1-xx)}$, ponatur $p = 2x\sqrt{(1-xx)}$, vt ſit $y = A\,\text{ſin}\, p$, eritque $dy = \frac{dp}{\sqrt{(1-pp)}}$. At eſt $dp = 2dx\sqrt{(1-xx)} - \frac{2xxdx}{\sqrt{(1-xx)}} = \frac{2dx(1-2xx)}{\sqrt{(1-xx)}}$, & $\sqrt{(1-pp)} = 1-2xx$, quibus valoribus ſubſtitutis, erit $dy = \frac{2dx}{\sqrt{(1-xx)}}$. Quod etiam inde patet, quod $2x\sqrt{(1-xx)}$ ſit ſinus arcus dupli, dum x eſt ſinus ſimpli, erit ergo $y = 2A\,\text{ſin}\, x$, ideoque $dy = \frac{2dx}{\sqrt{(1-xx)}}$.

II. Si ſit $y = A\,\text{ſin}\, \frac{1-xx}{1+xx}$; ponatur $\frac{1-xx}{1+xx} = p$, erit $dp = \frac{-4xdx}{(1+xx)^2}$ & $\sqrt{(1-pp)} = \frac{2x}{1+xx}$. Quare cum ſit $dy = \frac{dp}{\sqrt{(1-pp)}}$, erit $dy = \frac{-2dx}{1+xx}$.

III. Si ſit $y = \text{A ſin} \sqrt{\frac{1-x}{2}}$, ponatur $\sqrt{\frac{1-x}{2}} = p$, erit $\sqrt{(1-pp)} = \sqrt{\frac{1+x}{2}}$, & $dp = \frac{-dx}{4\sqrt{\left(\frac{1-x}{2}\right)}}$, vnde fit $dy = \frac{dp}{\sqrt{(1-pp)}} = \frac{-dx}{2\sqrt{(1-xx)}}$.

IV. Si ſit $y = \text{A tang} \frac{2x}{1-xx}$, facto $p = \frac{2x}{1-xx}$, erit $1+pp = \frac{(1+xx)^2}{(1-xx)^2}$, & $dp = \frac{2dx(1+xx)}{(1-xx)^2}$. Quare cum ſit $dy = \frac{dp}{1+pp}$ per regulam tangentium (197), erit $dy = \frac{2dx}{1+xx}$.

V. Si ſit $y = \text{A tang} \frac{\sqrt{(1+xx)}-1}{x}$, poſito $p = \frac{\sqrt{(1+xx)}-1}{x}$, fiet $pp = \frac{2+xx-2\sqrt{(1+xx)}}{xx}$, & $1+pp = \frac{2+2xx-2\sqrt{(1+xx)}}{xx} = \frac{2(\sqrt{(1+xx)}-1)\sqrt{(1+xx)}}{xx}$. Atqui $dp = \frac{-dx}{xx\sqrt{(1+xx)}} + \frac{dx}{xx} = \frac{dx(\sqrt{(1+xx)}-1)}{xx\sqrt{(1+xx)}}$. Quare cum ſit $dy = \frac{dp}{1+pp}$, fiet $dy = \frac{dx}{2(1+xx)}$; quod etiam inde intelligitur, quod ſit $\text{A tang} \frac{\sqrt{(1+xx)}-1}{x} = \frac{1}{2}\text{A tang}\, x$.

VI. Si ſit $y = e^{A\,\text{ſin}\,x}$, haec formula quoque per praecedentia differentiabitur: fiet enim $dy = e^{A\,\text{ſin}\,x} \frac{dx}{\sqrt{(1-xx)}}$. Hoc ergo modo omnes functiones ipſius x, in quas praeter logarithmos atque exponentiales quantitates etiam arcus circulares ingrediuntur, differentiari poterunt.

200. Quoniam differentialia arcuum per dx diuiſa ſunt quantitates algebraicae, eorum differentialia ſecunda & ſequentia per ea, quae de functionum algebraicarum differentiatione expoſuimus, inuenientur. Sit $y = A\,\text{ſin}\,x$, quia eſt $dy = \frac{dx}{\sqrt{(1-xx)}}$, erit $\frac{dy}{dx} = \frac{1}{\sqrt{(1-xx)}}$, cuius differentiale dabit valorem pro $\frac{ddy}{dx^2}$, ſi quidem dx ſumatur conſtans: vnde differentialia ipſius y cuiusuis ordinis ita ſe habebunt.

Si ſit $y = A\,\text{ſin}\,x$; erit

$$\frac{dy}{dx} = \frac{1}{\sqrt{(1-xx)}}; \text{ \& ſumto } dx \text{ conſtante}$$

$$\frac{ddy}{dx^2} = \frac{x}{(1-xx)^{\frac{3}{2}}}$$

$$\frac{d^3y}{dx^3} = \frac{1+2xx}{(1-xx)^{\frac{5}{2}}}$$

$$\frac{d^4y}{dx^4} = \frac{9x+6x^3}{(1-xx)^{\frac{7}{2}}}$$

d^5y

$$\frac{d^5y}{dx^5} = \frac{9 + 72x^2 + 24x^4}{(1 - xx)^{\frac{9}{2}}}$$

$$\frac{d^6y}{dx^6} = \frac{225x + 600x^3 + 120x^5}{(1 - xx)^{\frac{11}{2}}}$$

&c.

vnde concludimus vt ſupra §. 177 fore generaliter:

$$\frac{d^{n+1}y}{dx^{n+1}} = \frac{1.2.3 \ldots\ldots n}{(1 - xx)^{n+\frac{1}{2}}} \text{ in}$$

$$\left(x^n + \frac{1}{2} \cdot \frac{n(n-1)}{1.\ 2.} x^{n-2} + \frac{1.3}{2.4} \cdot \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4} x^{n-4}\right.$$

$$\left. + \frac{1.3.5}{2.4.6} \cdot \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1.\ 2.\ 3.\ 4.\ 5.\ 6} x^{n-6} + \&c.\right)$$

201. Superſunt quantitates, quae ex harum inuerſione naſcuntur, ſcilicet ſinus, tangentesue arcuum datorum, quas quomodo differentiare oporteat, oſtendamus. Sit igitur x arcus circuli, & $\text{ſin}\,x$ denotet eius ſinum, cuius differentiale inueſtigemus. Ponamus $y = \text{ſin}\,x$, ac poſito $x + dx$ loco x, quia y abit in $y + dy$, erit

$$y + dy = \text{ſin}(x + dx), \ \& \ dy = \text{ſin}(x + dx) - \text{ſin}\,x.$$

Eſt autem $\text{ſin}(x + dx) = \text{ſin}\,x.\ \text{coſ}\,dx + \text{coſ}\,x.\ \text{ſin}\,dx$, atque cum ſit, vti in introductione oſtendimus

$$\text{ſin}\,z = \frac{z}{1} - \frac{z^3}{1.2.3} + \frac{z^5}{1.2.3.4.5} - \&c.$$

$$\text{coſ}\,z = 1 - \frac{z^2}{1.2} + \frac{z^4}{1.2.3.4} - \&c.$$

 erit

erit reiectis terminis euaneſcentibus $\operatorname{coſ} dx = 1$ & $\operatorname{ſin} dx = dx$, vnde fit $\operatorname{ſin}(x+dx) = \operatorname{ſin} x + dx \operatorname{coſ} x$. Quare, poſito $y = \operatorname{ſin} x$, erit $dy = dx \operatorname{coſ} x$. *Differentiale ergo ſinus arcus cuiusuis aequatur differentiali arcus per coſinum multiplicato.* Si igitur fuerit p functio quaecunque ipſius x, erit ſimili modo $d.\operatorname{ſin} p = dp \operatorname{coſ} p$.

202. Similiter ſi proponatur $\operatorname{coſ} x$, ſeu coſinus arcus x, cuius differentiale inueſtigari oporteat. Ponatur $y = \operatorname{coſ} x$, & poſito $x+dx$ loco x, fiet $y+dy = \operatorname{coſ}(x+dx)$. Eſt vero $\operatorname{coſ}(x+dx) = \operatorname{coſ} x.\operatorname{coſ} dx - \operatorname{ſin} x.\operatorname{ſin} dx$, & quia vt modo vidimus eſt $\operatorname{coſ} dx = 1$ & $\operatorname{ſin} dx = dx$, erit $y+dy = \operatorname{coſ} x - dx \operatorname{ſin} x$, ideoque $dy = -dx \operatorname{ſin} x$. *Quare differentiale coſinus cuiusque arcus aequatur differentiali arcus negatiue ſumto per ſinum eiusdem arcus multiplicato.* Sic ſi p fuerit functio quaecunque ipſius x, erit $d.\operatorname{coſ} p = -dp \operatorname{ſin} p$. Hae differentiationes quoque ex antecedentibus elici poſſunt hoc modo: ſi fuerit $y = \operatorname{ſin} p$, erit $p = A \operatorname{ſin} y$; & $dp = \frac{dy}{\sqrt{(1-yy)}}$; at ob $y = \operatorname{ſin} p$, erit $\operatorname{coſ} p = \sqrt{(1-yy)}$, quo valore ſubſtituto erit $dp = \frac{dy}{\operatorname{coſ} p}$ & $dy = dp \operatorname{coſ} p$, vt ante. Pari modo ſi ſit $y = \operatorname{coſ} p$, erit $\sqrt{(1-yy)} = \operatorname{ſin} p$, & $p = A \operatorname{coſ} y$, ideoque $dp = \frac{-dy}{\sqrt{(1-yy)}} = \frac{-dy}{\operatorname{ſin} p}$, vnde fit vt ante $dy = -dp \operatorname{ſin} p$.

203. Si fuerit $y = \operatorname{tang} x$, erit $dy = \operatorname{tang}(x+dx) - \operatorname{tang} x$; at eſt $\operatorname{tang}(x+dx) = \frac{\operatorname{tang} x + \operatorname{tang}.dx}{1 - \operatorname{tang} x.\operatorname{tang} dx}$,

a qua

a qua fractione ſi tangens x ſubtrahatur, remanebit $dy = \frac{\text{tang}\, dx(1 + \text{tang}\, x.\, \text{tang}\, x)}{1 - \text{tang}\, x.\, \text{tang}\, dx}$. Verum arcus euaneſcentis dx tangens ipſi arcui eſt aequalis, ideoque tang $dx = dx$, & denominator $1 - dx\, \text{tang}\, x$, abit in vnitatem: quocirca fiet $dy = dx(1 + \text{tang}\, x^2)$. Eſt vero $1 + \text{tang}\, x^2 = \text{ſec.}\, x^2 = \frac{1}{\text{coſ}\, x^2}$, denotante coſ x^2 quadratum coſinus ipſius x: conſequenter ſi fuerit $y = \text{tang}\, x$, erit $dy = dx\, \text{ſec}\, x^2 = \frac{dx}{\text{coſ}\, x^2}$. Quod differentiale quoque per differentiationem ſinuum & coſinuum inueniri poteſt; cum enim ſit $\text{tang}\, x = \frac{\text{ſin}\, x}{\text{coſ}\, x}$, erit

$$dy = \frac{dx\, \text{coſ}\, x.\, \text{coſ}\, x + dx\, \text{ſin}\, x.\, \text{ſin}\, x}{\text{coſ}\, x^2} = \frac{dx}{\text{coſ}\, x^2},$$

ob $\text{ſin}\, x^2 + \text{coſ}\, x^2 = 1$.

204. Aliter etiam hoc differentiale inuenitur. Cum ſit $y = \text{tang}\, x$, erit $x = \text{A tang}\, y$, & per praecepta ſuperiora fiet $dx = \frac{dy}{1 + yy}$. At cum ſit $y = \text{tang}\, x$, erit $V(1 + yy) = \text{ſec}\, x = \frac{1}{\text{coſ}\, x}$, ideoque $dx = dy\, \text{coſ}\, x^2$, & $dy = \frac{dx}{\text{coſ}\, x^2}$, vt ante. *Tangentis ergo cuiusuis arcus differentiale aequatur differentiali arcus diuiſo per quadratum coſinus eiusdem arcus.* Simili modo ſi proponatur

 $y =$

$y = \cot x$, fiet $x = A \cot y$, & $dx = \frac{-dy}{1+yy}$. At vero erit $\sqrt{(1+yy)} = \text{coſec}\,x = \frac{1}{\text{ſin}\,x}$, vnde habebitur $dx = -dy\,\text{ſin}\,x^2$, & $dy = \frac{-dx}{\text{ſin}\,x^2}$. *Cotangentis ergo cuiusuis arcus differentiale aequatur differentiali arcus negatiue ſumto ac per quadratum ſinus eiusdem arcus diuiſo.* Vel quia eſt $\cot x = \frac{\text{coſ}\,x}{\text{ſin}\,x}$, fiet hanc fractionem differentiando:

$$dy = \frac{-dx\,\text{ſin}\,x^2 - dx\,\text{coſ}\,x^2}{\text{ſin}\,x^2} = \frac{-dx}{\text{ſin}\,x^2},$$

vti modo inuenimus.

205. Si proponatur ſecans arcus, vt ſit $y = \text{ſec.}\,x$, quia erit $y = \frac{1}{\text{coſ}\,x}$, erit $dy = \frac{dx\,\text{ſin}\,x}{\text{coſ}\,x^2} = dx\,\text{tang}\,x.\,\text{ſec.}\,x$. Simili modo ſi fuerit $y = \text{coſec.}\,x$, ob $y = \frac{1}{\text{ſin}\,x}$, erit $dy = \frac{-dx\,\text{coſ}\,x}{\text{ſin}\,x^2} = -dx\,\cot x\,\text{coſec.}\,x$, pro quibus caſibus peculiares regulas formare ſuperfluum foret. Si ſinus verſus arcus proponatur $y = \text{ſv.}\,x$, quia eſt $y = 1 - \text{coſ}\,x$, erit $dy = dx\,\text{ſin}\,x$. Omnes ergo caſus, quibus linea quaepiam recta ad arcum relata proponitur, quia ſemper per ſinum coſinumue exprimi poteſt, ſine difficultate differentiari poterunt. Neque vero tantum differentialia prima, ſed etiam ſecunda & ſequentia per regulas datas inuenientur. Ponamus eſſe $y = \text{ſin}\,x$ & $z = \text{coſ}\,x$, atque dx eſſe conſtans: erit vt ſequitur:

$y =$

$y = \text{ſin } x$	$z = \text{coſ } x$
$dy = dx \text{ coſ } x$	$dz = - dx \text{ ſin } x$
$ddy = - dx^2 \text{ ſin } x$	$ddz = - dx^2 \text{ coſ } x$
$d^3y = - dx^3 \text{ coſ} x$	$d^3z = dx^3 \text{ ſin } x$
$d^4y = dx^4 \text{ ſin } x$	$d^4z = dx^4 \text{ coſ } x$
&c.	&c.

206. Simili modo inueniri poterunt differentialia omnium ordinum tangentis arcus x. Sit enim $y = \text{tang } x = \frac{\text{ſin } x}{\text{coſ} x}$, & ponatur dx conſtans, erit

$$y = \frac{\text{ſin } x}{\text{coſ} x}$$

$$\frac{dy}{dx} = \frac{1}{\text{coſ } x^2}$$

$$\frac{ddy}{dx^2} = \frac{2 \text{ ſin} x}{\text{coſ } x^3}$$

$$\frac{d^3y}{dx^3} = \frac{6}{\text{coſ } x^4} - \frac{4}{\text{coſ } x^2}$$

$$\frac{d^4y}{dx^4} = \frac{24 \text{ ſin } x}{\text{coſ} x^5} - \frac{8 \text{ ſin} x}{\text{coſ } x^3}$$

$$\frac{d^5y}{dx^5} = \frac{120}{\text{coſ} x^6} - \frac{120}{\text{coſ} x^4} + \frac{16}{\text{coſ } x^2}$$

$$\frac{d^6y}{dx^6} = \frac{720\,\text{ſin}\,x}{\text{coſ}\,x^7} - \frac{480\,\text{ſin}\,x}{\text{coſ}\,x^5} + \frac{32\,\text{ſin}\,x}{\text{coſ}\,x^3}$$

$$\frac{d^7y}{dx^7} = \frac{5040}{\text{coſ}\,x^8} - \frac{6720}{\text{coſ}\,x^6} + \frac{2016}{\text{coſ}\,x^4} - \frac{64}{\text{coſ}\,x^2}$$

&c.

207. Functiones ergo quaecunque, in quas ſinus vel coſinus arcuum ingrediuntur, per haec praecepta differentiari poterunt, vti ex ſequentibus exemplis videre licet.

I. Si ſit $y = 2\,\text{ſin}\,x.\,\text{coſ}\,x = \text{ſin}\,2x$
Erit $dy = 2dx\,\text{coſ}\,x^2 - 2dx\,\text{ſin}\,x^2 = 2dx\,\text{coſ}\,2x$.

II. Si ſit $y = \sqrt{\frac{1-\text{coſ}\,x}{2}}$, vel $y = \text{ſin}\,\frac{1}{2}x$

Erit $dy = \frac{dx\,\text{ſin}\,x}{2\sqrt{2(1-\text{coſ}\,x)}}$. Cum autem ſit $\sqrt{2(1-\text{coſ}\,x)} = 2\,\text{ſin}\,\frac{1}{2}x$, & $\text{ſin}\,x = 2\,\text{ſin}\,\frac{1}{2}x.\,\text{coſ}\,\frac{1}{2}x$; fiet $dy = \frac{1}{2}dx.\,\text{coſ}\,\frac{1}{2}x$, vti ex forma $y = \text{ſin}\,\frac{1}{2}x$ immediate ſequitur.

III. Si ſit $y = \text{coſ}\,l\frac{1}{x}$; erit, poſito $l\frac{1}{x} = p$, $y = \text{coſ}\,p$, & $dy = -dp\,\text{ſin}\,p$. At, ob $p = l1 - lx$, erit $dp = \frac{-dx}{x}$; ideoque $dy = \frac{dx}{x}\,\text{ſin}\,l\frac{1}{x}$.

IV. Si ſit $y = e^{\text{ſin}\,x}$; erit $dy = e^{\text{ſin}\,x}dx\,\text{coſ}\,x$.

V. Si

V. Si ſit $y = e^{\frac{-n}{\operatorname{coſ} x}}$; erit $dy = - \frac{e^{\frac{-n}{\operatorname{coſ} x}} n\,dx \operatorname{ſin} x}{\operatorname{coſ} x^2}$.

VI. Si ſit $y = l\left(1 - \sqrt{\left(1 - e^{\frac{-n}{\operatorname{ſin} x}}\right)}\right)$; ponatur $e^{\frac{-n}{\operatorname{ſin} x}} = p$; atque ob $y = l(1 - \sqrt{(1-p)})$, erit $dy = \frac{dp}{2(1-\sqrt{(1-p)})\sqrt{(1-p)}}$. At eſt $dp = \frac{e^{\frac{-n}{\operatorname{ſin} x}} n\,dx \operatorname{coſ} x}{\operatorname{ſin} x^2}$.

Quo valore ſubſtituto prodibit

$$dy = \frac{-n\,e^{\frac{-n}{\operatorname{ſin} x}}\,dx \operatorname{coſ} x}{2 \operatorname{ſin} x^2 \left(1 - \sqrt{\left(1 - e^{\frac{-n}{\operatorname{ſin} x}}\right)}\right) \sqrt{\left(1 - e^{\frac{-n}{\operatorname{ſin} x}}\right)}}.$$

Z CAPUT

CAPUT VII.

DE DIFFERENTIATIONE FUNCTIONUM DUAS PLURESUE VARIABILES INUOLUENTIUM.

208.

Si duae pluresue quantitates variabiles x, y, z, a ſe inuicem prorſus non pendeant, fieri poteſt, vt etiamſi omnes ſint variabiles, tamen dum vna creſcit decreſcitue, reliquae maneant inuariatae: quia enim nullum nexum inter ſe habere ponuntur, immutatio vnius reliquas non afficit. Neque ergo differentialia quantitatum y & z pendebunt a differentiali ipſius x; ideoque dum x differentiali ſuo dx augetur, quantitates y & z, vel eaedem manere, vel quomodocunque pro lubitu variari poſſunt. Quodſi igitur differentiale quantitatis x ſtatuatur dx, reliquarum quantitatum differentialia dy & dz manent indeterminata, atque pro arbitrio noſtro vel prorſus nihil, vel infinite parua ad dx quamuis rationem tenentia denotabunt.

209. Plerumque autem litterae y & z functiones ipſius x vel incognitas, vel quarum relatio ad x non ſpectatur, ſignificare ſolent, hocque caſu earum differentialia dy & dz certam ad dx relationem habebunt. Siue autem y & z pendeant ab x ſiue ſecus, ratio differentiationis; quam hic ſpectamus, eodem redit. Quaerimus

enim

enim functionis, quae ex pluribus variabilibus x, y, & z vtcunque ſit formata, differentiale, quod accipit, dum ſingulae variabiles x, y, & z ſuis differentialibus dx, dy, & dz creſcunt. Ad hoc ergo inueniendum in functione propoſita vbique loco variabilium quantitatum x, y, z ſcribatur reſpectiue $x+dx$; $y+dy$; $z+dz$, & ab expreſſione hoc modo reſultante auferatur ipſa functio propoſita: reſiduum dabit ipſum differentiale, quod quaeritur, quemadmodum ex natura differentialium luculenter conſtat.

210. Sit X functio ipſius x, eiusque differentiale, ſeu augmentum, dum x differentiali ſuo dx creſcit, ſit $=Pdx$. Deinde ſit Y functio ipſius y, eiusque differentiale $=Qdy$, quod augmentum Y accipit, dum y abit in $y+dy$: atque Z ſit functio ipſius z, eiusque differentiale ſit $=Rdz$, quae differentialia Pdx, Qdy, Rdz ex natura functionum X, Y & Z ope praeceptorum ſupra datorum inueniri poterunt. Quod ſi ergo propoſita fuerit haec quantitas $X+Y+Z$, quae vtique erit functio trium variabilium x, y, & z, eius differentiale erit $=Pdx+Qdy+Rdz$. Vtrum autem haec tria differentialia ſint inter ſe homogenea nec ne? perinde eſt. Termini enim qui continent poteſtates ipſius dx prae Pdx aeque euaneſcunt, ac ſi reliqua membra Qdy & Rdz abeſſent, ſimilique eſt ratio terminorum, qui in differentiatione functionum Y & Z ſunt neglecti.

211. Retineant X, Y & Z easdem ſignificationes, ſitque propoſita iſta functio XYZ ipſarum x, y & z, cu-

ius differentiale inueſtigari oporteat. Quoniam, ſi $x+dx$ loco x, $y+dy$ loco y, & $z+dz$ loco z ſcribatur, abit X in $X+Pdx$; Y in $Y+Qdy$; & Z in $Z+Rdz$, ipſa functio propoſita XYZ abibit in

$$(X+Pdx)(Y+Qdy)(Z+Rdz)$$

$$= XYZ + YZPdx + XZQdy + XYRdz$$

$$+ ZPQdxdy + YPRdxdz + XQRdydz + PQRdxdydz.$$

At quia dx, dy, & dz ſunt infinite parua, ſiue inter ſe ſint homogenea ſiue non; vltimus terminus prae uno quoque praecedentium euaneſcit. Deinde terminus $ZPQdxdy$ tam prae $YZPdx$ quam prae $XZQdy$ euaneſcit; atque ob eandem rationem termini $YPRdxdz$ & $XQZdydz$ euaneſcent. Ablata ergo ipſa functione propoſita XYZ, erit eius differentiale $= YZPdx + XZQdy + XYRdz$.

212. Exempla haec functionum trium variabilium x, y, & z, quibus pro lubitu quisque plura adiicere poteſt, ſufficiunt ad oſtendendum, ſi functio quaecunque trium variabilium x, y, & z proponatur, vtcunque etiam hae variabiles inter ſe fuerunt permixtae, eius differentiale ſemper huiusmodi formam eſſe habiturum $pdx + qdy + rdz$: vbi p, q, & r futurae ſint ſingulae functiones, vel omnium trium variabilium x, y, & z, vel binarum, vel vnius tantum, prout ratio compoſitionis, qua functio propoſita ex variabilibus x, y, & z atque conſtantibus formatur, fuerit comparata. Simili modo, ſi proponatur functio quatuor pluriumue variabilium

ſium x, y, z, & v, eius differentiale ſemper huius modi formam habebit $pdx + qdy + rdz + sdv$.

213. Conſideremus primum functionem duarum tantum variabilium x & y, quae ſit $= V$, cuius ergo differentiale ita ſe habebit, vt ſit $dV = pdx + qdy$. Si igitur quantitas y aſſumeretur conſtans, foret $dy = 0$, ideoque functionis V differentiale eſſet pdx: ſin autem x ſtatueretur conſtans, vt eſſet $dx = 0$, ſolaque y maneret variabilis, tum ipſius V differentiale prodiret $= qdy$. Cum igitur vtraque quantitate x & y variabili poſita ſit $dV = pdx + qdy$, iſta regula pro differentianda functione V duas variabiles x & y inuoluente reſultabit: *Ponatur primum ſola* x *variabilis, altera vero* y *tanquam conſtans tractetur, & quaeratur ipſius* V *differentiale, quod ſit* $=$ pdx. *Deinde ponatur ſola quantitas* y *variabilis, altera* x *pro conſtanti habita, & quaeratur ipſius* V *differentiale, quod ſit* $=$ qdy. *Quibus factis, poſita vtraque quantitate* x *&* y *variabili, fiet* dV $=$ pdx $+$ qdy.

214. Simili modo, cum functionis trium variabilium x, y, & z, quae ſit $= V$, differentiale huiusmodi habeat formam $dV = pdx + qdy + rdz$, manifeſtum eſt, ſi ſola quantitas x fuiſſet variabilis poſita, reliquae vero y & z conſtantes manſiſſent, ob $dy = 0$ & $dz = 0$, prodiiſſet ipſius V differentiale $= pdx$. Pari modo inueniretur differentiale ipſius $V = qdy$, ſi x & z eſſent conſtantes ſolaque y poneretur variabilis; atque ſi x & y tanquam conſtantes tractarentur ſolaque z ſtatueretur variabilis, pro-

diret differentiale ipſius $V = r\,dz$. Quare ad functionem trium pluriumue variabilium differentiandam, conſideretur ſeorſim quaelibet quantitas variabilis, & functio pro qualibet differentietur, quaſi reliquae omnes eſſent conſtantes; tum ſingula haec differentialia, quae ex ſingulis quantitatibus variabilibus ſunt inuenta, colligantur, eritque aggregatum differentiale quaeſitum functionis propoſitae.

215. In hac regula, quam pro differentiatione functionis quotcunque variabilium inuenimus, continetur demonſtratio regulae ſupra §. 170 datae generalis, cuius ope functio quaecunque vnicam variabilem complectens differentiari poteſt. Si enim pro ſingulis partibus ibi commemoratis totidem litterae diuerſae collocentur, functio ſpeciem induet functionis totidem diuerſarum variabilium, atque adeo modo hic praeſcripto differentiabitur, ſucceſſiue vnamquamque partem, quaſi ſola eſſet variabilis, tractando, cunctaque differentialia, quae ex ſingulis partibus oriuntur, in vnam ſummam coniiciendo: quae ſumma erit differentiale quaeſitum, poſtquam pro ſingulis litteris valores fuerint reſtituti. Haec ergo regula latiſſime patet, atque etiam ad functiones plurium variabilium, quomodocunque fuerint comparatae, extenditur. Vnde eius vſus per vniuerſum calculum differentialem eſt ampliſſimus.

216. Inuenta ergo regula generali, cuius ope functiones quotcunque variabilium differentiari poſſunt, eius vſum in nonnullis exemplis oſtendiſſe iuuabit.

I. Si

I. Si fuerit $V = xy$; erit $dV = xdy + ydx$.

II. Si fuerit $V = \frac{x}{y}$; erit $dV = \frac{dx}{y} - \frac{xdy}{yy}$.

III. Si fuerit $V = \frac{y}{\sqrt{(aa - xx)}}$; erit

$$dV = \frac{dy}{\sqrt{(aa-xx)}} + \frac{yxdx}{(aa-xx)^{\frac{3}{2}}}.$$

IV. Si fuerit $V = (\alpha x + \beta y + \gamma)^m (\delta x + \varepsilon y + \zeta)^n$;

erit

$$dV = m(\alpha x + \beta y + \gamma)^{m-1}(\delta x + \varepsilon y + \zeta)^n(\alpha dx + \beta dy)$$
$$+ n(\alpha x + \beta y + \gamma)^m(\delta x + \varepsilon y + \zeta)^{n-1}(\delta dx + \varepsilon dy),$$

ſiue

$$dV = (\alpha x + \beta y + \gamma)^{m-1}(\delta + \varepsilon y + \zeta)^{n-1} \text{ in}$$
$$\left(\begin{matrix} m\alpha\delta \\ n\alpha\delta \end{matrix} xdx \begin{matrix} + m\beta\delta \\ + n\alpha\varepsilon \end{matrix} xdy \begin{matrix} + m\alpha\varepsilon \\ + n\beta\delta \end{matrix} ydx \right.$$
$$\left. \begin{matrix} + m\beta\varepsilon \\ + n\beta\varepsilon \end{matrix} ydy \begin{matrix} + m\alpha\zeta \\ + n\gamma\delta \end{matrix} dx \begin{matrix} + m\beta\zeta \\ + n\gamma\varepsilon \end{matrix} dy \right).$$

V. Si fuerit $V = ylx$; erit $dV = dylx + \frac{ydx}{x}$.

VI. Si fuerit $V = x^y$; erit $dV = yx^{y-1}dx + x^y dylx$.

VII. Si fuerit $V = \text{A tang} \frac{y}{x}$; erit $dV = \frac{xdy - ydx}{xx + yy}$.

VIII. Si fuerit $V = \text{ſin}\, x . \text{coſ}\, y$; erit

$$dV = dx \,\text{coſ}\, x \,\text{coſ}\, y - dy \,\text{ſin}\, x . \text{ſin}\, y.$$

IX.

IX. Si fuerit $V = \frac{e^z y}{\surd(xx+yy)}$; erit

$$dV = \frac{e^z y dz}{\surd(xx+yy)} + \frac{e^z(xx dy - yx dx)}{(xx+yy)\surd(xx+yy)}.$$

X. Si fuerit $V = e^z \text{A ſin} \frac{x - \surd(xx-yy)}{x + \surd(xx-yy)}$,

reperietur $dV = e^z dz \text{ A ſin} \frac{x - \surd(xx-yy)}{x + \surd(xx-yy)}$

$$+ e^z \frac{xydy - yydx}{(x + \surd(xx-yy))(xx-yy)^{\frac{3}{4}}\surd x}.$$

217. Quoniam vidimus, ſi V fuerit functio quaecunque binarum variabilium x & y, eius differentiale huiusmodi habiturum eſſe formam $dV = Pdx + Qdy$, in qua ſint P & Q functiones a functione V pendentes per eamque determinatae: ſequitur has duas quantitates P & Q certo quodam modo etiam a ſe inuicem pendere, propterea quod vtraque ab eadem functione V pendet. Quicunque igitur ſit iſte nexus inter quantitates finitas P & Q, quem deinceps inueſtigabimus, perſpicuum eſt, non omnes formulas differentiales huiusmodi $Pdx + Qdy$, in quibus P & Q pro lubitu ſint ex x & y formatae, poſſe eſſe differentialia cuiuspiam functionis finitae V ipſarum x & y. Niſi enim ea relatio inter functiones P & Q intercedat, quam natura differentiationis requirit, huiusmodi differentiale $Pdx + Qdy$ oriri plane per differentiationem non potuit, ideoque viciſſim integrale non habebit.

218.

218. In integratione igitur plurimum intereſt noſſe hanc relationem inter quantitates P & Q, vt differentialia, quae reuera ex differentiatione functionis cuiuspiam finitae ſunt orta, dignoſci queant ab iis, quae ad libitum ſunt formata, atque nulla integralia admittunt. Quanquam autem hic nondum integrationis negotium ſuſcipimus, tamen ad naturam differentialium realium penitius inſpiciendam conueniet hanc relationem inueſtigari; quippe cuius cognitio non ſolum ad calculum integralem, ad quem hic viam paramus, eſt maxime neceſſaria, ſed etiam in ipſo calculo differentiali inſignem lucem accendit. Primum igitur patet, ſi V ſit functio duarum variabilium x & y, in eius differentiali $Pdx + Qdy$ vtriusque differentiale dx & dy ineſſe oportere. Neque ergo poteſt eſſe $P = 0$ neque $Q = 0$. Hinc ſi P fuerit functio ipſarum x & y, formula Pdx nullius quantitatis finitae poterit eſt differentiale, ſeu nulla extat quantitas finita, cuius differentiale ſit Pdx.

219. Sic nulla datur quantitas finita V ſiue algebraica ſiue transcendens, cuius differentiale ſit $yxdx$, ſi quidem ſit y quantitas variabilis ab x non pendens. Si enim ponamus dari eiusmodi quantitatem finitam V, quia y in eius differentiale ingreditur, neceſſe eſt, vt y quoque in ipſa quantitate V inſit; verum ſi V contineret y, ob variabilitatem ipſius y neceſſario quoque in differentiali ipſius V differentiale dy ineſſe deberet. Quod tamen cum non adſit, fieri nequit, vt differentiale $yxdx$ ex cuiuspiam quantitatis finitae differentiatione ſit ortum. Cum

 igi-

igitur pateat formulam $Pdx + Qdy$, ſi Q ſit o, & P contineat y, differentiale reale eſſe non poſſe, ſimul intelligitur, quantitati Q non pro lubitu valorem tribui poſſe, ſed eum a valore ipſius P pendere.

220. Quo igitur hanc relationem inter P & Q in differentiali $dV = Pdx + Qdy$ inueſtigemus, ponamus primo V eſſe functionem nullius dimenſionis ipſarum x & y: a caſibus enim particularibus ad relationem generalem aſcendamus. Quod ſi ergo ponamus $y = tx$, ex functione V quantitas x prorſus euaneſcet, prodibitque functio ipſius t tantum, quae ſit $= T$, cuius differentiale erit $= \Theta dt$, exiſtente Θ functione ipſius t. Ponamus igitur quoque in differentiali $Pdx + Qdy$, vbique $y = tx$, & $dy = tdx + xdt$, quo facto prodibit

$$Pdx + Qtdx + Qxdt;$$

in quo cum dx non contineatur, neceſſe eſt vt ſit $P + Qt = 0$; ideoque $Q = -\frac{P}{t} = -\frac{Px}{y}$, ſeu erit $Px + Qy = 0$, vnde relatio inter P & Q pro hoc caſu innoteſcit. Deinde debet eſſe $\Theta = Qx$, ideoque $Qx =$ functioni ipſius t, hoc eſt functioni nullius dimenſionis ipſarum x & y. Atque ob $Q = \frac{\Theta}{x}$ fiet $P = -\frac{\Theta y}{xx}$, & tam Px quam Qy erunt functiones nullius dimenſionis ipſarum x & y.

221. Si igitur functio nullius dimenſionis ipſarum x & y, quae ſit $= V$, differentietur, eius differentiale $dV =$

$dV = P dx + Q dy$, ſemper ita erit comparatum, vt ſit $Px + Qy = 0$. Hoc eſt ſi in differentiali loco differentialium dx & dy ſcribantur x & y, reſultabit quantitas $= 0$: vti in his exemplis vſu venire patet:

I. Sit $V = \frac{x}{y}$; erit $dV = \frac{y dx - x dy}{yy}$, atque poſito x loco dx & y loco dy, erit $\frac{yx - xy}{yy} = 0$.

II. Sit $V = \frac{x}{\sqrt{(xx - yy)}}$, erit $dV = - \frac{yydx + yxdy}{(xx - yy)^{\frac{3}{2}}}$, vnde fit $\frac{-yyx + yyx}{(xx - yy)^{\frac{3}{2}}} = 0$.

III. Sit $V = \frac{y + \sqrt{(xx + yy)}}{-y + \sqrt{(xx + yy)}}$, quae eſt functio nullius dimenſionis ipſarum x & y; erit $dV = \frac{2xxdy - 2xydx}{(\sqrt{(xx + yy)} - y)^2 \sqrt{(xx + yy)}}$, quae forma poſitis x & y loco dx & dy fit $= 0$.

IV. Sit $V = l \frac{x + y}{x - y}$; erit $dV = \frac{2xdy - 2ydx}{xx - yy}$, atque $\frac{2xy - 2yx}{xx - yy} = 0$.

V. Sit $V = A \sin \frac{\sqrt{(x - y)}}{\sqrt{(x + y)}}$, erit $dV = \frac{ydx - xdy}{(x + y)\sqrt{2y(x - y)}}$, quae formula eadem proprietate gaudet.

222. Contemplemur nunc alias functiones homogeneas, ſitque V functio n dimenſionum ipſarum x & y. Quare ſi ponatur $y = tx$, induet V huiusmodi formam Tx^n, exiſtente T functione ipſius t, ſitque

$$dT = \Theta dt, \quad \text{erit} \quad dV = x^n \Theta dt + nTx^{n-1}dx.$$

Quodſi ergo ſtatuamus:

$$dV = Pdx + Qdy, \quad \text{ob} \quad dy = tdx + xdt,$$

fiet $$dV = Pdx + Qtdx + Qxdt:$$

quae forma quoniam cum illa congruere debet, erit

$$P + Qt = nTx^{n-1} = \frac{nV}{x}, \quad \text{ob} \quad V = Tx^n.$$

Hancobrem ob $t = \frac{y}{x}$, fiet $Px + Qy = nV$, quae aequatio relationem inter P & Q ita definit, vt ſi altera ſit cognita, altera facile inueniatur. Quia porro eſt $Qx = x^n\Theta$, erit Qx, ideoque etiam Qy & Px functio n dimenſionum ipſarum x & y.

223. Si ergo in differentiali cuiusuis functionis homogeneae ipſarum x & y, loco dx & dy, ponatur x & y, quantitas oriunda aequabitur ipſi functioni, cuius differentiale proponebatur, per numerum dimenſionum multiplicatae.

I. Si ſit $V = \surd(xx + yy)$; erit $n = 1$, & ob

$$dV = \frac{xdx + ydy}{\surd(xx + yy)}, \text{ fiet } \frac{xx + yy}{\surd(xx + yy)} = V = \surd(xx + yy).$$

II. Si

II. Si ſit $V = \frac{y^3 + x^3}{y - x}$; erit $n = 2$, &

$$dV = \frac{2y^3dy - 3yyxdy + 3yxxdx - 2x^3dx + y^3dx - x^3dy}{(y - x)^2}.$$

Ponatur x pro dx & y pro dy orietur:

$$\frac{2y^4 - 2y^3x + 2yx^3 - 2x^4}{(y - x)^2} = \frac{2y^3 + 2x^3}{y - x} = 2V.$$

III. Si ſit $V = \frac{1}{(yy + xx)^2}$, erit $n = -4$, atque $dV = -\frac{4ydy - 4xdx}{(yy + xx)^3}$. Quae formula poſitis x & y loco dx & dy abit in $-\frac{4yy - 4xx}{(yy + xx)^3} = -4V$.

IV. Si ſit $V = xxl\frac{y + x}{y - x}$; erit $n = 2$, atque $dV = 2xdxl\frac{y + x}{y - x} + \frac{2xx(ydx - xdy)}{yy - xx}$, facta autem memorata ſubſtitutione oritur $2xxl\frac{y + x}{y - x} = 2V$.

224. Similis proprietas obſeruabitur, ſi V fuerit functio homogenea plurium variabilium: ſit ergo V functio quantitatum x, y, z, quae coniunctim vbique n dimenſiones adimpleant; atque differentiale huiusmodi habebit formam $Pdx + Qdy + Rdz$. Ponatur iam $y = tx$ & $z = sx$, vt ſit $dy = tdx + xdt$, & $dz = sdx + xds$, atque functio V induet hanc formam Ux^n, exiſtente U

 func-

functione binarum variabilium t & s; hinc ergo ſi ſtatuatur $dU = p\,dt + q\,ds$, fiet

$$dV = x^n p\,dt + x^n q\,ds + nUx^{n-1}dx,$$

Prior autem forma dabit

$$dV = P\,dx + Qt\,dx + Qx\,dt + Rs\,dx + Rx\,ds:$$

quae cum illa collata praebet

$$P + Qt + Rs = nUx^{n-1} = \frac{nV}{x},$$

vnde obtinetur $Px + Qy + Rz = nV$; quae eadem proprietas ad quotcunque plures variabiles extenditur.

225. Si igitur propoſita fuerit functio homogenea quotcunque variabilium x, y, z, v, &c. eius differentiale perpetuo hanc habebit proprietatem, vt ſi loco differentialium dx, dy, dz, dv, &c. ſcribantur reſpective quantitates finitae x, y, z, v, &c. prodeat ipſa functio propoſita per numerum dimenſionum multiplicata. Haecque regula etiam valet, ſi V fuerit functio homogenea vnicae tantum variabilis x: Hoc enim caſu erit V poteſtas ipſius x, puta $V = ax^n$, quae eſt functio homogenea n dimenſionum: nulla ſcilicet alia datur functio ipſius x, in qua x vbique n dimenſiones conſtituat praeter poteſtatem x^n. Cum igitur ſit $dV = nax^{n-1}dx$, ponatur x loco dx, atque prodibit nax^n, hoc eſt nV. Iſta ergo functionum homogenearum inſignis proprietas diligenter notari meretur, cum in calculo integrali maximam afferat vtilitatem.

226. Quo nunc in genere in relationem inter quantitates P & Q, quae differentiale $Pdx + Qdy$ functionis cuiuſcunque V duarum variabilium x & y conſtituunt, inquiramus, ad ſequentia attendi oportebit. Sit igitur V functio quaecunque ipſarum x & y; atque ponamus V abire in R, ſi loco x ponatur $x + dx$; poſito autem $y + dy$ loco y abeat V in S: quodſi autem ſimul $x + dx$ loco x, & $y + dy$ loco y ſcribatur, mutetur V in V^I. Cum itaque R oriatur ex V, poſito $x + dx$ loco x, manifeſtum eſt ſi vlterius in R ponatur $y + dy$ loco y, tum prodire V^I; idem enim eſt, ac ſi in V ſtatim poneretur $x + dx$ loco x, & $y + dy$ loco y. Simili modo ſi in S ponatur $x + dx$ loco x, quia S iam orta eſt ex V poſito $y + dy$ loco y, denuo prodibit V^I; vti ex hoc ſchematiſmo clarius perſpicitur.

Quantitas	abit in	ſi loco	ponatur.
V	R	x	$x + dx$
V	S	y	$y + dy$
V	V^I	x y	$x + dx$ $y + dy$
R	V^I	y	$y + dy$
S	V^I	x	$x + dx$

227. Si igitur V ita differentietur, vt tantum x tanquam variabilis, y vero tanquam conſtans tractetur, quia poſito $x+dx$ loco x, functio V abit in R, eius differentiale erit $=R-V$; at ex forma $dV=Pdx+Qdy$, ſequitur idem differentiale fore $=Pdx$, vnde erit $R-V=Pdx$. Quod ſi iam loco y ponatur $y+dy$, x vero tanquam conſtans tractetur, quia R abit in V^{I} & V in S, quantitas $R-V$ abibit in $V^{I}-S$; ideoque ipſius $R-V=Pdx$ differentiale, quod oritur ſi ſola y variabilis aſſumatur, erit $V^{I}-R-S+V$. Simili modo, cum poſito $y+dy$ loco y, abeat V in S, erit $S-V$ differentiale ipſius V poſita ſola y variabili, eritque propterea $S-V=Qdy$; nunc quia loco x poſito $x+dx$, tranſit S in V^{I} & V in R, quantitas $S-V$ abibit in $V^{I}-R$; atque ipſius $S-V=Qdy$ differentiale, quod oritur ſi ſola x variabilis ſtatuatur, erit $=V^{I}-R-S+V$, quod prorſus congruit cum differentiali ante inuento.

228. Ex hac conuenientia deducitur ſequens concluſio: Si functionis V cuiuscunque binarum variabilium x & y differentiale fuerit $dV=Pdx+Qdy$, tum differentiale ipſius Pdx, quod oritur ſi ſola quantitas y tanquam variabilis, x vero tanquam conſtans tractetur, aequale erit differentiali ipſius Qdy, quod oritur ſi ſola quantitas x tanquam variabilis, y vero tanquam conſtans tractetur. Si ſcilicet poſita ſola y variabili fuerit $dP=Zdy$ erit differentiale ipſius Pdx praeſcripto modo ſumtum $=Z\,dx\,dy$; atque poſita ſola x variabili erit quoque $dQ=Zdx$; ſic enim differentiale ipſius Qdy praeſcripto modo

modo ſumtum fiet quoque $= Z\,dx\,dy$. Hacque ratione intelligitur relatio, quae inter quantitates P & Q intercedit, atque paucis verbis in hoc conſiſtit, vt differentiale ipſius $P\,dx$ poſito x conſtante aequale ſit differentiali ipſius $Q\,dy$ poſito y conſtante.

229. Iſta inſignis proprietas clarius perſpicietur, ſi eam nonnullis exemplis illuſtremus.

I. Sit igitur $V = yx$; erit $dV = y\,dx + x\,dy$, ideoque $P = y$ & $Q = x$; vnde poſito x conſtante erit $d.P\,dx = dx\,dy$, & poſito y conſtante erit $d.Q\,dy = dx\,dy$, ſicque haec duo differentialia inter ſe aequantur.

II. Sit $V = \surd(xx + 2xy)$; erit $dV = \frac{x\,dx + y\,dx + y\,dy}{\surd(xx + 2xy)}$, ideoque $P = \frac{x + y}{\surd(xx + 2xy)}$, & $Q = \frac{x}{\surd(xx + 2xy)}$, vnde poſito x conſtante erit $d.P\,dx = \frac{xy\,dx\,dy}{(xx + 2xy)^{\frac{3}{2}}}$, & poſito y conſtante erit $d.Q\,dy = \frac{xy\,dx\,dy}{(xx + 2xy)^{\frac{3}{2}}}$.

III. Sit $V = x \sin A\,y + y \sin A\,x$; eritque $dV = dx \sin A\,y + x\,dy \cos y + dy \sin A\,x + y\,dx \cos x$. Quare erit

$$P\,dx = dx \sin A\,y + y\,dx \cos x,$$

& $$Q\,dy = dy \sin A\,x + x\,dy \cos y.$$

Poſito ergo x conſtante erit

$$d.Pdx = dxdy \operatorname{coſ} y + dxdy \operatorname{coſ} x,$$

& poſito y conſtante erit

$$d.Qdy = dxdy \operatorname{coſ} y + dxdy \operatorname{coſ} x.$$

IV. Sit $V = x^y$; erit $dV = x^y dylx + yx^{y-1}dx$, atque

$$Pdx = yx^{y-1}dx, \quad \& \quad Qdy = x^y dylx.$$

Quamobrem poſito x conſtante habebitur

$$d.Pdx = x^{y-1}dxdy + yx^{y-1}dxdylx,$$

& poſito y conſtante erit

$$d.Qdy = yx^{y-1}dxdylx + x^{y-1}dxdy.$$

230. Iſta proprietas etiam hoc modo enunciari poteſt, vnde eximia omnium functionum, quae duas variabiles inuoluunt, indoles cognoſcetur. Si functio quaecunque V duarum variabilium x & y differentietur poſita ſola x variabili, hocque differentiale denuo differentietur poſita ſola y variabili, tum poſt duplicem hanc differentiationem idem prodibit, ac ſi ordine inuerſo functio V primum poſita ſola y variabili differentiaretur, hocque differentiale poſita ſola x variabili denuo differentiaretur: vtroque ſcilicet caſu prodibit eadem expreſſio huius formae $Z\,dx\,dy$. Ratio huius identitatis ex praecedente proprietate manifeſto ſequitur: ſi enim V differentietur poſita ſola x variabili, prodit Pdx; &, ſi V differentietur poſita ſola y variabili, prodit Qdy, horum differentia-

tialium vero differentialia modo indicato ſumta inter ſe aequalia eſſe, ante demonſtrauimus. Caeterum haec indoles immediate ſequitur ex ratiocinio (§. 227) allato.

231. Relatio inter P & Q, ſi $Pdx + Qdy$ fuerit differentiale functionis V ſequenti etiam modo indicari poteſt. Quoniam P & Q ſunt functiones ipſarum x & y, differentientur ambae poſita vtraque x & y variabili:

Si ſcilicet fuerit $dV = Pdx + Qdy$

ſit $dP = pdx + rdy$

& $dQ = qdx + sdy.$

Poſito ergo x conſtante erit

$$dP = rdy, \quad \& \quad d.Pdx = rdxdy.$$

Deinde poſito y conſtante erit

$$dQ = qdx, \quad \& \quad d.Qdy = qdxdy.$$

Cum igitur haec duo differentialia $rdxdy$ & $qdxdy$ ſint inter aequalia, ſequitur fore $q = r$. Functiones ergo P & Q ita inuicem connectuntur, vt ſi ambae differentientur, vti fecimus, quantitates q & r inter ſe fiant aequales. Breuitatis gratia autem hoc ſaltem capite quantitates r & q ita commode denotari ſolent, vt r indicetur per $\left(\frac{dP}{dy}\right)$, quae ſcriptura deſignatur P ita differentiari, vt ſola y tanquam variabilis tractetur, atque differentiale iſtud per dy diuidatur: ſic enim prodibit quantitas finita r. Simili modo ſignificabit $\left(\frac{dQ}{dx}\right)$ quantitatem fini-

finitam q, quia hac ratione indicatur functionem Q ſola x poſita variabili differentiari, tumque differentiale per dx diuidi debere.

232. Vtamur ergo hoc ſcribendi modo, etiamſi alias ambiguitatem afferre poſſit, quae tamen hic per clauſulas euitatur, vt ambages in deſcribendis differentiandi conditionibus euitemus, ſicque breuiter relationem inter P & Q ita verbis exprimere poterimus, vt dicamus eſſe $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$. In huiusmodi ſcilicet fractionibus denominator praeter propriam ſignificationem, qua numerator per eum diuidi debet, indicat numeratoris differentiale ita eſſe capiendum, vt ea ſola quantitas cuius differentiale denominatorem conſtituit, tanquam variabilis ſpectetur. Hoc enim modo per diuiſionem differentialia prorſus ex calculo egredientur, iſtaeque fractiones $\left(\frac{dP}{dy}\right)$ & $\left(\frac{dQ}{dx}\right)$ exhibebunt quantitates finitas, quae in praeſenti caſu erunt inter ſe aequales. Hoc itaque modo recepto quantitates quoque p & s ita denotare licebit, vt ſit $p = \left(\frac{dP}{dx}\right)$ & $s = \left(\frac{dQ}{dy}\right)$, ſi quidem vt monitum eſt, differentiatio numeratoris per denominatorem reſtringatur.

233. Conſentit haec proprietas mirifice cum proprietate, quam ante in functionibus homogeneis ineſſe oſtendimus. Sit enim V functio homogenea n dimenſio-

ſionum ipſarum x & y, ponaturque $dV = Pdx + Qdy$, atque demonſtrauimus fore $nV = Px + Qy$, ideoque

$$Q = \frac{nV}{y} - \frac{Px}{y}. \text{ Sit } dP = pdx + rdy;$$

eritque $\left(\frac{dP}{dy}\right) = r$, cui aequale eſſe $\left(\frac{dQ}{dx}\right)$ ita oſtendetur. Differentietur Q poſita ſola x variabili, & quia in hac hypotheſi eſt $dQ = \frac{nPdx}{y} - \frac{Pdx}{y} - \frac{xpdx}{y}$, fiet $\left(\frac{dQ}{dx}\right) = \frac{(n-1)P}{y} - \frac{px}{y}$, debebitque eſſe

$$\frac{(n-1)P}{y} - \frac{px}{y} = r \text{ ſeu } (n-1)P = px + ry.$$

Quae aequalitas inde fit perſpicua, quod P ſit functio homogenea $n-1$ dimenſionum ipſarum x & y, vnde eius differentiale $dP = pdx + rdy$, ob proprietatem functionum homogenearum, ita debet eſſe comparatum, vt ſit $(n-1)P = px + ry$.

234. Iſta proprietas, quod ſit $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$, quam omnibus functionibus duarum variabilium x & y communem eſſe oſtendimus, nobis quoque patefaciet naturam functionum trium pluriumue variabilium. Sit V functio quaecunque trium variabilium x, y, & z, ac ponatur $dV = Pdx + Qdy + Rdz$. Quod ſi igitur in hac differentiatione z tanquam conſtans tractaretur, foret vtique $dV = Pdx + Qdy$; hoc autem caſu per

 an-

antecedentia debet eſſe $\left(\frac{dP}{dy}\right)=\left(\frac{dQ}{dx}\right)$. Deinde ſi quantitas y conſtans aſſumeretur, foret $dV = Pdx + Rdz$, erit ergo $\left(\frac{dP}{dz}\right)=\left(\frac{dR}{dx}\right)$. Denique poſito x conſtante reperietur $\left(\frac{dQ}{dz}\right)=\left(\frac{dR}{dy}\right)$. In differentiali ergo $Pdx + Qdy + Rdz$ functionis V quantitates P, Q, & R ita a ſe inuicem pendent, vt ſit

$$\left(\frac{dP}{dy}\right)=\left(\frac{dQ}{dx}\right);\ \left(\frac{dP}{dz}\right)=\left(\frac{dR}{dx}\right);\ \&\left(\frac{dQ}{dz}\right)=\left(\frac{dR}{dy}\right).$$

235. Sequitur hinc iſta functionum, quae tres pluresue variabiles inuoluunt, proprietas analoga ei, quam ſupra (230) de functionibus duarum variabilium oſtendimus. Si fuerit V functio quaecunque trium variabilium x, y, & z, eaque continuo ter differentietur, ita vt primum vna quantitatum, puta x, ſola variabilis ponatur, in differentiatione ſecunda ſola y, atque in tertia ſola z variabilis aſſumatur, prodibit expreſſio huius formae $Zdxdydz$, quae eadem reperietur, quocunque alio ordine quantitates x, y, & z collocentur. Sex igitur diverſis modis poſt triplicem differentiationem ad eandem expreſſionem $Zdxdydz$ peruenietur, quoniam ordo quantitatum x, y, & z ſexies variari poteſt. Quicunque ergo ordo eligatur, ſi functio V differentietur poſita ſola prima variabili, hocque differentiale denuo differentietur poſita ſola ſecunda variabili, atque differentiale hoc iterum

rum differentietur poſita ſola tertia variabili, eadem prodibit expreſſio, vtcunque ordo quantitatum x, y, & z varietur.

236. Quo ratio huius proprietatis clarius perſpiciatur, ponamus eſſe $dV = Pdx + Qdy + Rdz$; deinde etiam quantitates P, Q, & R differentiemus, eruntque earum differentialia per ante demonſtrata, ita comparata:

$$dP = pdx + sdy + tdz$$
$$dQ = sdx + qdy + udz$$
$$dR = tdy + udy + rdz.$$

Differentietur nunc V poſito ſolo x variabili prodibit Pdx; quod differentiale iterum differentietur poſito ſolo y variabili atque habebitur $sdxdy$; quod ſi differentietur poſito ſolo z variabili, poſtquam per $dxdydz$ fuerit diuiſum, obtinebitur $\left(\frac{ds}{dz}\right)$. Collocentur nunc variabiles hoc ordine y, x, z, atque prima differentiatio dabit Qdy, ſecunda $sdxdy$, & tertia (facta diuiſione per $dxdydz$) dabit $\left(\frac{ds}{dz}\right)$ vt ante. Diſponantur variabiles hoc ordine z, y, x, ac prima differentiatio dabit Rdz ſecunda $udydz$, tertia vera poſt diuiſionem per $dxdydz$ praebet $\left(\frac{du}{dx}\right)$. At cum poſito y conſtante ſit $dQ = sdx + udz$; erit $\left(\frac{ds}{dz}\right) = \left(\frac{du}{dx}\right)$, vti pariter eſt demonſtratum.

237. Ponamus eſſe $V = \frac{xxy}{aa - zz}$; hancque functionum toties ter differentiemus, quoties ordo variabilium x, y, z variari poteſt:

	I. DIFFER.		II. DIFFER.		III. DIFFER.
poſito variabili	ſolo x		ſolo y		ſolo z
	$\frac{2xydx}{aa - zz}$	;	$\frac{2xdxdy}{aa - zz}$	;	$\frac{4xzdxdydz}{(aa - zz)^2}$
poſito variabili	ſolo x		ſolo z		ſolo y
	$\frac{2xydx}{aa - zz}$	;	$\frac{4xyzdxdz}{(aa - zz)^2}$	;	$\frac{4xzdxdydz}{(aa - zz)^2}$
poſito variabili	ſolo y		ſolo x		ſolo z
	$\frac{xxdy}{aa - zz}$	;	$\frac{2xdxdy}{aa - zz}$	;	$\frac{4xzdxdydz}{(aa - zz)^2}$
poſito variabili	ſolo y		ſolo z		ſolo x
	$\frac{xxdy}{aa - zz}$	;	$\frac{2xxzdydz}{(aa - zz)^2}$	;	$\frac{4xzdxdydz}{(aa - zz)^2}$
poſito variabili	ſolo z		ſolo x		ſolo y
	$\frac{2xxyzdz}{(aa - zz)^2}$	;	$\frac{4xyzdxdz}{(aa - zz)^2}$	;	$\frac{4xzdxdydz}{(aa - zz)^2}$
poſito variabili	ſolo z		ſolo y		ſolo x
	$\frac{2xxyzdz}{(aa - zz)^2}$	;	$\frac{2xxzdydz}{(aa - zz)^2}$	;	$\frac{4xzdxdydz}{(aa - zz)^2}$

ex quo exemplo patet, quocunque ordine tres variabiles fuerint aſſumtae, poſt triplicem differentiationem ſemper eandem prodire expreſſionem $\frac{4xzdxdydz}{(aa - zz)^2}$.

238. Vti autem poſt triplicem differentiationem ad eandem expreſſionem eſt peruentum, ita quoque conſenſus deprehenditur in differentialibus, quae ſecunda differentiatio ſuppeditauit. In iis ſcilicet expreſſio quaeuis bis occurrit; vnde patet, quae formulae iisdem differentialibus ſint affectae, easdem quoque inter ſe eſſe aequales, atque differentialia tertia ideo eſſe omnia inter ſe aequalia, quia iisdem differentialibus $dxdydz$ ſunt affecta. Hinc igitur concludimus, ſi V fuerit functio quotcunque variabilium x, y, z, v, u, &c. eaque ſucceſſiue aliquoties differentietur, vt ſemper vnica tantum quantitas variabilis aſſumatur; tum quoties ad expreſſiones perueniatur, quae iisdem differentialibus ſint affectae, eas quoque inter ſe aequales fore. Sic duplici differentiatione orietur huiusmodi expreſſio $Zdxdy$, dum in altera ſola x, in altera ſola y aſſumta eſt variabilis: perindeque eſt vtra prius, poſteriusue ſit variabilis aſſumta. Simili modo ſex variis modis per triplicem differentiationem eadem exſurget expreſſio $Zdxdydz$; atque viginti quatuor variis modis peruenietur poſt quadruplicem differentiationem ad eandem expreſſionem huius formae $Zdxdydzdv$, atque ita porro.

239. Veritatem horum Theorematum quilibet adhibita leui attentione ex ante explicatis principiis facile agnoſcet, atque propria meditatione facilius intuebitur, quam tantis verborum ambagibus, ſine quibus demonſtrationes proferri non poſſent. Quia vero harum proprietatum cognitio maximi eſt momenti in calculo integrali, Tyrones ſunt monendi, vt non ſolum has proprietates ipſi diligen-

ter meditentur, earumque veritatem ſcrutentur, ſed etiam pluribus exemplis comprobent; quo hoc pacto ſibi hanc materiam familiariorem reddant, fructusque inde natos poſtmodum facilius percipere queant. Neque vero ſolum tyrones, ſed etiam ii, qui principiis calculi differentialis iam ſunt imbuti, ad hoc ſunt cohortandi; quoniam in omnibus fere manuductionibus ad hanc Analyſeos partem hoc argumentum penitus praetermitti ſolet. Plerumque enim Auctores ſolas differentiationis regulas praeſcripſiſſe, earumque vſum in Geometria ſublimiori oſtendiſſe fuerunt contenti, neque in naturam atque proprietates differentialium inquiſiuerunt; vnde tamen maxima ſubſidia in calculum integralem redundant. Quam ob cauſam hoc argumentum fere nouum in iſto Capite fuſius perſequi viſum eſt, quo ſimul via ad integrationes alias difficiliores pararetur, atque negotium poſtea ſuſcipiendum ſubleuaretur.

240. Cognitis igitur his proprietatibus, quibus differentialia functionum duas pluresue variabiles inuoluentium gaudent, facile poterimus dignoſcere, vtrum formula differentialis propoſita, in qua occurrunt duae pluresue variabiles, ſit orta ex differentiatione cuiuspiam functionis finitae an ſecus? Si enim in formula $Pdx + Qdy$ non fuerit $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$, certo poterimus affirmare, nullam exiſtere functionem ipſarum x & y, cuius differentiale ſit $= Pdx + Qdy$: neque ergo infra in calculo integrali huiusmodi formulae integrale indagari poteſt. Sic cum

cum in $yxdx + xxdy$ requiſita conditio non adſit, nulla datur functio, cuius differentiale eſt $= yxdx + xxdy$. Vtrum autem ſemper, quoties eſt $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$, formula ex differentiatione cuiuspiam functionis ſit orta? quaeſtio eſt, quae demum ex integrationis principiis ſolide affirmari poterit.

241. Si in formula differentiali propoſita tres pluresue inſint variabiles, vti $Pdx + Qdy + Rdz$; tum ea ex differentiatione ortum traxiſſe omnino nequit, niſi tres iſtae conditiones in ea locum habeant, vt ſit

$$\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right);\ \left(\frac{dP}{dz}\right) = \left(\frac{dR}{dx}\right);\ \&\ \left(\frac{dQ}{dz}\right) = \left(\frac{dR}{dy}\right).$$

Quarum conditionum, ſi vna tantum deſit, certo affirmare debemus, nullam extare functionum ipſarum x, y, & z, cuius differentiale ſit $Pdx + Qdy + Rdz$; huius modi ergo formularum differentialium nequidem requiri poſſunt integralia, hincque integrationem prorſus non recipere dicuntur. Facile autem intelligitur in calculo integrali formulas differentiales ante dignoſci oportere, vtrum integrationis ſint capaces, quam inueſtigatio integralis actu ſuſcipiatur.

CAPUT VIII.

DE FORMULARUM DIFFERENTIALIUM VLTERIORI DIFFERENTIATIONE.

242.

Si vnica variabilis adſit, eiusque differentiale primum conſtans aſſumatur, ſupra iam methodus eſt tradita differentialia cuiusque gradus inueniendi. Scilicet ſi functionis cuiusuis differentiale denuo differentietur, oritur eius differentiale ſecundum, hocque iterum differentiatum dat functionis differentiale tertium, atque ita porro. Haec vero eadem regula locum quoque habet, ſiue functio plures inuoluat variabiles ſiue vnicam tantum, cuius differentiale primum non ponitur conſtans. Sit igitur V functio quaecunque ipſius x, neque vero dx ſit conſtans, ſed vtcunque variabile, ita vt ipſius dx differentiale ſit $= ddx$, huiusque differentiale $= d^3x$, & ita porro, atque inueſtigemus differentialia ſecundum & ſequentia functionis V.

243. Ponamus differentiale primum functionis V eſſe $= Pdx$, vbi erit P functio quaepiam ipſius x pendens ab V. Si iam functionis V differentiale ſecundum inuenire velimus, eius differentiale primum Pdx denuo differentiari oportet; quod cum ſit productum ex duabus quantitatibus variabilibus P & dx, quarum illius differentiale ſit $dP = pdx$, huius vero dx differentiale

ddx, per regulam de factoribus datam erit differentiale ſecundum $ddV = Pddx + pdx^2$. Deinde ſi ponatur $dp = qdx$, cum differentiale ipſius dx^2 ſit $2dxddx$, erit iterum differentiando

$$d^3V = Pd^3x + dPddx + 2pdxddx + dpdx^2,$$

iam ob $dP = pdx$ & $dp = qdx$; erit

$$d^3V = Pd^3x + 3pdxddx + qdx^3,$$

ſimilique modo vlteriora differentialia inuenientur.

224. Applicemus haec ad poteſtates ipſius x, quarum ſingula differentialia inueſtigemus, ſi dx non ponatur conſtans:

I. Sit igitur $V = x$; erit $dV = dx$; $d^2V = d^2x$; $d^3V = d^3x$; $d^4V = d^4x$;

&c.

II. Sit $V = x^2$; erit $dV = 2xdx$; &

$$ddV = 2xddx + 2dx^2$$
$$d^3V = 2xd^3x + 6dxddx$$
$$d^4V = 2xd^4x + 8dxd^3x + 6ddx^2$$
$$d^5V = 2xd^5x + 10dxd^4x + 20ddxd^3x$$

&c.

III. Si in genere fuerit $V = x^n$; erit

$$dV = nx^{n-1}dx$$
$$ddV = nx^{n-1}ddx + n(n-1)x^{n-2}dx^2$$
$$d^3V = nx^{n-1}d^3x + 3n(n-1)x^{n-2}dxddx$$
$$+ n(n-1)(n-2)x^{n-3}dx^3$$

$$d^4V = nx^{n-1}d^4x + 4n(n-1)x^{n-2}dx\,d^3x$$
$$+ 3n(n-1)x^{n-2}ddx^2$$
$$+ 6n(n-1)(n-2)x^{n-3}dx^2ddx$$
$$+ n(n-1)(n-2)(n-3)x^{n-4}dx^4$$
$$\&c.$$

Si igitur fuerit dx conſtans, ac propterea

$$ddx = 0, \quad d^3x = 0, \quad d^4x = 0, \quad \&c.$$

orientur eadem differentialia, quae iam ſupra pro hac hypotheſi ſunt inuenta.

245. Quoniam igitur differentialia cuiusque ordinis ipſius x eadem lege differentiantur, qua quantitates finitae, expreſſiones quaecunque, in quibus praeter quantitatem finitam eius differentialia occurrunt, ſecundum praecepta ſupra data differentiari poterunt. Quam operationem, cum nonnunquam occurrat, hic aliquot exemplis illuſtrabimus.

I. Si fuerit $V = \frac{x\,ddx}{dx^2}$, differentiando prodibit

$$dV = \frac{x\,d^3x}{dx^2} + \frac{ddx}{dx} - \frac{2x\,ddx^2}{dx^3}.$$

II. Si fuerit $V = \frac{x}{dx}$; erit $dV = 1 - \frac{x\,ddx}{dx^2}$, vbi nihil impedit, quod pro V quantitatem infinite magnam poſuimus.

III. Si fuerit $V = xxl\frac{ddx}{dx^2}$, quia transmutatur

V in

$$V \text{ in } xxlddx - 2xxldx;$$

erit ſecundum regulas conſuetas differentiando:

$$dV = 2xdxlddx + \frac{xxd^3x}{ddx} - 4xdxldx - \frac{2xxddx}{dx}.$$

Simili autem modo differentialia altiora ipſius V reperientur.

246. Si expreſſio propoſita duas variabiles inuolvat, nempe x & y, vel vnius differentiale ponitur conſtans vel neutrius; arbitrarium enim eſt alterutrius differentiale conſtans aſſumi, quia ab arbitrio noſtro pendet, quemadmodum vnius valores ſucceſſiuos creſcere ſtatuere velimus. Neque vero vtriusque variabilis differentialia ſimul ſtatui poſſunt conſtantia, hoc ipſo enim relatio inter variabiles x & y aſſumeretur, quae tamen vel nulla eſt, vel incognita ponitur. Si enim, dum x aequabiliter creſcere ponimus, y quoque aequalia incrementa capere ſtatueretur, tum eo ipſo indicaretur fore $y = ax + b$; ſicque y ab x penderet, quod tamen aſſumere non licet. Hancobrem vel vnius tantum variabilis differentiale conſtans aſſumi poteſt vel nullum. Quodſi autem differentiationes abſoluere nouerimus nullo differentiali aſſumto conſtante, ſimul quoque differentialia conſtabunt, ſi alterutrum differentiale ponatur conſtans: tantum enim opus eſt, vt ſi dx conſtans ſtatuatur, vbique termini continentes ddx, d^3x, d^4x, &c. deleantur.

247. Denotet ergo V functionem quamcunque finitam ipſarum x & y, ſitque $dV = Pdx + Qdy$. Ad diffe-

differentiale ipſius V ſecundum inueniendum aſſumamus vtrumque differentiale dx & dy variabile, & cum P & Q ſint functiones ipſarum x & y ſtatuamus:

$$dP = p\,dx + r\,dy$$
$$dQ = r\,dx + q\,dy$$

ſupra enim vidimus eſſe $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right) = r.$
His poſitis differentietur $dV = P dx + Q dy$, & repe- rietur:

$$ddV = P ddx + p dx^2 + 2 r dx dy + Q ddy + q dy^2.$$

Si igitur differentiale dx ſtatuatur conſtans, erit

$$ddV = p dx^2 + 2 r dx dy + Q ddy + q dy^2,$$

ſin autem differentiale dy ſtatueretur conſtans, foret

$$ddV = P ddx + p dx^2 + 2 r dx dy + q dy^2.$$

248. Si igitur functio quaecunque ipſarum x & y bis differentietur, nullo differentiali poſito conſtante, eius differentiale ſecundum ſemper huiusmodi formam habebit:

$$ddV = P ddx + Q ddy + R dx^2 + S dy^2 + T dx dy$$

pendebunt autem quantitates P, Q, R, S, & T ita a ſe inuicem, vt ſit ſignandi modo Capite praecedente adhibito:

$$\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right); \; R = \left(\frac{dP}{dx}\right); \; S = \left(\frac{dQ}{dy}\right);$$
$$\& \; T = 2\left(\frac{dQ}{dx}\right) = 2\left(\frac{dP}{dy}\right),$$

quarum conditionum ſi vel vnica deſit, certo affirmare po- teri-

terimus, formulam propoſitam nullius functionis eſſe differentiale ſecundum. Statim ergo dignoſci poterit, vtrum huiusmodi formula ſit cuiuspiam quantitatis differentiale ſecundum an minus?

242. Simili modo differentialia tertia ac ſequentia inuenientur, quod in exemplo particulari oſtendiſſe expediet, quam formulas generales adhibendo.

Sit igitur $V = xy$;

Erit $dV = y\,dx + x\,dy$

$$ddV = y\,ddx + 2\,dx\,dy$$

$$d^3V = y\,d^3x + 3\,dy\,ddx + x\,d^3y$$

$$d^4V = y\,d^4x + 4\,dy\,d^3x + 6\,ddx\,ddy + 4\,dx\,d^3y + x\,d^4y$$

&c.

in quo exemplo coefficientes numerici legem poteſtatum binomii ſequuntur, indeque quousque libuerit continuari poſſunt.

At ſi fuerit $V = \frac{y}{x}$;

Erit $dV = \frac{dy}{x} - \frac{y\,dx}{xx}$

$$ddV = \frac{ddy}{x} - \frac{2\,dx\,dy}{xx} + \frac{2\,y\,dx^2}{x^3} - \frac{y\,ddx}{x^2}$$

$$d^3V = \frac{d^3y}{x} - \frac{3\,dx\,ddy}{xx} + \frac{6\,dx^2\,dy}{x^3} - \frac{3\,dy\,ddx}{x^2}$$

$$+ \frac{6\,y\,dx\,ddx}{x^3} - \frac{6\,y\,dx^3}{x^4} - \frac{y\,d^3x}{x^2}$$

&c.

in quo exemplo progreſſio differentialium non tam facile patet quam in praecedente.

250. Neque vero tantum haec differentiandi methodus ad functiones finitas adſtringitur, ſed etiam eodem negotio cuiusuis expreſſionis, quae iam differentialia in ſe continet, differentiale inueniri poteſt, ſiue vnum quoddam differentiale aſſumitur conſtans ſiue minus. Cum enim ſingula differentialia aeque & eadem lege differentientur ac quantitates finitae, regulae in praecedentibus capitibus traditae, etiam hic valent atque obſeruari debent. Denotet igitur V eam expreſſionem, quam differentiari oportet, ſiue ſit finita, ſiue infinite magna ſiue infinite parua; atque ratio differentiationis ex his exemplis perſpicietur:

I. Sit $V = \sqrt{(dx^2 + dy^2)}$;

Erit $dV = \dfrac{dx\,ddx + dy\,ddy}{\sqrt{(dx^2 + dy^2)}}$.

II. Sit $V = \dfrac{y\,dx}{dy}$;

Erit $dV = dx + \dfrac{y\,ddx}{dy} - \dfrac{y\,dx\,ddy}{dy^2}$.

III. Sit $V = \dfrac{(dx^2 + dy^2)^{\frac{3}{2}}}{dx\,ddy - dy\,ddx}$;

Erit $dV = \dfrac{(3dx\,ddx + 3dy\,ddy)\sqrt{(dx^2 + dy^2)}}{dx\,ddy - dy\,ddx}$

$- \dfrac{(dx^2 + dy^2)^{\frac{3}{2}}(dx\,d^3y - dy\,d^3x)}{(dx\,ddy - dy\,ddx)^2}$.

quae

quae differentialia cum ſint generaliſſime ſumta, nullo differentiali pro conſtante habito, hinc facile ea differentialia deriuari poterunt, quae oriuntur, ſi vel dx vel dy ſtatuatur conſtans.

251. Quia nullo differentiali conſtante aſſumto, nulla etiam lex, ſecundum quam ſucceſſiui variabilium valores progrediantur, praeſcribitur, differentialia ſecunda & ſequentium ordinum non erunt determinata, neque quicquam certi ſignificabunt. Hinc formula, in qua differentialia ſecunda atque altiora continentur, nullum determinatum habebit valorem, niſi quodpiam differentiale conſtans ſit aſſumtum; ſed eius ſignificatio erit vaga, atque variabitur, prouti aliud atque aliud differentiale fuerit conſtans poſitum. Interim tamen dantur quoque eiusmodi expreſſiones differentialia ſecunda continentes, quae, etiamſi nullum differentiale poſitum ſit conſtans, tamen ſignificatum determinatum complectuntur, qui perpetuo idem maneat, quodcunque differentiale conſtans ſtatuatur. Huiusmodi autem formularum naturam infra diligentius ſcrutabimur, modumque trademus eas ab aliis, quae valores determinatos non includunt, dignoſcendi.

252. Quo haec ratio formularum, in quibus differentialia ſecunda vel altiora inſunt, facilius perſpiciatur, contemplemur primum formulas vnicam variabilem continentes, atque facile patet, ſi in quapiam formula inſit eius variabilis x differentiale ſecundum ddx, nullumque differentiale conſtans ſtatuatur, formulam nullum valo-

 rem

rem fixum habere poſſe. Si enim ſtatuatur differentiale ipſius x conſtans, fiet $ddx = 0$; ſin autem ipſius xx differentiale $2xdx$ ſeu xdx conſtans ponatur, cum ipſius xdx differentiale $xddx + dx^2$ ſit $= 0$, fiet $ddx = -\frac{dx^2}{x}$. Verum ſi poteſtatis cuiuscunque x^n differentiale $nx^{n-1}dx$ ſeu $x^{n-1}dx$ debeat eſſe conſtans; erit eius differentiale ſecundum $x^{n-1}ddx + (n-1)x^{n-2}dx^2 = 0$, ideoque $ddx = -\frac{(n-1)dx^2}{x}$. Alii valores pro ddx prodibunt, ſi aliarum ipſius x functionum differentialia conſtantia ponantur. Manifeſtum autem eſt, formulam, in qua ddx occurrat, diuerſiſſimos induere valores, prout loco ddx ſcribatur vel 0 vel $-\frac{dx^2}{x}$ vel $-\frac{(n-1)dx^2}{x}$ vel alia huiusmodi expreſſio. Scilicet ſi proponatur formula $\frac{xxddx}{dx^2}$, quae ob ddx & dx^2 infinite parua homogenea. finitum valorem habere deberet; ea poſito dx conſtante abit in 0, ſi ſit $d.x^2$ conſtans, ea abit in $-x$; ſi ſit $d.x^3$ conſtans, ea abit in $-2x$; ſi $d.x^4$ ſit conſtans, ea abit in $-3x$, & ita porro. Neque ergo determinatum valorem habere poteſt, niſi definiatur, cuiusmodi differentiale conſtans ſit aſſumtum.

253. Iſta inconſtantia ſignificationis ſimili ratione oſtenditur, ſi differentiale tertium in quapiam formula inſit. Conſideremus hanc formulam $\frac{x^3 d^3 x}{dxddx}$, quae pariter

riter finitum valorem prae ſe fert. Si differentiale dx ſit conſtans, abit ea in $\frac{0}{0}$, cuius valor mox patebit. Sit $d.x^2$ conſtans, erit $ddx = -\frac{dx^2}{x}$; & denuo differentiando $d^3x = -\frac{2\,dx\,ddx}{x} + \frac{dx^3}{x^2} = \frac{3dx^3}{x^2}$, ob $ddx = -\frac{dx^2}{x}$: hoc ergo caſu formula propoſita $\frac{x^3d^3x}{dx\,ddx}$ abit in $-3x^2$. At ſi fuerit $d.x^n$ conſtans, erit $ddx = -\frac{(n-1)dx^2}{x}$, hincque

$$d^3x = -\frac{2(n-1)dx\,ddx}{x} + \frac{(n-1)dx^3}{x^2} = \frac{2(n-1)^2dx^3}{xx}$$

$$+ \frac{(n-1)dx^3}{xx} = \frac{(2n-1)(n-1)dx^3}{xx}.$$

Hoc ergo caſu erit

$$\frac{d^3x}{ddx} = -\frac{(2n-1)dx}{x}, \quad \& \quad \frac{x^3d^3x}{dx\,ddx} = -(2n-1)x^2,$$

vnde patet ſi ſit $n=1$, ſeu dx conſtans, valorem formulae fore $=-x^2$. Ex quo manifeſtum eſt, ſi in quapiam formula differentialia tertia vel altiora occurrant, neque ſimul indicetur, cuiusmodi differentiale aſſumtum ſit conſtans, eam formulam nullum certum valorem habere, atque adeo nihil prorſus ſignificare; quamobrem tales expreſſiones in calculo occurrere non poſſunt.

254. Simili modo ſi formula contineat duas pluresue variabiles, in eaque occurrant differentialia ſecundi altiorisue gradus, intelligetur valorem determinatum locum habere non poſſe, niſi differentiale quodpiam conſtans ſtatuatur, iis tantum exceptis caſibus, quos mox perpendemus. Quum primum enim ddx in quapiam formula ineſt, quoniam pro variis differentialibus, quae conſtantia ponuntur, valor ipſius ddx perpetuo variatur, fieri nequit, vt formula ſtatum obtineat valorem; hocque idem valet de quouis differentiali altiori ipſius x, atque etiam de differentialibus reliquarum variabilium ſecundis & altioribus. Sin autem duarum pluriumue variabilium differentialia ſecunda inſint, fieri poteſt, vt inconſtantia ab vno oriúnda per inconſtantiam reliquarum deſtruatur; hincque naſcitur ille caſus, cuius meminimus, quo formula huiusmodi differentialia ſecunda duarum pluriumue variabilium inuoluens valorem definitum habere poteſt, non obſtante quod nullum differentiale conſtans ſit poſitum.

255. Haec igitur formula $\frac{yddx + xddy}{dxdy}$, ſtatam atque fixam ſignificationem habere nequit, niſi quodpiam differentiale primum conſtans ſtatuatur. Nam ſi dx conſtans ponatur habebitur $\frac{xddy}{dxdy}$; ſin autem dy conſtans ponatur, habebitur $\frac{yddx}{dxdy}$; manifeſtum autem eſt has formulas non neceſſario inter ſe eſſe aequales. Si enim ne-

neceſſario eſſent aequales, tales manere deberent, quaecunque functio ipſius x loco y ſubſtitueretur. Ponamus tantum eſſe $y = xx$, & cum poſito dx conſtante, ſit $ddy = 2dx^2$, formula $\frac{xddy}{dxdy}$ abibit in 1, ſin autem dy ſeu $2xdx$ ponatur conſtans, fiet
$ddy = 2xddx + 2dx^2 = 0$, ideoque $ddx = -\frac{dx^2}{x}$,
vnde formula $\frac{yddx}{dxdy}$ abit in $-\frac{1}{2}$. Cum igitur in vnico caſu reperiatur diſcrepantia, multo minus in genere erit $\frac{xddy}{dxdy}$ poſito dx conſtante aequalis $\frac{yddx}{dxdy}$ poſito dy conſtante. Deinde quia formula $\frac{yddx + xddy}{dxdy}$ ſibi non conſtat, dummodo vel dx vel dy conſtans ponatur, multo minus ſibi conſtabit, ſi functionis cuiusuis vel ipſius x vel ipſius y vel vtriusque differentiale conſtans ponatur.

256. Hinc apparet huiusmodi formulam ſtatum valorem habere non poſſe, niſi ita ſit comparata, vt poſtquam loco variabilium y, & z, quae praeter x inſunt, functiones quaecunque ipſius x fuerint ſubſtitutae, differentialia ſecunda & altiora ipſius x, nempe ddx, d^3x, &c. penitus ex calculo excedant. Si enim poſt talem ſubſtitutionem quamcunque in formula adhuc relinqueretur ddx, vel d^3x, vel d^4x, &c. quia haec differentialia, prout alia aliaque conſtantia aſſumuntur, ſignificationem ſuam variant, valor quoque ipſius formulae erit vagus. Sic comparata eſt

for-

formula ante propoſita $\frac{yddx + xddy}{dxdy}$, quae ſi ſtatum haberet valorem, quicquid y ſignificet, ſtatum quoque habere deberet valorem, ſi y denotaret functionem quampiam ipſius x. At ſi tantum ponamus $y = x$, formula abit in $\frac{2xddx}{dx^2}$, quae vtique ob ddx in ea contentum eſt vaga, atque alios aliosque valores induit, prouti alia atque alia differentialia conſtantia ponuntur, vti ex §. 252 ſatis eſt manifeſtum.

257. Dubium autem hic ſubnaſcetur, vtrum dentur tales formulae duo pluraue differentialia ſecundi altiorisue gradus continentia, quae hac proprietate gaudeant, vt ſi loco reliquarum variabilium quaecunque functiones vnius ſubſtituantur, differentialia ſecundi gradus prorſus ſe deſtruant. Huic dubio primum ita occurramus, vt huiusmodi formulam proponamus, quae iſta proprietate ſit praedita, quo per explorationem vis quaeſtionis melius percipiatur. Dico igitur hanc formulam $\frac{dyddx - dxddy}{dx^3}$ memoratam proprietatem poſſidere: quaecunque enim functio ipſius x loco y ſubſtituatur, ſemper differentialia ſecundi gradus penitus euaneſcent; quam proprietatem ſequentibus exemplis comprobemus.

I. Sit $y = x^2$; erit $dy = 2xdx$,

& $ddy = 2xddx + 2dx^2$,

qui

qui valores in formula $\frac{dy\,ddx - dx\,ddy}{dx^3}$ ſubſtituti dabunt,

$$\frac{2x\,dx\,ddx - 2x\,dx\,ddx - 2dx^3}{dx^3} = -2.$$

II. Sit $y = x^n$; erit $dy = nx^{n-1}dx$, &

$$ddy = nx^{n-1}ddx + n(n-1)x^{n-2}dx^2,$$

qui valores ſubſtituti formulam $\frac{dy\,ddx - dx\,ddy}{dx^3}$ transmutabunt in hanc

$$\frac{nx^{n-1}dx\,ddx - nx^{n-1}dx\,ddx - n(n-1)x^{n-2}dx^3}{dx^3} = -n(n-)x^{n-2}.$$

III. Sit $y = -\sqrt{(1-xx)}$; erit $dy = \frac{x\,dx}{\sqrt{(1-xx)}}$, &

$$ddy = \frac{x\,ddx}{\sqrt{(1-xx)}} + \frac{dx^2}{(1-xx)^{\frac{3}{2}}};$$

atque formula $\frac{dy\,ddx - dx\,ddy}{dx^3}$, abit in

$$\frac{x\,ddx}{dx^2\sqrt{(1-xx)}} - \frac{x\,ddx}{dx^2\sqrt{(1-xx)}} - \frac{1}{(1-xx)^{\frac{3}{2}}} = \frac{-1}{(1-xx)^{\frac{3}{2}}}.$$

In his igitur omnibus exemplis differentialia ſecunda *ddx* ſe mutuo tollunt, hocque ita eueniet, quaecunque aliae functiones loco *y* ſubſtituantur.

258. Cum iſta exempla iam probauerint veritatem noſtrae propoſitionis, quod formula $\frac{dyddx - dxddy}{dx^3}$ fixum habeat valorem, etiamſi nullum differentiale conſtans ſit aſſumtum, demonſtrationem eo facilius adornare poterimus. Sit y functio quaecunque ipſius x, eiusque differentiale dy huiusmodi erit, vt ſit $dy = pdx$, atque p erit functio quaepiam ipſius x, eiusque differentiale propterea huiusmodi formam habebit $dp = qdx$, eritque q iterum functio ipſius x. Cum igitur ſit $dy = pdx$, erit differentiando $ddy = pddx + qdx^2$, & $dyddx - dxddy = pdxddx - pdxddx - qdx^3 = -qdx^3$; in qua expreſſione cum nullum inſit differentiale ſecundum, habebit ea valorem fixum, atque $\frac{dyddx - dxddy}{dx^3}$ erit $= -q$. Quomodocunque igitur y pendeat ab x, differentialia ſecunda in hac formula ſemper ſe mutuo tollent, hancque ob cauſam eius valor, qui alioquin eſſet vagus, fiet ſtatus ac fixus.

259. Quanquam hic poſuimus y eſſe functionem ipſius x, tamen veritas aeque ſubſiſtit, ſi y ab x prorſus non pendeat, vti aſſumſimus. Dum enim pro y functionem quamcunque ſubſtituimus, neque qualis ſit determinauimus, nullam pendentiam ab x ipſi y tribuimus. Interim tamen ſine functionis mentione demonſtratio formari poteſt; quaecunque enim y ſit quantitas ſiue pendens ab x ſiue non pendens, eius differentiale dy homoge-

geneum erit cum dx, ſicque $\frac{dy}{dx}$ quantitatem finitam denotabit, cuius differentiale, quod capit, dum x in $x+dx$ & y in $y+dy$ abit, erit fixum, neque a differentialium ſecundorum lege pendebit. Sit igitur $\frac{dy}{dx}=p$; erit $dy=pdx$, & $ddy=pddx+dpdx$, vnde fit

$$dxddy-dyddx=dpdx^2,$$

cuius valor non eſt vagus, quia tantum differentialia prima continet; ac propterea idem manet, ſiue quodpiam differentiale conſtans accipiatur, qualecunque id demum ſit, ſiue nullum differentiale poſitum ſit conſtans.

260. Quia igitur $dyddx-dxddy$ non obſtantibus differentialibus ſecundis, quae potentia ſe mutuo deſtruere cenſeri poſſunt, ſignificationem habet fixam; expreſſio quaecunque, in qua nulla alia differentialia ſecunda praeter formulam $dyddx-dxddy$ inſunt, pariter ſignificationem habebit fixam. Seu ſi ponatur $dyddx-dxddy=\omega$, atque V fuerit quantitas ex x, y, earum differentialibus primis dx, dy atque ex ω vtcunque compoſita, ea valorem habebit fixum. Cum enim in differentialibus primis dx & dy nulla ratio habeatur eius legis arbitrariae, qua valores ſucceſſiui ipſius x creſcere ponuntur, in ω differentialia ſecunda ſe mutuo tollunt, etiam ipſa quantitas V non erit vaga ſed fixa. Sic iſta expreſſio $\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxddy-dyddx}$ valorem obtinet fixum,

 quam-

quamuis ea differentialibus ſecundis inquinata videatur, atque inſuper, quia numerator eſt homogeneus denominatori, valorem obtinet finitum, niſi is caſu vel infinite magnus vel infinite paruus euadat.

261. Quemadmodum formula $dxddy - dyddx$ valorem fixum habere oſtenſa eſt, ita quoque ſi tertia variabilis z accedat, hae formulae $dxddz - dzddx$ & $dyddz - dzddy$ valores fixos habebunt. Hinc expreſſiones, quas tres variabiles x, y, & z inuoluunt, ſi in eis nulla alia differentialia ſecunda occurrant, praeter haec aſſignata, tum perinde erunt fixae, ac ſi nulla plane differentialia ſecunda ineſſent. Ita haec expreſſio:

$$\frac{(dx^2 + dy^2 + dz^2)^{\frac{3}{2}}}{(dx + dz)\,ddy - (dy + dz)\,ddx + (dx - dy)\,ddz}$$

non obſtantibus differentialibus ſecundis, fixa gaudet ſignificatione. Similique modo formulae exhiberi poſſunt, plures variabiles continentes, in quibus differentialia ſecunda non impediunt, quominus earum ſignificatio ſit fixa.

262. Exceptis ergo huius generis formulis, quae differentialia ſecunda complectuntur, reliquae omnes ſignificationes habebunt vagas, neque propterea in calculo locum habere poſſunt, niſi quodpiam differentiale primum definiatur, quod conſtans ſit aſſumtum. Statim vero atque differentiale quodpiam primum conſtans aſſumitur, omnes expreſſiones quotcunque variabiles continе-

tineant, & cuiuscunque ordinis differentialia poſt primum in eas ingrediuntur, fixas obtinebunt ſignificationes, neque amplius ex calculo excluduntur. Si enim verbi gratia dx aſſumtum ſit conſtans, ipſius x differentialia ſecunda & ſequentia euaneſcunt; & quaecunque functiones ipſius x loco reliquarum variabilium y, z, &c. ſubſtituantur, earum differentialia ſecunda per dx^2, tertia per dx^3, &c. determinabuntur, ſicque inconſtantia a differentialibus ſecundis oriunda tollitur. Idem euenit, ſi alius variabilis ſeu functionis cuiuscunque differentiale primum conſtans ponatur.

263. Ex his igitur ſequitur differentialia ſecunda & altiorum ordinum reuera nunquam in calculum ingredi, atque ob vagam ſignificationem prorſus ad Analyſin eſſe inepta. Quando enim differentialia ſecunda adeſſe videntur, vel differentiale quodpiam primum conſtans aſſumitur, vel nullum. Priori caſu differentialia ſecunda prorſus ex calculo euaneſcunt, dum per differentialia prima determinantur. Poſteriori caſu autem niſi ſe mutuo deſtruant, ſignificatio erit vaga, & propterea in Analyſi locum nullum inueniunt; ſin autem ſe mutuo deſtruunt, tantum apparenter adſunt, & reuera ſolae quantitates finitae cum ſuis differentialibus primis adeſſe cenſendae ſunt. Quoniam tamen ſaepiſſime apparenter tantum in calculo vſurpantur, neceſſe fuit, vt methodus eas tractandi exponeretur. Modum autem mox oſtendemus, cuius ope differentialia ſecunda & altiora ſemper exterminari queant.

264. Si expreſſio vnicam contineat variabilem x, eiusque differentialia altiora ddx, d^3x, d^4x, &c. in ea occurrant, ea ſignificatum fixum habere nequit, niſi quodpiam differentiale primum conſtans ſit poſitum. Sit igitur t illa quantitas variabilis, cuius differentiale dt ſit conſtans poſitum, ita vt ſit $ddt = 0$, $d^3t = 0$, $d^4t = 0$, &c. Ponatur $dx = pdt$; eritque p quantitas finita, cuius differentiale vaga ſignificatione differentialium ſecundorum non afficietur, hincque etiam $\frac{dp}{dt}$ erit quantitas finita. Sit $dp = qdt$, ſimilique modo vlterius $dq = rdt$; $dr = sdt$; &c. erunt q, r, s, &c. quantitates finitae fixos ſignificatus habentes. Cum igitur ſit $dx = pdt$; erit

$$ddx = dpdt = qdt^2 \; ; \; d^3x = dqdt^2 = rdt^3 \; ;$$
$$d^4x = drdt^3 = sdt^4 \; ; \; \&c.$$

qui valores ſi loco ddx, d^3x, d^4x, &c. ſubſtituantur, tota expreſſio meras quantitates finitas cum differentiali primo dt continebit, ideoque non amplius vagam ſignificationem habebit.

265. Si x ſit functio ipſius t, poterit hoc modo quantitas x prorſus eliminari, ita vt ſola quantitas t cum ſuo differentiali dt in expreſſione remaneat: ſin autem t ſit functio ipſius x, viciſſim quoque x erit ipſius t functio. Interim tamen ipſa quantitas x cum ſuo differentiali primo dx, in calculo retineri poteſt, dummodo poſt ſubſtitutiones ante factas vbique loco t & dt earum valores per x & dx expreſſi reſtituantur. Quod quo pla-

planius fiat, ponamus t eſſe $= x^n$, ita vt differentiale primum ipſius x^n conſtans ſit poſitum. Quia igitur eſt

$$dt = nx^{n-1}dx \;;\quad \text{erit}\quad p = \frac{1}{nx^{n-1}}\;;\quad \&$$

$$dp = \frac{-(n-1)dx}{nx^n} = qdt = nqx^{n-1}dx\;;$$

$$\text{vnde fit}\quad q = \frac{-(n-1)}{nnx^{2n-1}}\;;\quad \&$$

$$dq = \frac{(n-1)(2n-1)dx}{nnx^{2n}} = rdt = nrx^{n-1}dx.$$

Hinc porro fit

$$r = \frac{(n-1)(2n-1)}{n^3x^{3n-1}}\;;\ \&\ s = -\frac{(n-1)(2n-1)(3n-1)}{n^4x^{4n-1}}.$$

Quare ſi differentiale ipſius x^n ponatur conſtans, erit:

$$ddx = -\frac{(n-1)dx^2}{x}$$

$$d^3x = \frac{(n-1)(2n-1)dx^3}{xx}$$

$$d^4x = -\frac{(n-1)(2n-1(3n-1)dx^4}{x^3}$$

&c.

266. Si expreſſio duas contineat variabiles x & y, earumque vnius x differentiale poſitum ſit conſtans, ob $ddx = 0$, alia differentialia ſecunda & altiora non inerunt, praeter ddy, d^3y, &c. Haec autem eodem modo, quo ante vſi ſumus, tolli poterunt ponendo

$dy =$

$dy = pdx$; $dp = qdx$; $dq = rdx$; $dr = sdx$; &c.
fiet enim

$ddy = qdx^2$; $d^3y = rdx^3$; $d^4y = sdx^4$ &c.

quibus ſubſtitutis expreſſio orietur, quae praeter quantitates finitas x, y, p, q, r, s, &c. nonniſi differentiale primum dx continebit. Sic ſi propoſita fuerit haec expreſſio

$$\frac{ydx^4 + xdyd^3y + xd^4y}{(xx + yy)\,ddy},$$

in qua dx eſt conſtans aſſumtum; ponatur

$dy = pdx$; $dp = qdx$; $dq = rdx$; & $dr = sdx$;

quibus valoribus ſubſtitutis expreſſio propoſita transmutabitur in hanc: $\frac{(y + xpr + xs)\,dx^2}{(xx + yy)\,q}$, quae nulla amplius differentialia ſecunda altioraue continet.

267. Simili modo differentialia ſecunda & altiora tollentur, ſi dy fuerit conſtans aſſumtum. Verum ſi aliud differentiale primum quodcunque dt ſtatuatur conſtans, tum primum modo ante indicato differentialia ipſius x altiora ex calculo tollantur, ponendo

$dx = pdt$; $dp = qdt$; $dq = rdt$; $dr = sdt$; &c.
vnde fit

$ddx = qdt^2$; $d^3x = rdt^3$; $d^4x = sdt^4$; &c.

Deinde ſimili modo differentialia altiora ipſius y ponendo

$dy =$

$dy = Pdt$; $dP = Qdt$; $dQ = Rdt$; $dR = Sdt$; &c.
vnde fiet

$ddy = Qdt^2$; $d^3y = Rdt^3$; $d^4y = Sdt^4$; &c.

quibus ſubſtitutis obtinebitur expreſſio, quaè praeter quantitates finitas, x, p, q, r, s, &c. y, P, Q, R, S, &c. ſolum differentiale dt complectetur, neque propterea vagam habebit ſignificationem.

268. Si differentiale primum, quod conſtans ponitur, vel ab x vel ab y vel ab vtroque ſimul pendet, tum non opus eſt, vt duplex quantitatum finitarum p, q, r, &c. ſeries introducatur. Si enim dt ab x tantum pendet, tum litterae p, q, r, &c. fient functiones ipſius x, ſolaeque litterae P, Q, R, &c. ingrediuntur; idemque euenit, ſi differentiale conſtans dt ab y tantum pendeat. At ſi dt ab vtraque pendeat, operatio aliquantum immutari debet. Ponamus exempli gratia hoc differentiale ydx conſtans eſſe aſſumtum, eritque $yddx + dxdy = 0$; vnde fit $ddx = -\frac{dxdy}{y}$. Sit nunc $dy = pdx$; $dp = qdx$; $dq = rdx$ &c. eritque $ddx = -\frac{pdx^2}{y}$; vlteriusque differentiando

$$d^3x = -\frac{qdx^3}{y} + \frac{ppdx^3}{yy} - \frac{2pdxddx}{y},$$

ſubſtituatur hic loco ddx eius valor $-\frac{pdx^2}{y}$; fiet

$$d^3x = -\frac{qdx^3}{y} + \frac{3ppdx^3}{yy}; \text{ porroque}$$

$$d^4x = -\frac{rdx^4}{y} + \frac{pqdx^4}{yy} + \frac{6pqdx^4}{yy} - \frac{6p^3dx^4}{y^3}$$
$$+ \left(\frac{3pp}{yy} - \frac{q}{y}\right) 3dx^2ddx;$$

& pro ddx ſubſtituto valore $-\frac{pdx^2}{y}$ emerget

$$d^4x = \left(\frac{-r}{y} + \frac{10pq}{yy} - \frac{15p^3}{y^3}\right) dx^4 \quad \text{\&c.}$$

Deinde cum ſit $dy = pdx$; erit

$$ddy = qdx^2 + pddx = \left(q - \frac{pp}{q}\right) dx^2;$$

& continuo pro ddx valore $-\frac{pdx^2}{y}$ ſubſtituendo fiet

$$d^3y = \left(r - \frac{4pq}{y} + \frac{3p^3}{yy}\right) dx^3, \quad \&$$

$$d^4y = \left(s - \frac{7pr}{y} - \frac{4qq}{y} + \frac{25ppq}{yy} - \frac{15p^4}{y^3}\right) dx^4 \quad \text{\&c.}$$

qui valores loco differentialium altiorum ipſarum x & y ſubſtituti mutabunt expreſſionem propoſitam in eiusmodi formam, quae nulla amplius differentialia altiora continebit, hincque conſideratione cuiuspiam differentialis conſtantis exuetur. Facta enim hac transformatione, quia differentialia ſecunda non inſunt, nequidem opus eſt, vt quale differentiale ſumtum ſit conſtans, commemoretur.

269. Saepiſſime autem in calculo ad lineas curuas applicato euenire ſolet, vt hoc differentiale primum $\sqrt{(dx^2 + dy^2)}$ conſtans aſſumatur: quare quemadmodum hoc caſu differentialia ſecunda & altiora eliminari de-

debeant, oſtendamus. Sic enim ſimul via patebit ad idem negotium abſoluendum, ſi aliud quodcunque differentiale aſſumendum ſit conſtans. Ponatur iterum

$dy = pdx$; $dp = qdx$; $dq = rdx$; $dr = sdx$; &c.

atque differentiale $\surd(dx^2 + dy^2)$ induet hanc formam $dx\surd(1+pp)$, quae cum ſit conſtans fiet

$$ddx\surd(1+pp) + \frac{pqdx^2}{\surd(1+pp)} = 0,$$

ideoque $$ddx = -\frac{pqdx^2}{1+pp};$$

vnde iam ipſius ddx valor habebitur: hinc porro erit

$$d^3x = -\frac{prdx^3}{1+pp} - \frac{qqdx^3}{1+pp} + \frac{2ppqqdx^3}{(1+pp)^2} - \frac{2pqdxddx}{1+pp}$$

$$= -\frac{prdx^3}{1+pp} - \frac{qqdx^3}{1+pp} + \frac{4ppqqdx^3}{(1+pp)^2}$$

$$= -\frac{prdx^3}{1+pp} + \frac{(3pp-1)qqdx^3}{(1+pp)^2}.$$

Deinde fiet

$$d^4x = -\frac{psdx^4}{1+pp} + \frac{(10pp-3)qrdx^4}{(1+pp)^2} - \frac{(15pp-13)pq^3dx^4}{(1+pp)^3}.$$

Quia autem aſſumſimus $dy = pdx$, fiet differentiando

$$ddy = qdx^2 + pddx = qdx^2 - \frac{ppqdx^2}{1+pp} = \frac{qdx^2}{1+pp},$$

$$d^3y = \frac{rdx^3}{1+pp} - \frac{2pqqdx^3}{(1+pp)^2} + \frac{2qdxddx}{1+pp},$$ ideoque

$$d^3y = \frac{rdx^3}{1+pp} - \frac{4pqqdx^3}{(1+pp)^2};$$

 por-

porroque differentiando:

$$d^4y = \frac{s\,dx^4}{1+pp} - \frac{13pqr\,dx^4}{(1+pp)^2} + \frac{4(6pp-1)q^3\,dx^4}{(1+pp)^3}.$$

Omnia ergo differentialia altiora vtriusque variabilis x & y per quantitates finitas & poteſtates ipſius dx exprimentur, atque poſt has ſubſtitutiones factas reſultabit expreſſio a differentialibus ſecundis prorſus libera.

270. Expoſito igitur modo differentialia ſecunda & altiora exuendi, conueniet hoc negotium aliquot exemplis illuſtrari.

I. Sit propoſita haec expreſſio $\frac{x\,ddy}{dx^2}$, in qua dx poſitum eſt conſtans. Poſito ergo $dy = p\,dx$, & $dp = q\,dx$, ob $ddy = q\,dx^2$, expreſſio propoſita abit in hanc finitam xq.

II. Sit propoſita haec expreſſio $\frac{dx^2 + dy^2}{ddx}$, in qua poſitum ſit dy conſtans. Ponatur $dx = p\,dy$; $dp = q\,dy$, ob $ddx = q\,dy^2$, orietur $\frac{1+pp}{q}$. Sin autem vt ante ſtatuere velimus $dy = p\,dx$, $dp = q\,dx$; ob dy conſtans erit $0 = p\,ddx + dp\,dx$ & $ddx = -\frac{q\,dx^2}{p}$; vnde expreſſio propoſita transibit in $\frac{-p(1+pp)}{q}$.

III. Sit propoſia haec expreſſio $\frac{y\,ddx - x\,ddy}{dx\,dy}$ in qua $y\,dx$ poſitum ſit conſtans. Ponatur $dy = p\,dx$ & $dp =$

$dp = q\,dx$, eritque ex §. 268: $ddx = -\frac{p\,dx^2}{y}$, $ddy = q\,dx^2 - \frac{pp\,dx^2}{y}$, quibus ſubſtitutis expreſſio propoſita transmutatur in hanc: $-1 - \frac{xq}{p} + \frac{xp}{y}$.

IV. Sit propoſita iſta expreſſio $\frac{dx^2 + dy^2}{ddy}$, in qua conſtans ſit poſitum $\sqrt{(dx^2 + dy^2)}$. Ponatur iterum $dy = p\,dx$, $dp = q\,dx$, & ex paragrapho praecedente erit $ddy = \frac{q\,dx^2}{1+pp}$; vnde expreſſio propoſita abibit in $\frac{(1+pp)^2}{q}$.

Ex his autem exemplis ſatis intelligitur, quemadmodum in quouis caſu oblato, quodcunque differentiale primum aſſumtum ſit conſtans, differentialia ſecunda atque altiora eliminari debeant.

271. Cum igitur hoc modo introducendis quantitatibus finitis p, q, r, s, &c. differentialia ſecunda & altiora ita eliminari queant, vt tota expreſſio praeter quantitates finitas x, y, p, q, r, s, &c. ſolum differentiale dx complectatur: viciſſim ſi huiusmodi expreſſio reducta proponatur, ea iterum in formam priorem transmutari poterit loco litterarum p, q, r, s, &c. introducendis differentialibus ſecundis & altioribus. Nunc autem perinde erit, quodnam differentiale primum conſtans aſſumatur; atque vel id ipſum, quod ante fuit aſſumtum conſtans

 po-

poni poteſt, vel aliud quodcunque. Quin etiam prorſus nullum differentiale conſtans aſſumi poterit, hocque modo prodibunt expreſſiones differentialia ſecunda altioraue continentes, quae etiamſi nullum differentiale conſtans ſit aſſumtum, tamen fixas ſignificationes obtineant, cuiusmodi expreſſiones dari ſupra oſtendimus.

272. Sit ergo propoſita expreſſio quaecunque continens litteras finitas x, y, p, q, r, &c. vna cum differentiali dx, in qua ſit $p = \frac{dy}{dx}$; $q = \frac{dp}{dx}$; $r = \frac{dq}{dx}$; &c. Si enim has litteras p, q, r, &c. ita eliminare velimus, vt earum loco introducamus differentialia ſecunda & altiora ipſarum x & y, nullo differentiali conſtante aſſumto: fiet $dp = \frac{dxddy - dyddx}{dx^2}$, hincque $q = \frac{dxddy - dyddx}{dx^3}$, quae formula differentiata dabit

$$dq = \frac{dx^2 d^3y - 3dxddxddy + 3dyddx^2 - dxdyd^3x}{dx^4},$$

vnde fit

$$r = \frac{dx^2 d^3y - 3dxddxddy + 3dyddx^2 - dxdyd^3x}{dx^5}.$$

Quod ſi inſuper littera s, quae denotat valorem $\frac{dr}{dx}$, inſit, pro ea ſubſtitui debebit hic valor $s =$

$$\frac{dx^3d^4y - 6dx^2ddxd^3y - 4dx^2ddyd^3x + 15dxddx^2ddy + 10dxdyddxd^3x - 15dyddx^3 - dx^2dyd^4x}{dx^7}$$

His

His igitur valoribus loco quantitatum p, q, r, s, &c. ſubſtitutis expreſſio propoſita transmutabitur in aliam differentialia altiora ipſarum x & y continentem, quae etiamſi nullum differentiale primum conſtans ſit aſſumtum, tamen non vagam ſed fixam habebit ſignificationem.

273. Hoc ergo modo quaeuis formula differentialis altioris gradus, in qua quodpiam differentiale primum aſſumtum eſt conſtans, transmutari poterit in aliam formam, in qua nullum differentiale conſtans ponitur, quae hoc non obſtante eundem valorem fixum habeat. Primum ſcilicet ope methodi ante traditae aſſumtis valoribus $dy = pdx$; $dp = qdx$; $dq = rdx$; $dr = sdx$; &c. differentalia altiora eliminentur, tum loco p, q, r, s, &c. valores nunc inuenti ſubſtituantur; atque orietur expreſſio priori aequalis nullum differentiale conſtans inuoluens: quam transformationem exempla ſequentia illuſtrabunt.

I. Sit propoſita haec expreſſio $\frac{xddy}{dx^2}$, in qua dx poſitum conſtans, quae transmutari debeat in aliam formam nullum differentiale conſtans inuoluentem.
Ponatur $dy = pdx$; $dp = qdx$; atque vt ante (270) vidimus expreſſio propoſita transibit in hanc: qx. Nunc loco q ſubſtituatur valor, quem obtinet nullo differentiali conſtanti aſſumto $q = \frac{dxddy - dyddx}{dx^3}$ atque reperietur haec expreſſio $\frac{xdxddy - xdyddx}{dx^3}$ propoſitae aequalis, & nullum amplius differentiale conſtans inuoluens.

II. Sit

II. Sit propoſita haec expreſſio $\frac{dx^2 + dy^2}{ddx}$, in qua dy aſſumtum eſt conſtans. Ponatur $dy = pdx$ & $dp = qdx$; eaque transibit in hanc: $-\frac{p(1+pp)}{q}$, ſtatuatur nunc $p = \frac{dy}{dx}$ & $q = \frac{dxddy - dyddx}{dx^3}$, atque inuenietur: $\frac{dy(dx^2 + dy^2)}{dyddx - dxddy}$ quae nullo differentiali aſſumto eundem fixum habet valorem, quem propoſita.

III. Sit propoſita haec expreſſio: $\frac{yddx - xddy}{dxdy}$, in qua differentiale ydx conſtans eſt aſſumtum. Ponatur $dy = pdx$, atque vti ſupra (270) vidimus haec expreſſio transmutatur in hanc: $-1 - \frac{xy}{p} + \frac{xp}{y}$, quae nullo differentiali conſtante aſſumto transformabitur in iſtam:

$$-1 - \frac{xdxddy + xdyddx}{dx^2dy} + \frac{xdy}{ydx}$$

$$= \frac{xdxdy^2 - ydx^2dy - yxdxddy + yxdyddx}{ydx^2dy}$$

IV. Sit propoſita haec expreſſio $\frac{dx^2 + dy^2}{ddy}$, in qua conſtans aſſumtum eſt differentiale $\sqrt{(dx^2 + dy^2)}$. Poſito $dy = pdx$, & $dp = qdx$, orietur haec expreſſio $\frac{(1+pp)^2}{q}$, (loco citato). Statuatur nunc $p = \frac{dy}{dx}$, &

$q =$

$q = \frac{dxddy - dyddx}{dx^3}$, atque nullo aſſumto differentiali conſtante nanciſcemur iſtam expreſſionem $\frac{(dx^2 + dy)^2}{dx^2 ddy - dxdyddx}$ propoſitae aequiualentem.

V. Sit propoſita haec expreſſio $\frac{dxd^3y}{xddy}$, in qua differentiale dx conſtans ſit aſſumtum. Ponatur

$$dy = pdx \;;\; dp = qdx \;\;\&\;\; dq = rdx \;;$$

atque ob

$$ddy = qdx^2 \;\;\&\;\; d^3y = rdx^3$$

formula propoſita abibit in hanc $\frac{rdx^2}{xq}$. Nunc loco q & r ſubſtituantur valores, quos nullo differentiali conſtante aſſumto recipiunt ſcilicet: $q = \frac{dxddy - dyddx}{dx^3}$, &

$$r = \frac{dx^2d^3y - 3dxddxddy + 3dyddx^2 - dxdyd^3x}{dx^5},$$

atque obtinebitur ſequens expreſſio propoſitae aequiualens:

$$\frac{dx^2d^3y - 3dxddxddy + 3dyddx^2 - dxdyd^3x}{dxddy - dyddx}$$

$$= \frac{dx(dxd^3y - dyd^3x)}{dxddy - dyddx} - 3ddx.$$

274. Si has transformationes diligentius intueamur, methodum eas perficiendi colligere poterimus expeditiorem, ita vt non opus ſit litteras p, q, r, &c. introdu-

cere. Varii autem modi hoc opus abſoluendi occurrent, prout aliud atque aliud differentiale in formula propoſita conſtans fuerit aſſumtum. Ponamus primum in formula propoſita differentiale dx conſtans eſſe aſſumtum; & quia loco dy poſuimus pdx, rurſusque $\frac{dy}{dx}$ loco p: differentialia prima dx & dy, vbicunque in expreſſione occurrunt, ſine alteratione relinquuntur. Vbi autem occurrit ddy, quia eius loco ſcribitur qdx^2, & porro loco q valor $\frac{dxddy - dyddx}{dx^3}$, transmutatio abſoluetur, ſi vbique loco ddy ſtatim ponatur $\frac{dxddy - dyddx}{dx}$ ſeu $ddy - \frac{dyddx}{dx}$. Si inſuper in expreſſione propoſita occurrat d^3y, quia eius loco ponitur rdx^3, ob valorem ipſius r ante inuentum, vbique loco d^3y ſcribi debebit

$$d^3y - \frac{3ddxddy}{dx} + \frac{3dyddx^2}{dx^2} - \frac{dyd^3x}{dx},$$

quo facto expreſſio propoſita transmutabitur in aliam, quae nullum differentiale conſtans inuoluit. Sic ſi proponatur iſta expreſſio $\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxddy}$, in qua dx poſitum eſt conſtans, ei aequalis erit poſito $ddy - \frac{dyddx}{dx}$ loco ddy, haec nullum differentiale conſtans inuoluens:

$$\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxddy - dyddx}.$$

275.

275. Hinc facile colligitur, ſi in expreſſione quapiam propoſita aſſumtum fuerit differentiale dy conſtans, tum vbique loco ddx ſcribi debere $ddx - \frac{dxddy}{dy}$, & loco d^3x hoc $d^3x - \frac{3ddxddy}{dy} + \frac{3dxddy^2}{dy^2} - \frac{dxd^3y}{dy}$; vt obtineatur expreſſio aequiualens, in qua nullum differentiale conſtans ponatur. Sin autem in expreſſione propoſita conſtans fuerit aſſumtum ydx, quoniam fit

$$ddx = -\frac{pdx^2}{y}, \quad \& \quad ddy = qdx^2 - \frac{ppdx^2}{y};$$

loco ddx vbique ſcribi debebit $-\frac{dxdy}{y}$, & loco ddy vbique $ddy - \frac{dyddx}{dx} - \frac{dy^2}{y}$: ad altiora differentialia, quia in hoc negotio rariſſime occurrere ſolent, non progredior. Quod ſi vero in expreſſione propoſita hoc differentiale $\sqrt{(dx^2 + dy^2)}$ aſſumtum fuerit conſtans, quia inuenimus $ddx = -\frac{pqdx^2}{1+pp}$ & $ddy = \frac{qdx^2}{1+pp}$: pro ddx vbique ſcribi debet $\frac{dy^2ddx - dxdyddy}{dx^2 + dy^2}$, & loco ddy vbique $\frac{dx^2ddy - dxdyddx}{dx^2 + dy^2}$. Sic ſi propoſita fuerit expreſſio $\frac{dy\sqrt{(dx^2+dy^2)}}{ddx}$, in qua $\sqrt{(dx^2+dy^2)}$ aſſumtum ſit conſtans, ea transmutabitur in hanc:

$\frac{(dx^2+dy^2)^2}{dyddx-dxddy}$, in qua nullum differentiale conſtans aſſumitur.

276. Quo iſtae reductiones facilius ad vſum accommodari queant, eas in ſequenti tabella complecti viſum eſt.

Formula igitur differentialis altioris gradus in aliam nullum differentiale conſtans inuoluentem transmutabitur ope ſubſtitutionum ſequentium :

I. Si differentiale dx fuerit conſtans aſſumtum

loco	ſcribatur
ddy	$ddy - \frac{dy\,ddx}{dx}$
d^3y	$d^3y - \frac{3\,ddx\,ddy}{dx} + \frac{3\,dy\,ddx^2}{dx^2} - \frac{dy\,d^3x}{dx}$

II. Si differentiale dy fuerit conſtans aſſumtum

loco	ſcribatur
ddx	$ddx - \frac{dx\,ddy}{dy}$
d^3x	$d^3x - \frac{3\,ddx\,ddy}{dy} + \frac{3\,dx\,ddy^2}{dy^2} - \frac{dx\,d^3y}{dy}$

III.

III. Si differentiale ydx fuerit conſtans aſſumtum

loco	ſcribatur
ddx	$-\frac{dx\,dy}{y}$
ddy	$ddy - \frac{dy\,ddx}{dx} - \frac{dy^2}{y}$
d^3x	$\frac{dy\,ddx}{y} - \frac{dx\,ddy}{y} + \frac{3\,dx\,dy^2}{yy}$
d^3y	$d^3y - \frac{3\,ddx\,ddy}{dx} + \frac{3\,dy\,ddx^2}{dx^2} - \frac{dy\,d^3x}{dx}$
	$-\frac{4\,dy\,ddy}{y} + \frac{4\,dy^2\,ddx}{y\,dx} + \frac{3\,dy^3}{yy}$.

IV. Si differentiale $\sqrt{(dx^2+dy^2)}$ fuerit conſtans aſſumtum.

loco	ſcribatur
ddx	$\frac{dy^2\,ddx - dx\,dy\,ddy}{dx^2+dy^2}$
ddy	$\frac{dx^2\,ddy - dx\,dy\,ddx}{dx^2+dy^2}$
d^3x	- - - - $\frac{dy^2\,d^3x - dx\,dy\,d^3y}{dx^2+dy^2}$
	$+\frac{(dx\,ddy - dy\,ddx)(3\,dy^2\,ddy - dx^2\,ddy + 4\,dx\,dy\,ddx)}{(dx^2+dy^2)^2}$
d^3y	- - - - $\frac{dx^2\,d^3y - dx\,dy\,d^3x}{dx^2+dy^2}$
	$+\frac{(dy\,ddx - dx\,ddy)(3\,dx^2\,ddx - dy^2\,ddx + 4\,dx\,dy\,ddy)}{(dx^2+dy^2)^2}$

277. Expreſſiones ergo iſtae, quae nullum differentiale conſtans includunt, ita erunt comparatae, vt pro lubitu quoduis differentiale conſtans aſſumi queat. Hincque expreſſiones differentiales altiorum graduum, in quibus nullum differentiale conſtans aſſumtum perhibetur, examinari poſſunt, vtrum ſignificatio earum ſit vaga an fixa? Ponatur enim pro lubitu quodpiam differentiale puta dx conſtans, tum per regulam §. praeced. priorem reducatur expreſſio iterum ad formam, in qua nullum differentiale conſtans ſit aſſumtum, quae ſi cum propoſita conueniat, ea erit fixa, neque ab inconſtantia differentialium ſecundorum pendebit: ſin autem expreſſio prodeat diuerſa, tum propoſita vagam habet ſignificationem. Sic ſi ponatur haec expreſſio $yddx - xddy$, in qua nullum differentiale poſitum ſit conſtans; ad inueſtigandum, vtrum ſignificationem fixam habeat an vagam? ponatur dx conſtans, eaque abibit $-xddy$: nunc per regulam primam §. praeced. loco ddy ponat,

$$ddy - \frac{dyddx}{dx} \text{ ac prodibit } -xddy + \frac{xdyddx}{dx},$$

cuius a propoſita diſcrepantia indicat, propoſitam expresſionem fixam ſtatamque ſignificationem non habere.

278. Simili modo ſi proponatur expreſſio generalis huiusmodi $Pddx + Qdxdy + Rddy$, conditio definiri poterit, ſub qua ea nullo differentiali conſtante aſſumto valorem fixum habeat. Ponatur enim dx conſtans, atque expreſſio propoſita abibit in hanc $Qdxdy + Rddy$: nunc haec iterum transformetur in aliam formam, vt

eius

eius ſignificatus idem maneat, etiamſi nullum differentiale conſtans fingatur, ſicque prodibit $Qdxdy + Rddy - \frac{Rdyddx}{dx}$, quae forma cum propoſita congruet, ſi fuerit $Pdx + Rdy = o$; hocque ſolo caſu valor eius erit fixus. Verum ſi non fuerit $P = -\frac{Rdy}{dx}$ ſeu $R = -\frac{Pdx}{dy}$ tum expreſſio propoſita $Pddx + Qdxdy + Rddy$ valorem fixum non habebit, ſed eius ſignificatio erit vaga atque diuerſa, prout aliud atque aliud differentiale conſtans aſſumitur.

279. Ex his principiis etiam facile erit expreſſionem differentialem, in qua quodpiam differentiale conſtans eſt poſitum, transmutare in aliam formam, in qua aliud differentiale conſtans aſſumatur. Reducatur enim primum ad eiusmodi formam, quae nullum differentiale conſtans inuoluat, quo facto illud alterum differentiale conſtans ponatur. Sic ſi in expreſſione propoſita differentiale dx aſſumtum ſit conſtans, eaque transmutanda ſit in aliam, quae differentiale dy conſtans implicet: in formulis ſupra loco ddy & d^3y ſubſtituendis ob dy conſtans ponatur $ddy = o$, $d^3y = o$, atque quaeſito ſatisfiet, ſi loco ddy ſubſtituatur $-\frac{dyddx}{dx}$ & $\frac{3dyddx^2}{dx^2} - \frac{dyddx}{dx}$ loco d^3y. Hoc modo iſta formula $-\frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dxddy}$, in qua dx poſitum eſt conſtans, transmutabitur in hanc $\frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dyddx}$, in qua dy ponitur conſtans.

280. Si contra formula, in qua dy conſtans eſt poſitum, transmutari debeat in aliam, in qua dx ſit conſtans, tum loco ddx ſubſtitui debet $-\frac{dxddy}{dy}$ & loco d^3x haec expreſſio $\frac{3dxddy^2}{dy^2} - \frac{dxd^3y}{dy}$. Simili modo ſi formula, in qua $V(dx^2 + dy^2)$ poſitum eſt conſtans, transmutari debeat in aliam, in qua dx ſit conſtans, tum loco ddx ſcribatur $-\frac{dxdyddy}{dx^2+dy^2}$ & $\frac{dx^2ddy}{dx^2+dy^2}$ loco ddy. At ſi formula, qua dx conſtans eſt aſſumtum, transmutari debeat in aliam, in qua $V(dx^2 + dy^2)$ ſit conſtans, quia ob $dx^2 + dy^2$ conſtans fit $dxddx + dyddy = 0$, & $ddx = -\frac{dyddy}{dx}$, hoc valore loco ddx aſſumto, pro ddy ſcribi debebit $ddy + \frac{dy^2ddy}{dx^2} = \frac{(dx^2+dy^2)ddy}{dx^2}$. Sic haec formula $-\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxddy}$, in qua dx eſt conſtans, transmutabitur in aliam, in qua $V(dx^2 + dy^2)$ ponitur conſtans, quae erit $-\frac{dxV(dx^2+dy^2)}{ddy}$.

CAPUT

CAPUT IX.

DE AEQUATIONIBUS DIFFERENTIALIBUS.

281.

In hoc Capite imprimis eſt propoſitum earum functionum ipſius x, quae non explicite, ſed implicite per aequationem, qua relatio functionis iſtius y ad x continetur, definiuntur, differentiationem explicare: quo facto naturam aequationum differentialium in genere perpendemus, & quemadmodum ex aequationibus finitis oriantur, oſtendemus. Cum enim in calculo integrali ſummum negotium conſiſtat in integratione aequationum differentialium, ſeu in inuentione eiusmodi aequationum finitarum, qua cum differentialibus conueniant; neceſſe eſt, vt hoc loco indolem ac proprietates aequationum differentialium, quae ex earum origine ſequuntur, diligentius ſcrutemur, ſicque viam ad calculum integralem praeparemus.

282. Vt igitur hoc negotium abſoluamus, ſit y functio eiusmodi ipſius x, quae per hanc aequationem quadratam $yy + Py + Q = 0$ definiatur. Cum ergo haec expreſſio $yy + Py + Q$ ſit $= 0$, quicquid x ſignificet, nihilo quoque aequalis erit, ſi loco x ſcribatur $x + dx$, quo caſu y abit in $y + dy$. Facta autem hac ſubſtitutione, ſi a quantitate reſultante ſubtrahatur prior

$yy + Py + Q$, remanebit eius differentiale, quod propterea quoque erit $= 0$. Hinc patet ſi expreſſio quaecunque fuerit $= 0$, eius etiam differentiale fore aequale 0; atque ſi duae quaecunque expreſſiones inter ſe fuerint aequales, earum quoque differentialia fore aequalia. Cum igitur ſit $yy + Py + Q = 0$, erit quoque

$$2ydy + Pdy + ydP + dQ = 0:$$

quia vero P & Q ſunt functiones ipſius x, earum differentialia huiusmodi formam habebunt,

$$dP = pdx, \quad \& \quad dQ = qdx;$$

vnde fiet

$$2ydy + Pdy + ypdx + qdx = 0$$

ex qua oritur $\frac{dy}{dx} = -\frac{yp - q}{2y + P}$.

283. Quemadmodum ergo aequatio finita $yy + Py + Q = 0$ exponit relationem inter y & x, ita aequatio differentialis exprimit relationem ſeu rationem, quam dy tenet ad dx. Quoniam vero eſt $\frac{dy}{dx} = -\frac{yp - q}{2y + P}$, haec ratio $dy : dx$ cognoſci non poteſt, niſi ipſa functio y ſit cognita: neque vero res aliter ſe habere poteſt; cum enim ex aequatione finita y geminum obtineat valorem, vterque ſuum peculiare habebit differentiale, & vtriusque differentiale reperietur, prouti hic vel ille valor in expreſſione $-\frac{yp - q}{2y + P}$ loco y ſubſtituatur. Simili modo functio y per aequatio-

tionem cubicam definiatur, valor functionis $\frac{dy}{dx}$ erit triplex; triplici ſcilicet ipſius y valori reſpondens. Si in aequatione propoſita finita y quatuor pluresue habeat dimenſiones, neceſſe eſt vt $\frac{dy}{dx}$ totidem ſignificationes ſortiatur.

284. Interim tamen ipſa functio y ex aequatione eliminari poterit, cum duae habeantur aequationes y continentes, finita ſcilicet & differentialis: tum autem eius differentiale dy ad totidem dimenſiones aſſurget, quot ante habuerat y, ſicque iſta aequatio omnes diuerſas rationes ipſius dy ad dx ſimul complectetur. Sumamus praecedens exemplum aequationis $yy + Py + Q = 0$, cuius differentialis eſt:

$$2ydy + Pdy + ydP + dQ = 0,$$

ex qua fit $y = -\frac{Pdy - dQ}{2dy + dP}$, qui valor loco y in priori aequatione ſubſtitutus dabit:

$$(4Q-PP)dy^2 + (4Q-PP)dPdy + QdP^2 - PdPdQ + dQ^2 = 0,$$

cuius radices ſunt:

$$dy = -\tfrac{1}{2}dP \pm \frac{(\tfrac{1}{2}PdP - dQ)}{\sqrt{(PP - 4Q)}},$$

quae ſunt bina differentialia binorum ipſius y valorum ex aequatione finita:

$$y = -\tfrac{1}{2}P \pm \tfrac{1}{2}\sqrt{(PP - 4Q)}.$$

285. Inuento valore ipſius dy per repetitam differentiationem reperietur valor ipſius ddy, porroque ipſorum d^3y, d^4y, &c. qui autem, cum determinati non ſint, niſi aliquod differentiale primum conſtans ſtatuatur. Ponamus commoditatis ergo dx conſtans, atque ad hoc oſtendendum ſumamus hoc exemplum $y^3 + x^3 = 3axy$, vnde per differentiationem oritur

$$3yydy + 3xxdx = 3axdy + 3aydx,$$

hincque $\frac{dy}{dx} = \frac{ay - xx}{yy - ax}$, ſumantur denuo differentialia poſito dx conſtante atque inuenietur $\frac{ddy}{dx} =$

$$- \frac{ayydy - aaxdy + 2xxydy - 2xyydx + aaydx + axxdx}{(yy - ax)^2}$$

ſubſtituatur loco dy eius valor modo inuentus $\frac{aydx - xxdx}{yy - ax}$, atque diuiſione per dx facta habebitur

$$\frac{ddy}{dx^2} = \frac{(ay - xx)(2xxy - ayy - aax)}{(yy - ax)^3}$$

$$+ \frac{axx + aay - 2xyy}{(yy - ax)^2}$$

ſeu

$$\frac{ddy}{dx^2} = \frac{6axxyy - 2x^4y - 2xy^4 - 2a^3xy}{(yy - ax)^3}$$

$$= - \frac{2a^3xy}{(yy - ax)^3}$$

cum ex aequatione finita ſit $2x^4y + 2xy^4 = 6axxyy$: hocque modo ope aequationis finitae his valores in innumeras formas transmutari poſſunt.

286.

286. Aequatio etiam differentialis prima infinitis modis poteſt variari, dum cum aequatione finita permiſcetur. Sic cum exemplo praecedente inuenta eſſet aequatio differentialis

$$yy\,dy + xx\,dx = ax\,dy + ay\,dx,$$

ſi ea multiplicetur per y, orietur

$$y^3dy + xxy\,dx = axy\,dy + ayy\,dx,$$

in qua ſi loco y^3 ſubſtituatur eius valor $3axy - x^3$ orietur haec aequatio noua

$$2axy\,dy - x^3dy + xxy\,dx = ayy\,dx;$$

quae denuo per y multiplicata, poſtquam loco y^3 eius valor fuerit ſubſtitutus, praebebit

$$2axy^2dy - x^3y\,dy + xxyy\,dx = 3aaxy\,dx - ax^3dx.$$

Generaliter autem ſi P, Q, R, denotent functiones quascunque ipſarum x & y. Si aequatio differentialis multiplicetur per P erit

$$Pyy\,dy + Pxx\,dx = aPx\,dy + aPy\,dx.$$

Tum cum ſit $x^3 + y^3 - 3axy = 0$ erit quoque

$$(x^3 + y^3 - 3axy)(Q\,dx + R\,dy) = 0,$$

quae aequationes inuicem additae dabunt aequationem differentialem generalem ex propoſita aequatione finita natam

$$\begin{aligned}&Pyydy - aPxdy + Rx^3dy + Ry^3dy - 3aRxydy + \\ &Pxxdx - aPydx + Qx^3dx + Qy^3dx - 3aQxydx = 0.\end{aligned}$$

287. Poſſunt vero etiam per ipſam differentiationem infinitae aequationes differentiales ex eadem aequatio-

tione finita inueniri, dum ea, antequam differentietur, per quantitatem quamcunque aut multiplicatur aut diuiditur. Sic ſi P fuerit functio quaecunque ipſarum x & y, vt ſit $dP = pdx + qdy$, ſi aequatio finita per P multiplicetur, atque tum demum differentietur, obtinebitur aequatio differentialis generalis, quae infinitas formas diuerſas induet, prouti pro P aliae atque aliae functiones aſſumuntur. Tum vero multiplicitas adhuc in infinitum augebitur, ſi ad hanc aequationem differentialem inuentam addatur ipſa aequatio finita per huiusmodi formulam $Qdx + Rdy$ multiplicata, vbi pro Q & R functiones quascunque ipſarum x & y aſſumere licet. Quanquam autem in his omnibus aequationibus relatio inter dy & dx, quam differentiale functionis y aequatione finita per x determinatae ad dx tenet, comprehenditur; tamen plerumque multo latius patent, & differentiale ipſius y per alias aequationes finitas determinati exprimit; cuius rei ratio in calculo integrali potiſſimum explicabitur.

288. Non ſolum autem ex eadem aequatione finita innumerabiles aequationes differentiales deduci poſſunt, ſed etiam plures imo infinitae exhiberi poſſunt aequationes finitae, quae ad easdem aequationes differentiales deducantur. Sic hae duae aequationes $yy = ax + ab$ & $yy = ax$ omnino ſunt diuerſae, dum in priori quaecunque quantitas conſtans in locum ipſius b collocatur. Interim tamen hae ambae aequationes differentiatae eandem dant aequationem differentialem $2ydy = adx$; quin etiam omnes aequationes in hac forma $yy = ax$ con-

ten-

tentae, quicunque valor ipſi a tribuatur, in vna aequatione differentiali, in qua a non inſit, comprehendi poſſunt. Diuidatur enim aequatio illa per x vt ſit $\frac{yy}{x} = a$, haecque differentia dabit $2xdy - ydx = 0$. Poſſunt quoque aequationes transcendentes & algebraicae ad eandem aequationem differentialem perduci, vti fit in iſtis aequationibus

$$yy - ax = 0 \quad \& \quad yy - ax = bbe^{\frac{x}{a}},$$

ſi enim vtraque per $e^{\frac{x}{a}}$ diuidatur, vt habeantur iſtae aequationes:

$$e^{-\frac{x}{a}}(yy - ax) = 0 \quad \& \quad e^{-\frac{x}{a}}(yy - ax) = bb,$$

ex vtriusque differentiatione orietur eadem differentialis

$$2ydy - adx - \frac{yydx}{a} + xdx = 0.$$

289. Ratio huius diuerſitatis in hoc conſiſtit, quod quantitatis conſtantis differentiale ſit $= 0$. Quodſi ergo aequatio finita ad eiusmodi formam reducatur, vt quantitas quaepiam conſtans ſola adſit, neque per variabiles vel multiplicetur vel diuidatur; tum per differentiationem eruetur aequatio, in qua illa quantitas conſtans prorſus non adſit. Hoc modo quaelibet quantitas conſtans, quae in aequationem finitam ingreditur, per differentiationem tolli poteſt. Sic ſi propoſita fuerit aequatio $x^3 + y^3 = 3axy$; ſi ea per xy diuidatur vt habeatur

x^3

$\frac{x^3 + y^3}{xy} = 3a$, haec aequatio differentiata dabit:

$$2x^3 y dx + 2xy^3 dy - x^4 dy - y^4 dx = 0,$$

quam conſtans a amplius non ingreditur.

290. Si plures quantitates conſtantes, quae in aequatione finita inſunt, tolli debeant, id fiet per differentiationem bis pluriesue repetitam; ſicque tandem obtinebuntur aequationes differentiales altiorum graduum iis conſtantibus prorſus carentes. Sit propoſita haec aequatio $yy = maa - nxx$, ex qua per differentiationem conſtantes maa & n tolli debeant. Prima quidem tolletur prima differentiatione, vnde fit $ydy + nxdx = 0$, hinc porro formetur aequatio $\frac{ydy}{xdx} + n = 0$, quae ſumto dx conſtante, per differentiationem dabit:

$$xyddy + xdy^2 - ydxdy = 0,$$

quae etſi nullam conſtantem complectitur, tamen omnes aequationes in hac forma $yy = maa - nxx$ contentas, quicunque valores litteris m, n & aa tribuantur, in ſe aeque comprehendit.

291. Non ſolum vero quantitates conſtantes, quae in aequationem finitam ingrediuntur, per differentiationem tolli poſſunt, ſed etiam altera variabilis, eius ſcilicet, cuius differentiale conſtans aſſumitur, per differentiationem eliminari poterit. Ex aequatione enim inter x & y propoſita quaeratur valor x, vt ſit $x = Y$ denotante Y func-

functionem ipſius y; eritque $dx = dY$, & ſumto dx conſtante, fiet differentiando $0 = ddY$. Sin autem fuerit $xx + ax + b = Y$, fiet ter differentiando $0 = d^3Y$, & aequatio $x^3 + axx + bx + c = Y$ quater differentiata dat $0 = d^4Y$. Quanquam autem in his aequationibus vna tantum variabilis ineſſe videtur, quae propterea variabilis eſſe ceſſaret, dum vnica variabilis in nulla aequatione adeſſe poteſt; tamen quia differentiale dx conſtans eſt aſſumtum, eiusque ratio in aequatione haberi debet, reuera in aequationem ingredi cenſendum eſt. Hinc mirandum non eſt, ſi ſaepius aequationes differentiales ſecundi altiorisue gradus occurrant, in quibus vnica tantum variabilis ineſſe videatur.

292. Praecipue autem notandum eſt, per differentiationem quantitates irrationales ac transcendentes ex aequatione tolli poſſe. Quod quidem ad irrationales attinet, quoniam per reductiones cognitas irrationalitas eliminari poteſt, hoc facto, per differentiationem aequatio obtinetur ab irrationalitate libera. Verum hoc ſaepenumero commodius ſine iſta reductione fieri poteſt, dum per comparationem aequationis differentialis cum finita formula irrationalis, ſi vna tantum inſit, eliminari poteſt. Sin autem duae pluresue partes irrationales in aequatione finita contineantur, tum eius aequatio differentialis denuo differentietur, ſicque aequationes differentiales altiorum graduum tot quaerantur, quot requiruntur ad ſingulas partes irrationales eliminandas. Hoc modo etiam exponentes indefiniti pariter atque fracti tolli poterunt.

runt. Vti ſi fuerit $y^m = (aa - xx)^n$, poſt differentiationem habebitur

$$my^{m-1}dy = -2n(aa - xx)^{n-1}xdx,$$

quae per finitam diuiſa dat $\frac{mdy}{y} = -\frac{2nxdx}{aa-xx}$, in qua nullus amplius exponens indefinitus occurrit. Hinc ergo patet aequationem differentialem ab omni irrationalitate liberam ortam eſſe poſſe ex aequatione finita irrationali, atque adeo quantitates transcendentes inuolvente.

293. Vt autem intelligatur, quomodo per differentiationem quantitates transcendentes eliminentur, incipiamus a logarithmis, quorum differentialia cum ſint algebraica, negotium ſine difficultate abſoluetur. Sit enim $y = xlx$: erit $\frac{y}{x} = lx$, vnde differentiando fit $\frac{xdy - ydx}{xx} = \frac{dx}{x}$, ideoque $xdy - ydx = xdx$. Si bini inſint logarithmi duplici differentiatione erit opus: ſit enim $ylx = xly$; erit $\frac{ylx}{x} = ly$, & differentiando, $\frac{xdylx + ydx - ydxlx}{xx} = \frac{dy}{y}$, ex qua concluditur fore $lx = \frac{xxdy - yydx}{yxdy - yydx}$. Haec aequatio iam iterum differentietur poſito dx conſtante, atque prodibit

$$\frac{dx}{x} = \frac{xxddy + 2xdxdy - 2ydxdy}{yxdy - yydx}$$

$$+ \frac{(yydx - xxdy)(yxddy + xdy^2 - ydxdy)}{(yxdy - yydx)^2}$$

ſeu $\frac{dx}{x} =$

$$\frac{y^3xdxddy - yyxxdxddy + 3yxxdxdy^2 - y^2xdxdy^2 + y^3dx^2dy - 2xyydx^2dy - x^3dy^3}{(yxdy - yydx)^2}$$

quae reducta dabit:

$$y^3xdxddy - yyxxdxddy + 3yxxdxdy^2 - 2xyydxdy^2$$

$$+ 3y^3dx^2dy - 2xyydx^2dy - x^3dy^3 - \frac{y^4dx^3}{x} = 0$$

ſeu

$$yyxx(y - x)dxddy + 3yxdxdy(xxdy + yydx)$$

$$- 2yyxxdxdy(dx + dy) = x^4dy^3 + y^4dx^2.$$

294. Quantitates exponentiales ex aequatione eodem modo, quo logarithmi per differentiationem tolluntur. Si enim huiusmodi propoſita fuerit $P = e^Q$, vbi P & Q functiones quascunque ipſarum x & y denotent; ea aequatio transmutari poterit in hanc logarithmicam $lP = Q$; cuius differentialis eſt $\frac{dP}{P} = dQ$ ſeu $dP = PdQ$. Neque obſtat, ſi quantitates exponentiales magis fuerint complicatae, tum enim ſi vna differentiatio non ſufficit, duabus pluribusue negotium abſoluetur.

I. Sit $y = \frac{e^x + e^{-x}}{e^x - e^{-x}}$; multiplicetur huius fractionis numerator ac denominator per e^x eritque $y = \frac{e^{2x}+1}{e^{2x}-1}$, vnde fit

$$e^{2x} = \frac{y+1}{y-1} \quad \& \quad 2x = l\frac{y+1}{y-1},$$

cuius differentiale eſt

$$dx = -\frac{dy}{yy-1} = \frac{dy}{1-yy}.$$

II. Sit $y = l\frac{e^x + e^{-x}}{2}$, fiet per primam differentiationem $dy = \frac{(e^x - e^{-x})dx}{e^x + e^{-x}}$, ſeu $\frac{dy}{dx} = \frac{e^{2x}-1}{e^{2x}+1}$, atque $e^{2x} = \frac{dy+dx}{dx-dy}$. Ergo $2x = l\frac{dy+dx}{dx-dy}$.

Sumto ergo dx conſtante erit

$$dx = \frac{dx\,ddy}{dx^2 - dy^2} \quad \text{ſeu} \quad dx^2 = ddy + dy^2.$$

295. Simili modo quantitates transcendentes a circulo pendentes ex aequatione ope differentiationis tollentur, vti ex his exemplis intelligetur.

I. Sit $y = a\,\text{A ſin}\,\frac{x}{a}$; erit $dy = \frac{a\,dx}{\sqrt{(aa-xx)}}$.

II. Sit $y = a\,\text{coſ}\,\frac{y}{x}$; erit

$$\frac{y}{a} = \text{coſ}\,\frac{y}{x}, \quad \& \quad \frac{dy}{a} = -\frac{x\,dy + y\,dx}{xx}\,\text{ſin}\,\frac{y}{x}.$$

At

At cum ſit

$$\text{coſ}\,\frac{y}{x} = \frac{y}{a};\quad \text{erit}\quad \text{ſin}\,\frac{y}{x} = \frac{\sqrt{(aa - yy)}}{a},$$

quo valore ſubſtituuo habebitur

$$\frac{dy}{a} = \frac{(ydx - xdy)\sqrt{(aa - yy)}}{axx}$$

$$\text{ſeu}\quad xxdy = (ydx - xdy)\sqrt{(aa - yy)}.$$

III. Sit $y = m\,\text{ſin}\,x + n\,\text{coſ}\,x$, erit poſt differentiationem primam $dy = mdx\,\text{coſ}\,x - ndx\,\text{ſin}\,x$: quae denuo differentiata poſito dx conſtante dabit

$$ddy = -mdx^2\,\text{ſin}\,x - ndx^2\,\text{coſ}\,x,$$

haec autem per primam diuiſa dat

$$\frac{ddy}{y} = -dx^2 \quad \text{ſeu}\quad ddy + ydx^2 = 0,$$

ex qua non ſolum ſinus & coſinus, ſed etiam conſtantes m & n euanuerunt.

IV. Sit $y = \text{ſin}\,lx$; erit $A\,\text{ſin}\,y = lx$, vnde per differentiationem fit $\frac{dy}{\sqrt{(1-yy)}} = \frac{dx}{x}$; quae ſumtis quadratis dat $xxdy^2 = dx^2 - yydx^2$, haecque poſito dx conſtante vlterius differentiata praebet,

$$2xxdyddy + 2xdxdy^2 = -2ydx^2dy$$

$$\text{ſeu}\quad xxddy + xdxdy + ydx^2 = 0.$$

V. Sit $y = ae^{mx}\,\text{ſin}\,nx$, erit differentiando

$$dy = mae^{mx}dx\,\text{ſin}\,nx + nae^{mx}dx\,\text{coſ}\,nx,$$

 quae

quae per propoſitam diuiſa dat:

$$\frac{dy}{y} = m\,dx + \frac{n\,dx\,\text{coſ}\,nx}{\text{ſin}\,nx} = m\,dx + n\,dx\,\text{cot}\,nx.$$

Erit ergo $\text{A cot}\left(\frac{dy}{ny\,dx} - \frac{m}{n}\right) = nx.$ Quae aequatio poſito dx conſtante differentiata dat:

$$n\,dx = \frac{n\,dx\,dy^2 - ny\,dx\,ddy}{m^2y^2dx^2 + n^2y^2dx^2 - 2my\,dx\,dy}$$

ſeu $(m^2 + n^2)y^2dx^2 - 2my\,dx\,dy = dy^2 - y\,ddy.$

Perſpicuum igitur eſt, etiamſi in aequatione differentiali nullae quantitates transcendentes inſint, eam tamen ex aequatione finita oriri potuiſſe, quae a quantitatibus transcendentibus vtcunque ſit affecta.

296. Quoniam igitur aequationes differentiales ſiue primi ſiue altioris gradus, quae duas variabiles x & y continent, ex aequationibus finitis oriuntur; iis etiam relatio inter binas iſtas variabiles exprimitur. Propoſita ſcilicet aequatione differentiali quacunque binas variabiles x & y complectente, ea ſignificatur certa quaedam relatio inter x & y, qua y fit functio quaedam ipſius x. Hinc natura aequationis differentialis perſpicitur, ſi loco y ea ipſius x functio aſſignari poterit, quae per aequationem illam indicatur; ſeu quae ſit ita comparata, vt ſi ea vbique loco y, eiusque differentiale loco dy, atque eius altiora differentialia loco ddy, d^3y, &c. ſubſtituantur, aequatio reſultet identica. In huius autem functionis

nis inueſtigatione verſatur calculus integralis, cuius finis eo tendit, vt propoſita aequatione differentiali quacunque, functio illa ipſius x, cui altera variabilis y eſt aequalis, definiatur; ſeu quod eodem redit, vt aequatio finita inveniatur, qua relatio inter x & y contineatur.

297. Si exempli gratia proponatur aequatio haec

$$2ydy - adx - \frac{yydx}{a} + xdx = 0$$

ad quam ſupra §. 288 peruenimus, eiusmodi relatio inter x & y ea definitur, quae ſimul hac aequatione finita $yy - ax = bbe^{\frac{x}{a}}$ continetur. Cum igitur hinc ſit

$$yy = ax + bbe^{\frac{x}{a}}, \text{ patet } \sqrt{(ax + bbe^{\frac{x}{a}})} = y$$

eam eſſe functionem ipſius x, cui variabilis y vi propoſitae aequationis differentialis ſit aequalis. Namque ſi in aequatione loco yy, hunc valorem $ax + bbe^{\frac{x}{a}}$ & loco $2ydy$ eius differentiale $adx + \frac{bb}{a}e^{\frac{x}{a}}dx$ ſubſtituamus, orietur aequatio identica:

$$adx + \frac{bb}{a}e^{\frac{x}{a}}dx - adx - xdx - \frac{bb}{a}e^{\frac{x}{a}}dx + xdx = 0.$$

Sicque patet omnem aequationem differentialem aeque ac finitam certam relationem inter variabiles x & y exhibere, quae autem ſine ſubſidio calculi integralis reperiri nequeat.

298. Quo haec facilius intelligantur, ponamus cognitam eſſe eam functionem ipſius x, quae ipſi y vi cuiuscunque aequationis differentialis ſiue primi ſiue altioris gradus, conueniat; ſitque

$$dy = pdx;\quad dp = qdx;\quad dq = rdx;\quad \&c.$$

atque ſi in aequatione differentiale dx aſſumtum ſit conſtans, erit $ddy = qdx^2$, $d^3y = rdx^3$, &c. qui valores poſtquam in aequatione erunt ſubſtituti, ob omnes eius terminos homogeneos, differentialia dx per diuiſionem euaneſcent, orieturque aequatio finitas tantum quantitates x, y, p, q, r, &c. complectens. Cum igitur ſint p, q, r, &c. quantitates a natura functionis y pendentes, aequatio reuera tantum inter duas variabiles x & y ſubſiſtet; ſicque viciſſim conſtat, omni aequatione differentiali certam quantam relationem inter variabiles x & y determinari. Quamobrem ſi in ſolutione cuiusuis problematis ad aequationem differentialem inter x & y perveniatur, per eam aeque relatio inter x & y exprimi cenſenda eſt, ac ſi ad aequationem finitam eſſet perventum.

299. Hoc igitur modo aequatio quaeuis differentialis ita ad formam finitam reduci poteſt, vt in ea nonniſi quantitates finitae contineantur, differentialia autem ſeu infinite parua prorſus excedant. Cum enim ſit y certa functio ipſius x, ſi ponatur

$$dy = pdx;\quad dp = qdx;\quad dq = rdx;\quad \&c.$$

quodcunque differentiale fuerit conſtans acceptum, differentia-

tialia ſecunda & altiora per poteſtates ipſius dx exprimentur, quae deinceps per diuiſionem penitus tollentur. Vt ſi proponeretur haec aequatio

$$xyd^3y + xxdyddy + yydxddy - xydx^3 = 0$$

in qua dx ponitur conſtans; facto

$$dy = pdx, \quad dp = qdx, \quad \& \quad dq = rdx,$$

ea abibit in

$$xyr + xxpq + yyq - xy = 0,$$

poſtquam ſcilicet tota aequatio per dx^3 eſt diuiſa. Haecque aequatio finita relationem inter x & y determinat.

300. Omnes ergo aequationes differentiales, cuiuscunque ſint ordinis, his ſubſtitutionibus

$$dy = pdx; \quad dp = qdx; \quad dq = rdx; \quad \&c.$$

ad meras quantitates finitas reducuntur. Atque ſi aequatio differentialis fuerit primi ordinis, ita vt differentialia prima eam tantum ingrediantur, per iſtam reductionem praeter variabiles y & x inſuper quantitas p introducetur. Sin autem aequatio differentialis fuerit ſecundi ordinis continens differentialia ſecunda, praeterea quantitas q; ac, ſi fuerit differentialis tertii ordinis, introducetur inſuper quantitas r, ſicque porro. Quoniam igitur hoc modo differentialia prorſus ex calculo exterminantur, ratio illa differentialis conſtantis penitus ceſſat; neque amplius, etiamſi inſint quantitates q, r, ex differentialibus ſecundis oriundae, opus erit indicare, an quodpiam differentiale conſtans ſit aſſumtum. Perinde enim eſt, vtrum

in euolutione aliquod differentiale pro lubitu conſtans ſtatuatur, an nullum.

301. Si igitur aequatio differentialis ſecundi vel altioris gradus proponatur, in qua nullum differentiale primum conſtans eſſe aſſumtum perhibetur, hoc modo ſtatim explorari poterit, vtrum ea determinatam relationem inter variabiles x & y contineat, nec ne? Quia enim nullum differentiale conſtans aſſumitur, in arbitrio noſtro relinquitur, quodnam differentiale conſtans ponere velimus; hincque tantum erit diſpiciendum, vtrum diverſis differentialibus conſtantibus poſitis aequatio eandem relationem inter x & y exhibeat. Quodſi non eueniat, certum eſt ſignum, aequationem nullam determinatam relationem exprimere, ideoque in ſolutione nullius problematis locum habere poſſe. Tutiſſimus autem modus ſimulque facillimus hoc explorandi erit is ipſe, quem ſupra in ſimili negotio pro expreſſionibus differentialibus altiorum ordinum, num fixos habeant ſignificatus? dignoſcendis tradidimus.

302. Propoſita ergo huiusmodi aequatione differentiali ſecundi altiorisue ordinis, in qua nullum differentiale conſtans ſit poſitum, ſtatuatur differentiale dx conſtans; deinde haec aequatio, vti ſupra de expreſſionibus differentialibus oſtendimus, iterum reducatur ad eiusmodi formam, quae nullum differentiale conſtans ſupponat, ſtatuendo ſcilicet $ddy - \frac{dyddx}{dx}$ loco ddy;

&

& $d^3y - \frac{3ddxddy}{dx} + \frac{3dyddx^2}{dx^2} - \frac{dyd^3x}{dx}$ loco d^3y,

&c.

Quo facto diſpiciatur, vtrum aequatio hoc modo reſultans conueniat cum aequatione propoſita; quod ſi eueniat, aequatio propoſita determinatam relationem inter x & y complectetur; ſin autem ſecus accidat, aequatio erit vaga, neque definitam rationem inter variabiles x & y exprimet: quemadmodum hoc iam ante fuſius eſt demonſtratum.

303. Sit, quo hoc plenius explicetur, haec aequatio propoſita, quae nullo differentiali conſtante poſito reperta eſſe perhibeatur.

$$Pddx + Qddy + Rdx^2 + Sdxdy + Tdy^2 = 0.$$

Ponatur dx conſtans, atque ea tranſibit in hanc:

$$Qddy + Rdx^2 + Sdxdy + Tdy^2 = 0.$$

Ex hac nunc iterum conſideratio differentialis conſtantis exuatur, modo ante praeſcripto, & obtinebitur:

$$-\frac{Qdyddx}{dx} + Qddy + Rdx^2 + Sdxdy + Tdy^2 = 0,$$

quae, quoniam a propoſita tantum ratione primi termini diſcrepat, videndum eſt, vtrum ſit $P = -\frac{Qdy}{dx}$. Quod ſi deprehendatur, aequatio propoſita fixam relationem inter x & y exhibebit, quae per regulas in calculo integrali tradendas reperietur, quodcunque differentiale primum

mum conſtans accipiatur. At, ſi fieri nequeat $P = -\frac{Qdy}{dx}$, aequatio propoſita erit impoſſibilis.

304. Niſi igitur haec propoſita aequatio:

$$Pddx + Qddy + Rdx^2 + Sdxdy + Tdy^2 = 0$$

ſit abſurda, neceſſe eſt vt ſit $Pdx + Qdy = 0$, quod duplici modo euenire poteſt: vel enim actu erit

$$P = -\frac{Qdy}{dx}, \text{ ſeu aequatio } Pdx + Qdy = 0$$

identica; vel erit $Pdx + Qdy = 0$ ipſa illa aequatio differentialis primi gradus, ex cuius differentiatione propoſita eſt orta: quo poſteriore caſu aequatio $Pdx + Qdy = 0$ congruet cum propoſita, eandemque relationem inter x & y continebit, ſicque ſine auxilio calculi integralis haec relatio erui poterit. Cum enim ſit $Pdx + Qdy = 0$, erit differentiando

$$Pddx + Qddy + dPdx + dQdy = 0,$$

quae ab aequatione propoſita ſubtracta relinquet:

$$Rdx^2 + Sdxdy + Tdy^2 = dPdx + dQdy.$$

Cum autem ſit $dy = -\frac{Pdx}{Q}$, differentialia prorſus extingui poterunt, naſceturque aequatio finita inter x & y earum relationem indicans.

305. Ponamus in ſolutione problematis nullo differentiali conſtante aſſumto peruentum eſſe ad hanc aequationem:

$$x^3ddx + xxyddy - yydx^2 + xxdy^2 + aadx^2 = 0.$$

Erit

Erit ergo, cum aequationem abſurdum non continere conſtet: $x^3 dx + xxydy = 0$, ſeu $xdx + ydy = 0$: cuius differentiale erit

$$x^3 ddx + xxyddy + 3xxdx^2 + 2xydxdy + xxdy^2 = 0$$

quae aequatio a propoſita ſubtracta relinquit:

$$aadx^2 - yydx^2 - 3xxdx^2 - 2xydxdy = 0, \text{ ſeu}$$
$$aadx - yydx - 3xxdx - 2xydy = 0.$$

Cum autem ſit

$$xdx + ydy = 0; \text{ erit } 2xydy = -2xxdx;$$

ideoque

$$aadx - yydx - xxdx = 0 \text{ ſeu } yy + xx = aa;$$

quae aequatio veram relationem inter x & y exprimit, ſiquidem ea conſentit cum differentiali primum inuenta $xdx + ydy = 0$. Qui conſenſus, niſi ſe manifeſtaſſet, aequatio propoſita pro impoſſibili eſſet habenda; cum autem hoc caſu locum habuerit, aequationem finitam $xx + yy = aa$ ſine calculo integrali elicere licuit.

306. Vt vero etiam exemplum aequationis impoſſibilis afferamus, propoſita ſit haec aequatio:

$$yyddx - xxddy + ydx^2 - xdy^2 + adxdy = 0,$$

in qua nullum differentiale conſtans ſit aſſumtum. Foret ergo $yydx - xxdy = 0$, ideoque differentiando

$$yyddx - xxddy + 2ydxdy + 2xdxdy = 0,$$

quae propoſitae aequalis poſita dabit:

$$ydx^2 - xdy^2 + adxdy = 2ydxdy - 2xdxdy.$$

 Cum

Cum vero ſit $dy = \frac{yydx}{xx}$, extinguendis differentialibus obtinebitur:

$$0 = y - \frac{y^4}{x^3} + \frac{ayy}{xx} = \frac{2y^3}{xx} - \frac{2yy}{x} \quad \text{ſeu}$$

$$x^3 - y^3 + axy = 2xyy - 2xxy,$$

quae vtrum cum differentiali $yydx - xxdy = 0$ conſentiat, eam differentiando, facile patebit, fiet enim:

$$3xxdx - 3yydy + axdy + aydx = 2yydx + 4xydy - 2xxdy - 4xydx$$

$$\text{ſeu} \quad \frac{dy}{dx} = \frac{3xx + ay - 2yy + 4xy}{9yy - ax + 4xy - 2xx},$$

$$\text{at ex illa eſt} \quad \frac{dy}{dx} = \frac{yy}{xx}, \quad \text{foretque ergo}$$

$$3x^4 + 4x^3y + axxy = 3y^4 + 4xy^3 - axyy$$

$$\text{ſeu} \quad axy = \frac{3y^4 + 4xy^3 - 4x^3y - 3x^4}{x + y} =$$

$$= 3y^3 + xyy - xxy - 3x^3.$$

Verum ex aequatione finita primum inuenta eſt

$$axy = y^3 + 2xyy - 2xxy - x^3,$$

quae ab iſta ſubtracta relinquit:

$$0 = 2y^3 - xyy + xxy - 2x^3,$$

quae reſoluitur in has:

$$0 = y - x; \quad \& \quad 2yy + yx + 2xx = 0.$$

Quarum illa $y = x$ quidem cum differentiali $dy = \frac{yydx}{xx}$ conſtare poteſt, at vero aequationi finitae primum inuen-

tae

tae aduerſatur, niſi ſtatuatur $a = 0$, vel niſi vtraque variabilis x & y conſtans ſtatuatur, quo quidem caſu ob $dx = 0$ & $dy = 0$ omnibus aequationibus differentialibus ſatisfit, aequatio propoſita ſubſiſtere nequit.

307. Conſideremus nunc etiam aequationes differentiales tres variabiles x, y, & z inuoluentes, quae erunt vel primi, vel ſecundi, vel altioris gradus. Ad quarum naturam ſcrutandam notari oportet, aequationem finitam tres variabiles complectentem determinare relationem, quam vnaquaeque ad binas reliquas teneat; definitur ergo, qualis functio ſit z ipſarum x & y. Quemadmodum igitur aequatio huiusmodi finita reſoluitur, ſi reperiatur qualis functio ipſarum x & y loco z ſubſtitui debeat, vt aequationi ſatisfiat, ita quoque aequatio differentialis tres variabiles complectens determinabit, qualis functio vna ſit reliquarum; isque huiusmodi aequationem reſoluiſſe cenſendus eſt, qui indicauerit eam binarum variabilium x & y functionem, quae loco tertiae z ſubſtituta aequationi ſatisfaciat, ſeu eam identicam reddat. Aequatio ergo differentialis reſoluitur, ſi vel functio ipſarum x & y valorem ipſius z exhibens definiatur, vel aequatio finita aſſignetur, qua idem debitus ipſius z valor exprimatur.

308. Quanquam autem omnis aequatio differentialis duas tantum variabiles complectens, ſemper determinatam relationem inter eas exprimit; tamen hoc non ſemper euenit in aequationibus differentialibus trium variabilium. Dantur enim eiusmodi aequationes, quibus

pla-

plane nullo modo ſatisfieri poterit, quaecunque functio ipſarum x & y in locum ipſius z ſubſtituatur. Vti ſi propoſita fuerit haec aequatio $zdy = ydx$ facile patet, nullam prorſus dari functionem ipſarum x & y, quae loco z ſubſtituta reddat $zdy = ydx$, differentialia enim dx & dy nullo modo extinguentur. Simili modo apparet nullam dari functionem ipſarum x & z, quae loco y ſubſtituta eidem aequationi ſatisfaciat. Quaecunque enim pro y concipiatur functio ipſarum x & z, in eius differentiali dy ineſt dz, quod quia in aequatione non ineſt, deſtrui non poterit. Hancobrem nulla aequatio finita inter x, y, & z dari poteſt, quae aequationi differentiali $zdy = ydx$ conueniat.

309. Hinc aequationes differentiales tres variabiles continentes diſtribui oportet in imaginarias & reales. Huiusmodi autem aequatio erit imaginaria ſeu abſurda, cui per nullam aequationem finitam ſatisfieri poteſt, cuiusmodi erat illa $zdy = ydx$, quam modo conſideravimus. Aequatio autem erit realis, cui aequiualens aequatio finita exhiberi poteſt, quod euenit, ſi vna variabilis aequalis ſit certae cuipiam functioni binarum reliquarum. Cuiusmodi eſt haec aequatio:

$$zdy + ydz = xdz + zdx + xdy + ydx$$

congruit enim haec cum iſta aequatione finita:

$$yz = xz + xy \quad \text{ſitque} \quad z = \frac{xy}{y - x}.$$

Iſtud ergo diſcrimen inter huiusmodi aequationes imagina-

ginarias & reales diligentiſſime eſt obſeruandum; praecipue in calculo integrali, quia ridiculum foret, cuiuspiam aequationis differentialis velle integralem, hoc eſt aequationem finitam ſatisfacientem quaerere, quae plane nullam habeat.

310. Primum igitur patet, omnes aequationes differentiales trium variabilium, in quibus tantum binarum differentialia occurrant, eſſe imaginarias & abſurdas. Ponamus enim in aequatione, quae contineat variabilem z, tantum ineſſe differentialia dx & dy, differentiale autem dz prorſus abeſſe; atque manifeſtum erit nullam exhiberi poſſe functionem ipſarum x & y, quae loco z ſubſtituta aequationem identicam producat; differentialia enim dx & dy nullo modo tollentur. His ergo caſibus omnino nulla datur aequatio finita ſatisfaciens: niſi forte eiusmodi relatio inter x & y aſſignari queat, quae quicquid ſit z ſubſiſtere poſſit, vti fit in hac aequatione:

$$z\,dy - z\,dx = y\,dy - x\,dx,$$

cui ſatisfacit aequatio $y = x$. Facile autem inueſtigatur, quibus caſibus hoc eueniat, quaerendo relationem inter x & y primo ſi $z = 0$, & tum an iſta relatio aequationi pro quocunque ipſius z valore ſatisfaciat.

311. Neque vero ſolum aequatio tres variabiles inuoluens eſt abſurda, ſi duo tantum continet differentialia, ſed etiam ſi in ea omnia tria differentialia occurrant, talis eſſe poterit. Quos caſus vt euoluamus, ponamus

P & Q eſſe functiones ipſarum x & y tantum, atque haberi hanc aequationem

$$dz = P\,dx + Q\,dy,$$

quae ſi non eſt abſurda, erit z functio quaepiam ipſarum x & y, cuius differentiale ſit

$$dz = p\,dx + q\,dy, \text{ eritque } P = p \;\&\; Q = q.$$

At ſupra demonſtrauimus $p\,dx + q\,dy$ non eſſe poſſe differentiale cuiusquam functionis ipſarum x & y, niſi ſit $\left(\frac{dp}{dy}\right) = \left(\frac{dq}{dx}\right)$, denotante, vti ante aſſumſimus $\left(\frac{dp}{dy}\right)$ differentiale ipſius p poſita ſola y variabili, per dy diuiſum, atque $\left(\frac{dq}{dx}\right)$ differentiale ipſius q, poſita ſola x variabili, diuiſum per dx. Quocirca aequatio $dz = P\,dx + Q\,dy$ realis eſſe nequit, niſi ſit

$$\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right).$$

312. Similis omnino erit ratio huius aequationis

$$dZ = P\,dx + Q\,dy$$

ſi Z denotet functionem quamcunque ipſius z, P vero & Q ſint functiones ipſarum x & y, tertiam variabilem z non complectentes. Vt enim Z aequalis fieri poſſit functioni ipſarum x & y, neceſſe eſt vt ſit $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$. Ex hoc ergo criterio aequatio differentialis quaeque propoſita, quae quidem in hac forma generali contineatur,

diiu-

diiudicari poteſt, vtrum ſit realis an abſurda. Sic patebit hanc aequationem $zdz = ydx + xdy$ eſſe realem, nam ob

$$P = y \;\; \& \;\; Q = x, \text{ fit } \left(\frac{dP}{dy}\right) = 1 = \left(\frac{dQ}{dx}\right) = 1.$$

Haec vero aequatio $azdz = yydx + xxdy$ eſt abſurda, fit enim $\left(\frac{dP}{dy}\right) = 2y$ & $\left(\frac{dQ}{dx}\right) = 2x$; qui valores ſunt inaequales.

313. Vt autem criterum latiſſime patens inueſtigemus, ſint P, Q, & R functiones quaecunque ipſarum x, y, & z; atque omnis aequatio differentialis trium variabilium, ſiquidem ſit primi gradus, continebitur in hac forma:

$$Pdx + Qdy + Rdz = 0.$$

Quoties ergo haec aequatio eſt realis, z aequabitur functioni cuipiam ipſarum x & y; eiusque adeo differentiale erit huius formae $dz = pdx + qdy$. Quare ſi in aequatione propoſita iſta functio ipſarum x & y loco z, & $pdx + qdy$ loco dz ſubſtituatur, neceſſe eſt, vt prodeat aequatio identica $0 = 0$. Atque cum ex aequatione propoſita fiat:

$$dz = -\frac{Pdx}{R} - \frac{Qdy}{R},$$

ſi in P, Q, & R valor ille loco z ſubſtituatur, neceſſe eſt vt fiat

$$p = -\frac{P}{R}, \;\; \& \;\; q = -\frac{Q}{R}.$$

314. Quoniam vero eſt $dz = p\,dx + q\,dy$, erit per ante demonſtrata $\left(\frac{dp}{dy}\right) = \left(\frac{dq}{dx}\right)$. Cum igitur ſubſtituto loco z ipſius valore in x & y ſit

$$p = -\frac{P}{R} \quad \& \quad q = -\frac{Q}{R},$$

erit $$\left(\frac{dp}{dy}\right) = \left(-\frac{R\,dP + P\,dR}{RR\,dy}\right)$$

& $$\left(\frac{dq}{dx}\right) = \left(-\frac{R\,dQ + Q\,dR}{RR\,dx}\right)$$

ideoque habebitur per RR multiplicando haec aequatio:

$$P\left(\frac{dR}{dy}\right) - R\left(\frac{dP}{dy}\right) = Q\left(\frac{dR}{dx}\right) - R\left(\frac{dQ}{dx}\right);$$

vbi denominatores dy & dx iterum indicant, in differentialibus numeratorum eam ſolam quantitatem variabilem aſſumi debere, cuius differentiale denominatorem conſtituit. Haec autem differentialia dP, dQ, dR ante cognoſci non poſſunt, quam in ipſis quantitatibus P, Q, & R valor debitus loco z fuerit ſubſtitutus, qui autem cum ſit incognitus, ſequenti modo erit procedendum.

315. Quia P, Q, & R ſunt functiones ipſarum x, y, & z, ponamus

$$dP = \alpha\,dx + \beta\,dy + \gamma\,dz$$
$$dQ = \delta\,dx + \varepsilon\,dy + \zeta\,dz$$
$$dR = \eta\,dx + \theta\,dy + \iota\,dz$$

vbi

vbi α, β, γ, δ, ϵ, &c. denotant eas functiones, quae ex differentiatione oriuntur. Concipiamus nunc loco z vbique eius valorem in x & y expreſſum ſubſtitui, & loco dz, ponamus valorem $p\,dx + q\,dy$; fietque

$$dP = (\alpha + \gamma p)dx + (\beta + \gamma q)dy$$
$$dQ = (\delta + \zeta p)dx + (\epsilon + \zeta q)dy$$
$$dR = (\eta + \iota p)dx + (\theta + \iota q)dy.$$

Ex his ergo valoribus erit:

$$\left(\frac{dR}{dy}\right) = \theta + \iota q \;;\; \left(\frac{dR}{dx}\right) = \eta + \iota p$$
$$\left(\frac{dP}{dy}\right) = \beta + \gamma q \;;\; \left(\frac{dQ}{dx}\right) = \delta + \zeta p.$$

316. Cum igitur ad realitatem aequationis requiratur, vt ſit:

$$P\left(\frac{dR}{dy}\right) - R\left(\frac{dP}{dy}\right) = Q\left(\frac{dR}{dx}\right) - R\left(\frac{dQ}{dx}\right),$$

fiet ſi inuenti valores ſubſtituantur:

$$P(\theta + \iota q) - R(\beta + \gamma q) = Q(\eta + \iota p) - R(\delta + \zeta p).$$

At ante inuenimus eſſe $p = -\frac{P}{R}$ & $q = -\frac{Q}{R}$ qui valores, cum differentialia non amplius in computum veniant, adhiberi poterunt, etiamſi loco z eius valor in x & y non ſubſtituatur. Eritque ergo

$$P\theta - \frac{PQ\iota}{R} - R\beta + Q\gamma = Q\eta - \frac{PQ\iota}{R} - R\delta + P\zeta$$

ſeu $\quad 0 = P(\zeta - \theta) + Q(\eta - \gamma) + R(\beta - \delta).$

Quia autem quantitates β, δ, γ, η, ζ, θ, per differentiationem inueniuntur, erit ſuperiori notandi modo adhibito:

$$0 = P\left(\frac{dQ}{dz} - \frac{dR}{dy}\right) + Q\left(\frac{dR}{dx} - \frac{dP}{dz}\right) + R\left(\frac{dP}{dy} - \frac{dQ}{dx}\right).$$

Quae proprietas, niſi in aequatione locum habeat, aequatio non erit realis, ſed imaginaria & abſurda.

317. Quanquam hanc regulam ex conſideratione variabilis z elicuimus, tamen quia omnes quantitates aeque ingrediuntur, manifeſtum eſt, & reliquarum conſideratione, eandem expreſſionem prodituram fuiſſe. Propoſita ergo aequatione differentiali primi gradus, quae tres variabiles inuoluat, quacunque, ſtatim diiudicari poterit vtrum ſit realis an imaginaria. Comparetur enim cum hac forma generali:

$$P\,dx + Q\,dy + R\,dz = 0,$$

atque quaeratur valor huius formulae:

$$P\left(\frac{dQ}{dz} - \frac{dR}{dy}\right) + Q\left(\frac{dR}{dx} - \frac{dP}{dz}\right) + R\left(\frac{dP}{dy} - \frac{dQ}{dx}\right),$$

qui ſi fuerit $= 0$, aequatio erit realis, ſin autem non fuerit $= 0$, certum hoc eſt ſignum, aequationem eſſe imaginariam ſeu abſurdam.

318. Aequatio propoſita per diuiſionem quoque ſemper ad huiusmodi formam reduci poteſt:

$$P\,dx + Q\,dy + dz = 0,$$

in

in quam, cum prior abeat ſi fiat $R = 1$, criterium ſimplicius exprimetur, hoc modo:

$$P\left(\frac{dQ}{dz}\right) - Q\left(\frac{dP}{dz}\right) + \left(\frac{dP}{dy}\right) - \left(\frac{dQ}{dx}\right) = 0.$$

Quoties enim haec expreſſio reuera nihilo aequalis reperitur, toties aequatio propoſita erit realis; ſin autem contrarium eueniat, aequatio erit imaginaria. Poſterius quidem ex iis, quae demonſtrauimus, eſt certum; de priori autem adhuc dubitari poſſit, vtrum aequatio ſemper ſit realis, quoties quidem hoc criterium id indicat. Quod cum hoc loco pleniſſime demonſtrari nequeat, ſed in calculo demum integrali demonſtratione confirmari poſſit, hic tantum id affirmamus; neque autem periculum inde eſt metuendum, ſi quis tantisper de eius veritate dubitare voluerit.

319. Ex hoc ergo criterio primum patet, ſi in aequatione $P\,dx + Q\,dy + R\,dz = 0$, fuerit P functio ipſius x, Q functio ipſius y, & R functio ipſius z tantum, aequationem ſemper fore realem. Fit enim

$$\left(\frac{dP}{dy}\right) = 0\ ;\ \left(\frac{dP}{dz}\right) = 0\ ;\ \left(\frac{dQ}{dz}\right) = 0\ ;$$
$$\left(\frac{dQ}{dx}\right) = 0\ ;\ \left(\frac{dR}{dx}\right) = 0\ \&\ \left(\frac{dR}{dy}\right) = 0\ ;$$

ideoque tota expreſſio criterii ſponte euaneſcet.

320. Si fuerit vt ante P ipſius x, & Q ipſius y functio tantum, R autem functio quaecunque ipſarum x, y & z, aequatio erit realis ſi fuerit:

$$P\left(\frac{dR}{dy}\right) = Q\left(\frac{dR}{dx}\right) \quad \text{ſeu} \quad \left(\frac{dR}{dx}\right) : \left(\frac{dR}{dy}\right) = P : Q$$

Sic ſi propoſita fuerit haec aequatio:

$$\frac{2\,dx}{x} + \frac{3\,dy}{y} + \frac{x^2 y^3\,dz}{z^6} = 0.$$

Quia hic eſt $P = \frac{2}{x}$; $Q = \frac{3}{y}$, & $R = \frac{x^2 y^3}{z^6}$

hinc $\left(\frac{dR}{dx}\right) = \frac{2xy^3}{z^6}$; atque $\left(\frac{dR}{dy}\right) = \frac{3xxyy}{z^6}$;

erit $P\left(\frac{dR}{dy}\right) = Q\left(\frac{dR}{dx}\right) = \frac{6xyy}{z^6}$; ideoque aequatio propoſita erit realis.

321. Si fuerint P & Q functiones ipſarum x & y, at R functio ipſius z tantum, ob

$$\left(\frac{dP}{dz}\right) = 0 \;;\; \left(\frac{dQ}{dz}\right) = 0 \;;\; \left(\frac{dR}{dx}\right) = 0 \;\&\; \left(\frac{dR}{dy}\right) = 0,$$

aequatio erit realis ſi fuerit $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$. Haec eadem vero conditio requiritur, ſi $P\,dx + Q\,dy$ debeat eſſe differentiale determinatum, ſeu ex differentiatione cuiuspiam functionis finitae ipſarum x & y ortum. Hucque redit quod ſupra §. 312 iam obſeruauimus, aequatio-

tionem $dZ = Pdx + Qdy$, ſi Z ſit functio ipſius z tantum at P & Q functiones ipſarum x & y, realem eſſe non poſſe, niſi ſit $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$. Ambo autem iſti caſus inter ſe prorſus conueniunt: nam loco $R\,dz$, ſi R eſt functio ipſius z tantum, poni poteſt dZ exiſtente Z functione ipſius z.

322. Vt hoc criterium inuentum exemplo illuſtremus, conſideremus hanc aequationem:

$$(6xy^2z - 5yz^3)\,dx + (5x^2yz - 4xz^3)\,dy$$
$$+ (4x^2y^2 - 6xyz^2)\,dz = 0,$$

qua cum forma generali comparata fit:

$$P = 6xy^2z - 5yz^3\;;\quad \left(\frac{dP}{dy}\right) = 12xyz - 5z^3\;;$$
$$\left(\frac{dP}{dz}\right) = 6xy^2 - 15yz^2$$
$$Q = 5x^2yz - 4xz^3\;;\quad \left(\frac{dQ}{dx}\right) = 10xyz - 4z^3\;;$$
$$\left(\frac{dQ}{dz}\right) = 5x^2y - 12xzz$$
$$R = 4x^2y^2 - 6xyz^2\;;\quad \left(\frac{dR}{dx}\right) = 8xy^2 - 6yz^2\;;$$
$$\left(\frac{dR}{dy}\right) = 8x^2y - 6xz^2.$$

His inuentis valoribus aequatio iudicium continens erit haec:

$$+ (6xy^4z - 5yz^3)(-3xxy - 6xzz)$$
$$+ (5x^2yz - 4xz^3)(2xyy + 9yzz)$$
$$+ (4x^2y^2 - 6xyz^2)(2xyz - z^3) = 0.$$

Haec autem expreſſio ſi euoluatur, omnes termini actu ſe mutuo deſtruunt, fitque $0 = 0$, quod indicat aequationem propoſitam eſſe realem.

323. Quando autem expreſſio hoc modo ex criterio eruta non euaneſcit, tum id ſignum eſt aequationem propoſitam eſſe imaginariam. Quoniam vero hoc pacto ex criterio aequatio finita inuenitur, ea, ſi quidem aequationi differentiali conueniat, ſimul relationem indicabit, quam variabiles inter ſe tenent. Atque hoc modo ii caſus, quorum ſupra meminimus (310), euoluuntur. Sit enim propoſita iſta aequatio:

$$(z - x)\,dx + (y - z)\,dy = 0,$$

fiet $P = z - x$; $Q = y - z$; & $R = 0$,

porro $\left(\frac{dP}{dz}\right) = 1$, & $\left(\frac{dQ}{dz}\right) = -1$.

Aequatio iudicium exhibens fit $P\left(\frac{dQ}{dz}\right) = Q\left(\frac{dP}{dz}\right)$

ſeu $z - x = z - y$; vnde fit $y = x$.

Quoniam igitur hic caſu euenit, vt aequatio $y = x$ ſimul aequationi differentiali ſatisfaciat, dicendum eſt propoſitam aequationem nil aliud ſignificare, niſi eſſe $y = x$.

324.

324. Propoſita ergo aequatione differentiali tres variabiles continente:

$$P\,dx + Q\,dy + R\,dz = 0,$$

tres conſiderandi erunt caſus ſequentes, ad quos haec aequatio deducit:

$$P\left(\frac{dQ}{dz}-\frac{dR}{dy}\right)+Q\left(\frac{dR}{dx}-\frac{dP}{dz}\right)+R\left(\frac{dP}{dy}-\frac{dQ}{dx}\right)=0.$$

Primus eſt ſi haec expreſſio reuera fit $=0$, tumque aequatio propoſita erit realis. Sin autem haec aequatio finita non ſit identica, tum diſpiciendum eſt, vtrum ea aequationi propoſitae ſatisfaciat: quodſi euenit, habebitur aequatio finita, qui eſt caſus ſecundus. Tertius autem caſus locum habet, ſi aequatio finita cum propoſita differentiali ſubſiſtere nequeat, atque tum aequatio propoſita erit imaginaria; neque enim vlla aequatio finita exhiberi poterit, quae ipſi ſatisfaciat.

325. Caſus primus ac tertius per ſe ſunt perſpicui, ſecundus autem, etſi rariſſime occurrit, probe tamen notari meretur: & cum eius exemplum iam ſupra in aequatione, quae duo tantum continet differentialia, exhibuerimus, etiam aequationem afferamus, in qua omnia tria differentialia inſint:

$$(z-y)\,dx + x\,dy + (y-z)\,dz = 0.$$

 Erit

Erit ergo :

$$P = z - y \;;\; \left(\frac{dQ}{dz}\right) = 0 \;;\; \left(\frac{dR}{dy}\right) = 1$$

$$Q = x \;;\; \left(\frac{dR}{dx}\right) = 0 \;;\; \left(\frac{dP}{dz}\right) = 1$$

$$R = y - z \;;\; \left(\frac{dP}{dy}\right) = -1 \;;\; \left(\frac{dQ}{dx}\right) = 1$$

vnde aequatio finita criterium continens euadet:

$$z - x - y = 0, \text{ ſeu } z = x + y$$

ſubſtituatur hic valor pro z in aequatione differentiali fietque $x\,dx + x\,dy - x(dx + dy) = 0$;
quae aequatio, cum ſit identica, ſequitur aequationem differentialem nil aliud ſignificare, niſi $z = x + y$.

326. Quoniam diximus omnes aequationes differentiales primi ordinis, in quibus tres variabiles inſunt contineri in hac forma:

$$P\,dx + Q\,dy + R\,dz = 0,$$

dubium hic naſci poterit circa eas aequationes, in quibus differentialia prima duas pluresue dimenſiones conſtituunt, cuiusmodi eſt haec:

$$P\,dx^2 + Q\,dy^2 + R\,dz^2 = 2S\,dx\,dy + 2T\,dx\,dz + V\,dx\,dz.$$

Ve-

Verum de huiusmodi aequationibus notandum eſt, eas nullo modo reales eſſe poſſe, niſi habeant diuiſores prioris formae, qui propterea aequationes ſimplices conſtituent. Cum enim ex hac aequatione fiat:

$$dz = \frac{-Tdx - Vdy \pm V(dx^2(T^2-PR)+2dxdy(TV+RS)+dy^2(V^2-QR))}{R}$$

facile patet z functioni cuipiam ipſarum x & y, ſeu dz huiusmodi expreſſioni $p\,dx + q\,dy$ aequale fieri non poſſe, niſi quantitas irrationalis euadat rationalis, quod eueniet ſi fuerit:

$$(T^2 - PR)(V^2 - QR) = (TV + RS)^2$$

ſeu

$$R = \frac{PVV + 2STV + QTT}{PQ - SS}.$$

Niſi ergo haec aequatio finita ipſa aequationi propoſitae ſatisfaciat, haec erit imaginaria.

327. Supereſſet vt in hoc Capite quoque aequationes differentiales altiorum ordinum, quae tres variabiles complectuntur, perpenderemus, caſusque definiremus, quibus eae vel reales vel imaginariae euadunt; verum quia criteria nimis fierent intricata, hunc laborem hic praetermittimus, praeſertim, cum ex iisdem fonti-

bus, quos hic aperuimus, ſequantur. Ceterum ſi in calculo integrali his criteriis erit opus, tum ea facile erui poterunt. Ob eandem cauſam hic quoque aequationes, quae plures variabiles complectuntur, non contemplamur, cum fere nunquam occurrant, atque, ſi vnquam occurrerent, ex principiis hic traditis ſine negotio examinari poſſent. Quare his expoſitis Inſtitutioni Calculi Differentialis hic finem imponimus progreſſuri ad inſignes vſus oſtendendos, quos iſte calculus cum in ipſa Analyſi, tum in Geometria ſublimiori affert.

INSTI-

INSTITUTIONUM CALCULI DIFFERENTIALIS

PARS POSTERIOR

CONTINENS

VSUM HUIUS CALCULI IN ANALYSI FINITORUM, NEC NON IN DOCTRINA SERIERUM.

CAPUT I.

DE TRANSFORMATIONE SERIERUM.

1.

Cum nobis propoſitum ſit vſum Calculi differentialis tam in vniuerſa Analyſi, quam in doctrina de ſeriebus oſtendere; nonnulla ſubſidia ex Algebra communi, quae vulgo tractari non ſolent, hic erunt repetenda. Quae quamuis maximam partem iam in Introductione ſumus complexi, tamen quaedam ibi ſunt praetermiſſa, vel ſtudio quod expediat ea tum demum explicari, quando neceſſitas id exigat, vel quia cuncta, quibus opus ſit futurum, praeuideri non poterant. Huc pertinet transformatio ſerierum, cui hoc Caput deſtinauimus, qua quaeuis ſeries in innumerabiles alias ſeries transmutatur, quae omnes eandem habeant ſummam communem; ita vt, ſi ſeriei propoſitae ſumma ſit cognita, reliquae ſeries omnes ſimul ſummari queant. Hoc autem capite praemiſſo, eo vberius doctrinam ſerierum per calculum differentialem & integralem amplificare poterimus.

2. Conſiderabimus autem potiſſimum eiusmodi ſeries, quarum ſinguli termini per poteſtates ſucceſſiuas quantitatis cuiusdam indeterminatae ſunt multiplicati: quo-

niam hae latius patent, maioremque vtilitatem afferent.

Sit igitur propoſita ſequens ſeries generalis, cuius ſummam, ſiue ſit cognita ſiue ſecus, ponamus $= S$, ſitque

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \&c.$$

Ponatur iam $x = \frac{y}{1+y}$, & cum ſit per ſeries infinitas

$$\begin{aligned} x &= y - y^2 + y^3 - y^4 + y^5 - y^6 + \&c. \\ x^2 &= y^2 - 2y^3 + 3y^4 - 4y^5 + 6y^6 - 7y^7 + \&c. \\ x^3 &= y^3 - 3y^4 + 6y^5 - 10y^6 + 15y^7 - 21y^8 + \&c. \\ x^4 &= y^4 - 4y^5 + 10y^6 - 20y^7 + 35y^8 - 56y^9 + \&c. \end{aligned}$$
&c.

hi valores ſubſtituti, ſerieque ſecundum poteſtates ipſius y diſpoſita, dabunt

$$\begin{array}{llllll} S = ay & - ay^2 & + ay^3 & - ay^4 & + ay^5 & \&c. \\ & + b & - 2b & + 3b & - 4b & \\ & & + c & - 3c & + 6c & \\ & & & + d & - 4d & \\ & & & & + e. & \end{array}$$

3. Quoniam poſuimus $x = \frac{y}{1+y}$; erit $y = \frac{x}{1-x}$; quo valore loco y ſubſtituto, ſeries propoſita

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \&c.$$

transmutabitur in hanc:

$$S = a.\frac{x}{1-x} + (b-a)\frac{x^2}{(1-x)^2} + (c-2b+a)\frac{x^3}{(1-x)^3} + \&c.$$

in

in qua coefficiens ſecundi termini $b - a$ eſt differentia prima ipſius a ex ſerie a, b, c, d, e, &c. quam ſupra per Δa expoſuimus; coefficiens tertii termini $c - 2b + a$ eſt differentia ſecunda $\Delta^2 a$; coefficiens quarti eſt differentia tertia $\Delta^3 a$, &c. Hinc differentiis ipſius a continuis, quae formantur ex ſerie a, b, c, d, e, &c. adhibendis propoſita Series transmutabitur in hanc

$$S = \frac{x}{1-x} a + \frac{x^2}{(1-x)^2} \Delta a + \frac{x^3}{(1-x)^3} \Delta^2 a + \frac{x^4}{(1-x)^4} \Delta^3 a + \&c.$$

cuius ergo ſeriei ſumma habebitur, ſi propoſitae ſumma fuerit cognita.

4. Si igitur ſeries a, b, c, d, &c. ita fuerit comparata, vt tandem differentias habeat conſtantes, quod euenit, ſi eius terminus generalis fuerit functio rationalis integra, tum ſeries poſterior $\frac{x}{1-x} a + \frac{x^2}{(1-x)^2} \Delta a + \&c.$ tandem habebit terminos euaneſcentes, ſicque eius ſumma per expreſſionem finitam exhiberi poterit. Ita ſi ſeriei a, b, c, d, &c. differentiae primae iam fuerint conſtantes, tum ſeriei huius:

$$ax + bx^2 + cx^3 + dx^4 + \&c.$$

ſumma erit $= \frac{x}{1-x} a + \frac{x^2}{(1-x)^2} \Delta a$. At ſi illius ſeriei coefficientium differentiae ſecundae fiant conſtantes, tum ipſius ſeriei propoſitae erit

$$= \frac{x}{1-x} a + \frac{x^2}{(1-x)^2} \Delta a + \frac{x^3}{(1-x)^3} \Delta\Delta a.$$

 Vnde

Vnde ſummae huiusmodi ſerierum ex differentiis coefficientium facile inuenientur.

I. *Quaeratur ſumma huius ſeriei:*

$$1x + 3x^2 + 5x^3 + 7x^4 + 9x^5 + \&c.$$

Diff. I. 2, 2, 2, 2, &c.

Cum ergo differentiae primae ſint conſtantes, ob $a = 1$ & $\Delta a = 2$, erit ſeriei propoſitae ſumma

$$= \frac{x}{1-x} + \frac{2xx}{(1-x)^2} = \frac{x+xx}{(1-x)^2}.$$

II. *Quaeratur ſumma huius ſeriei:*

$$1x + 4xx + 9x^3 + 16x^4 + 25x^5 + \&c.$$

Diff. I. 3, 5, 7, 9, &c.

Diff. II. 2, 2, 2, &c.

Quia itaque eſt

$$a = 1; \quad \Delta a = 3; \quad \Delta^2 a = 2;$$

erit ſeriei propoſitae ſumma

$$= \frac{x}{1-x} + \frac{3xx}{(1-x)^2} + \frac{2x^3}{(1-x)^3} = \frac{x+xx}{(1-x)^3}.$$

III. *Quaeratur ſumma huius ſeriei:*

$$S = 4x + 15x^2 + 40x^3 + 85x^4 + 156x^5 + 259x^6 + \&c.$$

Diff. I. 11, 25, 45, 71, 103

Diff. II. 14, 20, 26, 32

Diff. III. 6, 6, 6,

Quia

Quia eſt

$$a = 4;\quad \Delta a = 11;\quad \Delta^2 a = 14;\quad \Delta^3 a = 6;$$

erit ſumma

$$S = \frac{4x}{1-x} + \frac{11xx}{(1-x)^2} + \frac{14x^3}{(1-x)^3} + \frac{6x^4}{(1-x)^4},$$

ſiue

$$S = \frac{4x - xx + 4x^3 - x^4}{(1-x)^4} = \frac{x(1+xx)(4-x)}{(1-x)^4}.$$

5. Quanquam hoc modo iſtarum ſerierum in infinitum progredientium ſummae inueniuntur; tamen ex iisdem principiis hae Series quoque ad datum quemuis terminum ſummari poſſunt. Propoſita enim ſit haec ſeries

$$S = ax + bx^2 + cx^3 + dx^4 + \ldots\ldots + ox^n,$$

& quaeratur eius ſumma, ſi in infinitum progrediatur, quae erit $= \frac{x}{1-x}a + \frac{x^2}{(1-x)^2}\Delta a + \frac{x^3}{(1-x)^3}\Delta^2 a + \&c.$ Nunc conſiderentur eiusdem ſeriei termini poſt vltimum ox^n ſequentes, qui ſint

$$px^{n+1} + qx^{n+2} + rx^{n+3} + sx^{n+4} + \&c.$$

cuius ſeriei, ſi per x^n diuidatur, ſumma, vt ante inueniri poterit; quae rurſus per x^n multiplicata erit

$$\frac{x^{n+1}}{1-x}p + \frac{x^{n+2}}{(1-x)^2}\Delta p + \frac{x^{n+3}}{(1-x)^3}\Delta^2 p + \&c.$$

quae ſumma ſi a totius ſeriei in infinitum continuatae

 ſum-

ſumma ſubtrahatur, remanebit ſumma portionis propoſitae quaeſita: S =

$$\frac{x}{1-x}(a-x^n p)+\frac{x^2}{(1-x)^2}(\Delta a-x^n\Delta p)+\frac{x^3}{(1-x)^3}(\Delta^2 a-x^n\Delta^2 p)$$
&c.

I. *Quaeratur ſumma huius ſeriei finitae.*

$$S=1x+2x^2+3x^3+4x^4+\ldots\ldots+nx^n.$$

Tam horum coefficientium, quam terminum vltimum ſequentium quaerantur differentiae:

1, 2, 3, 4, &c.	$n+1$, $n+2$, $n+3$, &c.
1, 1, 1	1, 1;

eritque

$$a=1,\ \Delta a=1,\ p=n+1,\ \Delta p=1,$$

unde ſumma quaeſita eſt:

$$S=\frac{x}{1-x}(1-(n+1)x^n)+\frac{x^2}{(1-x)^2}(1-x^n),$$

ſeu

$$S=\frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}.$$

II. *Quaeratur ſumma huius ſeriei finitae.*

$$S=1x+4x^2+9x^3+16x^4+\ldots\ldots+n^2x^n.$$

Inueſtigentur primum differentiae hoc modo:

1, 4, 9, 16, &c.	$(n+1)^2$, $(n+2)^2$, $(n+2)^2$, &c.
3, 5, 7	$2n+3$, $2n+5$
2, 2,	2

qui-

quibus inuentis erit ſumma quaeſita S $=$

$$\frac{x}{1-x}(1-(n+1)^2x^n)+\frac{x^2}{(1-x)^2}(3-(2n+3)x^n)+\frac{x^3}{(1-x)^3}(2-2x^n),$$

ſeu S $=$

$$\frac{x+xx-(n+1)^2x^{n+1}+(2nn+2n-1)x^{n+2}-nnx^{n+3}}{(1-x)^3}.$$

6. Quodſi autem ſeries propoſita non eiusmodi habeat coefficientes, qui tandem ad differentias conſtantes deducantur, tum transmutatio hic exhibita nihil confert ad eius ſummam determinandam. Neque vero etiam eius ope ſumma proxime definiri poterit commodius, quam per ipſam ſeriei propoſitae additionem fieri licet. Si enim in ſerie $ax+bx^2+cx^3+dx^4+$ &c. fuerit $x<1$ quo ſolo caſu ſummatio proprie ſic dicta locum habere poteſt, erit $\frac{x}{1-x}>x$, ideoque non ſeries minus convergit quam propoſita. Sin autem in ſerie propoſita fuerit $x=1$ tum nouae ſeriei omnes plane termini fiunt infiniti, quo ergo caſu iſta transmutatio nullius prorſus erit vſus.

7. Conſideremus autem ſeriem, in qua ſigna $+$ & $-$ alternatim ſe excipiant, quae ex praecedente deducetur ponendo x negatiuum. Si itaque fuerit

$$S=ax-bx^2+cx^3-dx^4+ex^5-\&c.$$

cuius ſeriei negatiua oritur, ſi in praecedente ſtatua-

tur

tur x negatiuum. Sumantur ergo vt ante differentiae Δa, $\Delta^2 a$, $\Delta^3 a$, &c. ex ſerie coefficientium a, b, c, d, e, &c. ſignis ad ſolas ipſius x poteſtates relatis, atque ſeries propoſita transformabitur in hanc: S $=$

$$\frac{x}{1+x}a - \frac{x^2}{(1+x)^2}\Delta a + \frac{x^3}{(1+x)^3}\Delta^2 a - \frac{x^4}{(1+x)^4}\Delta^3 a + \&c.$$

vnde perſpicitur aequationem propoſitam iisdem caſibus ſummari poſſe quibus praecedens. Scilicet ſi ſeries a, b, c, d, &c. tandem ad differentias conſtantes deducatur.

8. Hoc autem caſu iſta transformatio commodam praebet approximationem ad valorem ſeriei propoſitae:

$$ax - bx^2 + cx^3 - dx^4 + ex^5 - fx^6 + \&c.$$

quantuscunque enim x fuerit numerus, fractio $\frac{x}{1+x}$, ſecundum cuius poteſtates altera ſeries progreditur, fit vnitate minor: atque ſi ſit $x = 1$, erit $\frac{x}{1+x} = \frac{1}{2}$. Sin autem ſit $x < 1$ puta $x = \frac{1}{n}$ fiet $\frac{x}{1+x} = \frac{1}{n+1}$, ideoque ſeries per transformationem inuenta ſemper magis conuergit quam propoſita. Conſideremus imprimis caſum, quo $x = 1$, qui ad ſeries ſummandas ingens affert ſubſidium, ſitque

$$S = a - b + c - d + e - f + \&c.$$

ac denotentur differentiae primae, ſecundae & ſequentes ipſius a, quas progreſſio a, b, c, d, e, &c. praebet

per

per Δ, $\Delta^2 a$, $\Delta^3 a$, &c. quibus inuentis erit

$$S = \tfrac{1}{2} a - \tfrac{1}{4} \Delta a + \tfrac{1}{8} \Delta^2 a - \tfrac{1}{16} \Delta^3 a + \&c.$$

quae niſi actu terminatur, ſummam vero proximam ſatis commode exhibet.

9. Vſum igitur huius vltimae transmutationis, qua ſumſimus $x = 1$, in aliquot exemplis oſtendamus, ac primo quidem in eiusmodi, quibus vera ſumma finite exprimi poteſt. Tales ſunt ſeries diuergentes, quibus numeri a, b, c, d, &c. tandem ad differentias conſtantes deducant, quarum ſummae, cum recepto huius vocis ſignificatu, proprie non exhiberi queant, vocem ſummae hic eo ſenſu, quem ſupra tribuimus, accipimus, ita vt denotet valorem expreſſionis finitae, ex cuius euolutione propoſita ſeries naſcatur.

I. *Sit igitur propoſita haec ſeries Leibnitzii:*

$$S = 1 - 1 + 1 - 1 + 1 - 1 + \&c.$$

in qua cum omnes termini ſint aequales, fient omnes differentiae $= 0$, ideoque ob $a = 1$, erit $S = \tfrac{1}{2}$.

II. *Sit propoſita iſta ſeries:*

$$S = 1 - 2 + 3 - 4 + 5 - 6 + \&c.$$

Diff. 1 = 1, 1, 1, 1, 1, &c,

Cum ergo ſit $a = 1$, $\Delta a = 1$, erit $S = \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{1}{4}$.

III. *Sit propoſita haec ſeries:*

$$S = 1 - 3 + 5 - 7 + 9 - \&c.$$

Diff. 1 = 2, 2, 2, 2, &c.

Ob $a = 1$ & $\Delta a = 2$ fit $S = \tfrac{1}{2} - \tfrac{2}{4} = 0$.

IV. *Sit propoſita haec ſeries trigonalium numerorum.*

$$S = 1 - 3 + 6 - 10 + 15 - 21 + \&c.$$

Diff. 1 = 2, 3, 4, 5, 6, &c.

Diff. 2 = 1, 1, 1, 1, &c.

Hic ergo ob $a = 1$, $\Delta a = 2$, & $\Delta\Delta a = 1$; erit

$$S = \tfrac{1}{2} - \tfrac{2}{4} + \tfrac{1}{8} = \tfrac{1}{8}.$$

V. *Sit propoſita ſeries quadratorum:*

$$S = 1 - 4 + 9 - 16 + 25 - 36 + \&c.$$

Diff. 1 = 3, 5, 7, 9, 11, &c.

Diff. 2 = 2, 2, 2, 2, &c.

Ob $a = 1$; $\Delta a = 3$; $\Delta\Delta a = 2$; erit $S = \tfrac{1}{2} - \tfrac{3}{4} + \tfrac{2}{8} = 0$.

VI. *Sit propoſita ſeries biquadratorum:*

$$S = 1 - 16 + 81 - 256 + 625 - 1296 + \&c.$$

Diff. 1 = 15, 65, 175, 369, 671

Diff. 2 = 50, 110, 194, 302

Diff. 3 = 60, 84, 108

Diff. 4 = 24, 24

Erit ergo $S = \tfrac{1}{2} - \tfrac{15}{4} + \tfrac{50}{8} - \tfrac{60}{16} + \tfrac{24}{32} = 0$.

10. Si ſeries magis diuergant vti geometriae aliaeque ſimiles, eae hoc modo ſtatim in ſeriem magis convergentem transmutantur, quae niſi adhuc ſatis conuergat, eodem modo in aliam magis conuergentem convertetur.

I. *Sit*

I. *Sit propoſita haec ſeries geometrica:*

$$S = 1 - 2 + 4 - 8 + 16 - 32 + \&c.$$

Diff. 1 = 1, 2, 4, 8, 16, &c.
Diff. 2 = 1, 2, 4, 8, &c.
Diff. 3 = 1, 2, 4, &c.

Cum igitur in omnibus differentiis primus terminus ſit $= 1$. Summa ſeriei exprimetur hoc modo

$$S = \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{8} - \tfrac{1}{16} + \tfrac{1}{32} - \tfrac{1}{64} + \&c.$$

cuius ſumma eſt $= \frac{1}{3}$, oritur enim ex euolutione fractionis $\frac{1}{2+1}$, dum propoſita oritur ex $\frac{1}{1+2}$.

II. *Sit propoſita haec ſeries recurrens:*

$$S = 1 - 2 + 5 - 12 + 29 - 70 + 169 - \&c.$$

Diff. 1 = 1, 3, 7, 17, 41, 99 &c.
Diff. 2 = 2, 4, 10, 24, 58 &c.
Diff. 3 = 2, 6, 14, 34 &c.
Diff. 4 = 4, 8, 20 &c.
Diff. 5 = 4, 12 &c.
Diff. 6 = 8 &c.
&c.

Continuarum ergo differentiarum termini primi conſtituunt hanc progreſſionem geometricam geminatam:

1, 1, 2, 2, 4, 4, 8, 8, 16, 16, &c.

vnde erit

$$S = \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{2}{8} - \tfrac{2}{16} + \tfrac{4}{32} - \tfrac{4}{64} + \tfrac{8}{128} \&c.$$

 cum

cum igitur praeter primum terminum reliqui bini ſe continuo deſtruant, erit $S = \frac{1}{2}$. Oritur autem ſeries propoſita ex euolutione fractionis $\frac{1}{1+2-1} = \frac{1}{2}$, vti in expreſſione naturae ſerierum recurrentium oſtendimus.

III. *Sit propoſita ſeries hypergeometrica:*

$$S = 1 - 2 + 6 - 24 + 120 - 720 + 5040 - \&c.$$

cuius differentias continuas hoc modo commodius inueſtigabimus:

	Diff. 1.	Diff. 2.	Diff. 3.	
1	1	3	11	
2	4	14	64	
6	18	78	426	
24	96	504	3216	&c.
120	600	3720	27240	
720	4320	30960	256320	
5040	35280	287280	2656080	
40320	322560	2943360		
362880	3265920			
3628800				

Quibus differentiis vlterius continuatis erit:

$$S = \frac{1}{2} - \frac{1}{4} + \frac{3}{8} - \frac{11}{16} + \frac{53}{32} - \frac{309}{64} + \frac{2119}{128} - \frac{16687}{256} + \frac{148329}{512} - \frac{1468457}{1024} + \frac{16019531}{2048} - \frac{190899411}{4096} + \&c.$$

Col-

Colligantur duo termini initiales, eritque $S = \frac{1}{4} + A$ exiſtente

$$A = \frac{3}{8} - \frac{11}{16} + \frac{53}{32} - \frac{309}{64} + \frac{2119}{128} - \&c.$$

Si nunc eodem modo differentiae capiantur, erit

$$A = \frac{3}{2^4} - \frac{5}{2^6} + \frac{21}{2^8} - \frac{99}{2^{10}} + \frac{615}{2^{12}} - \frac{4401}{2^{14}} + \frac{36585}{2^{16}}$$
$$- \frac{342207}{2^{18}} + \frac{3565323}{2^{20}} - \frac{40866525}{2^{22}} + \&c.$$

Colligantur duo termini initiales, quia conuergunt, fietque

$$A = \frac{7}{2^6} + B \quad \text{exiſtente} \quad B = \frac{21}{2^8} - \frac{99}{2^{10}} + \&c.$$

cuius ſeriei differentiis denuo ſumendis fiet:

$$B = \frac{21}{29} - \frac{15}{2^{12}} + \frac{159}{2^{15}} - \frac{429}{2^{18}} + \frac{5241}{2^{21}} - \frac{26283}{2^{24}}$$
$$+ \frac{338835}{2^{27}} - \frac{2771097}{2^{30}} + \&c.$$

Colligantur quatuor termini initiales in vnum & ſtatuatur $B = \frac{153}{2^{12}} + \frac{843}{2^{18}} + C$ exiſtente $C = \frac{5241}{2^{21}} - \frac{26283}{2^{24}} + \&c.$ fietque aliquot terminis actu colligendis proxime: $C = \frac{15645}{2^{24}} - \frac{60417}{2^{30}}$. Ex his ergo tandem concludetur ſumma ſeriei propoſitae: $S = 0,40082038$, quae tamen vix vltra tres quatuorue figuras pro accurata haberi

beri poteſt ob nimiam ſeriei diuergentiam; eſt tamen certe iuſto minor. Aliunde enim inueni hanc ſummam eſſe $= 0,4036524077$, vbi ne vltima quidem nota a vero aberrat.

11. Imprimis autem haec transmutatio ingentem affert vtilitatem ad ſeries iam quidem, ſed lente conuergentes in alias, quae multo promtius conuergant, transmutandas. Quoniam vero termini ſequentes minores ſunt quam praecedentes, differentiae primae fiunt negatiuae; vnde in ſequentibus ſignorum ratio probe eſt habenda.

I. *Sit propoſita haec ſeries:*

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \&c.$$

$$\text{Diff. } 1 = -\frac{1}{2};\ -\frac{1}{2.3};\ -\frac{1}{3.4};\ -\frac{1}{4.5};\ -\frac{1}{5.6}$$

$$\text{Diff. } 2 = +\frac{1}{3};\ \frac{2}{2.3.4};\ \frac{2}{3.4.5};\ \frac{2}{4.5.6}$$

$$\text{Diff. } 3 = -\frac{1}{4};\ -\frac{2.3}{2.3.4.5};\ -\frac{2.3}{3.4.5.6}$$

$$\text{Diff. } 4 = +\frac{1}{5};\ \&c.$$

Hinc ergo erit

$$S = \frac{1}{2} + \frac{1}{2.4} + \frac{1}{3.8} + \frac{1}{4.16} + \frac{1}{5.32} + \&c.$$

vtramque autem hanc ſeriem logarithmum hyperbolicum binarii exhibere, iam in Introductione oſtendimus.

II. *Sit*

II. *Sit propoſita iſta ſeries pro circulo :*

$$S = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \text{ \&c.}$$

$$\text{Diff. } 1 = -\frac{2}{1.3};\ -\frac{2}{3.5};\ -\frac{2}{5.7};\ -\frac{2}{7.9};\ -\frac{2}{9.11} \text{ \&c.}$$

$$\text{Diff. } 2 = +\frac{2.4}{1.3.5};\ \frac{2.4}{3.5.7};\ \frac{2.4}{5.7.9};\ \frac{2.4}{7,9.11} \text{ \&c.}$$

$$\text{Diff. } 3 = -\frac{2.4.6}{1.3.5.7};\ -\frac{2.4.6}{3.5.7.9};\ - \text{ \&c.}$$

&c.

Hinc ergo concluditur fore ſummam ſeriei :

$$S = \frac{1}{2} + \frac{1}{3.2} + \frac{1.2}{3.5.2} + \frac{1.2.3}{3.5.7.2} + \text{ \&c.}$$

ſeu

$$2S = 1 + \frac{1}{3} + \frac{1.2}{3.5} + \frac{1.2.3}{3.5.7} + \frac{1.2.3.4}{3.5.7.9} + \text{ \&c.}$$

III. *Quaeratur valor huius ſeriei infinitae :*

$$S = l2 - l3 + l4 - l5 + l6 - l7 + l8 - l9 \text{ \&c.}$$

Quia differentiae ab initio nimis fiunt inaequales, colligantur actu termini vsque ad $l10$ ex tabulis, quorum valor reperietur $= -0,3911005$; eritque

$$S = -0,3911005 + l10 - l11 + l12 - l13 + l14 - l15 + \text{ \&c.}$$

in infinitum.

De

Deſumantur hi logarithmi ex tabulis, eorumque differentiae quaerantur hoc modo

	Diff. 1.	Diff. 2.	Diff. 3.	Diff. 4.	Diff. 5.
$l10 = 1,0000000$	+	—	+	—	+
$l11 = 1,0413927$	413927	36042	5779	1292	368
$l12 = 1,0791812$	377885	30263	4487	924	
$l13 = 1,1139434$	347622	25776	3563		
$l14 = 1,1461280$	321846	22213			
$l15 = 1,1760913$	299633				

Ex quibus reperitur

$$l10 - l11 + l12 - l13 + \&c. =$$

$$\frac{1,0000000}{2} - \frac{413927}{4} - \frac{36042}{8} - \frac{5779}{16} - \frac{1292}{32} - \frac{368}{64}$$

$$= 0,4891606.$$

Hinc valor ſeriei propoſitae erit

$$S = l2 - l3 + l4 - l5 + \&c. = 0,0980601;$$

cui logarithmo reſpondet numerus $1,253315$.

12. Quemadmodum has transmutationes obtinuimus ponendo in ſerie loco x hanc fractionem $\frac{y}{1 \pm y}$, ita innumerabiles aliae transmutationes orientur, ſi loco x aliae functiones ipſius y ſubſtituantur. Sit iterum propoſita iſta ſeries:

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + fx^6 + \&c.$$

atque ponatur $x = y(1 - y)$, quo facto ſeries orietur ſequens

S =

$$S = ay - ayy$$
$$+ byy - 2by^3 + by^4$$
$$+ cy^3 - 3cy^4 + 3cy^5 - cy^6$$
$$+ dy^4 - 4dy^5 + 6dy^6$$
$$+ ey^5 - 5ey^6 \quad \&c.$$
$$+ fy^6$$

Quodſi ergo altera harum ſerierum fuerit ſummabilis, ſimul alterius ſumma erit cognita. Ita ſi ſtatuatur

$S = x + x^2 + x^3 + x^4 + x^5 + \&c. = \frac{x}{1-x}$, erit

$S = y - y^3 - y^4 + y^6 + y^7 - y^9 - y^{10} + \&c.$

Cuius ergo ſeriei ſumma erit $= \frac{y - yy}{1 - y + yy}$.

13. Si altera ſeries alicubi abrumpatur, tum ſumma prioris abſolute exhiberi poterit. Ponamus eſſe $a = 1$, & in ſerie inuenta omnes terminos poſt primum euaneſcere, vt ſit $S = y$; ideoque ob $x = y - yy$, erit ſumma prioris $= \frac{1}{2} - V(\frac{1}{4} - x)$.

Fiet autem ob $a = 1$; vt ſequitur:

$$b = 1 = \frac{1}{4} . 2^2$$

$$c = 2 = \frac{1.3}{4.6} . 2^4$$

$$d = 5 = \frac{1.3.5}{4.6.8} . 2^6$$

$$e = 14 = \frac{1.3.5.7}{4.6.8.10} 2^8$$

$$f = 42 = \frac{1.3.5.7.9}{4.6.8.10.12} 2^{10}$$

$$g = 132 = \frac{1.3.5.7.9.11}{4.6.8.10.12.14} 2^{12} \quad \&c.$$

Pp vnde

vnde prior ſeries abibit in hanc: $S =$ $\frac{1}{2} - \sqrt{(\frac{1}{4} - x)} = x + x^2 + 2x^3 + 5x^4 + 14x^5 + 42x^6 + 132x^7 + \&c.$ quae eadem inuenitur, ſi quantitas ſurda $\sqrt{(\frac{1}{4} - x)}$ in ſeriem euoluatur, atque ab $\frac{1}{2}$ ſubtrahatur.

14. Statuamus, quo transmutatio latius pateat $x = y(1 + ny)^{\nu}$, atque ſeries propoſita:

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \&c.$$

transmutabitur in ſequentem:

$$\begin{array}{llllll}
S = ay & + \frac{\nu}{1} nay^2 & & & & \\
 & + by^2 & & & & \\
 & & + \frac{\nu(\nu-1)}{1.2} n^2 ay^3 & + \frac{\nu(\nu-1)(\nu-2)}{1.2.3} n^3 ay^4 & + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{1.2.3.4} n^4 ay^5 & \\
 & & + \frac{2\nu}{1} nby^3 & + \frac{2\nu(2\nu-1)}{1.2} n^2 by^4 & + \frac{2\nu(2\nu-1)(2\nu-2)}{1.2.3} n^3 by^5 & \\
 & & + cy^3 & + \frac{3\nu}{1} ncy^4 & + \frac{3\nu(3\nu-1)}{1.2} n^2 cy^5 & \\
 & & & + dy^4 & + \frac{4\nu}{1} ndy^5 & \\
 & & & & + ey^5 & \\
 & & & \&c. & &
\end{array}$$

Si ergo illius ſeriei ſumma fuerit cognita, & huius ſimul ſumma habebitur, ac viciſſim. Quoniam vero n & ν pro lubitu accipi poſſunt, hinc ex vna ſerie ſummabili innumerae aliae ſummabiles inueniri poſſunt.

15. Poſſunt etiam eiusmodi transmutationes fieri, vt ſeriei inuentae ſumma fiat irrationalis, hoc modo.

Sit

Sit propoſita iſta ſeries:

$$S = ax + bx^3 + cx^5 + dx^7 + ex^9 + fx^{11} + \&c.$$

erit

$$Sx = ax^2 + bx^4 + cx^6 + dx^8 + ex^{10} + fx^{12} + \&c.$$

Iam ſtatuatur

$$x = \frac{y}{\sqrt{(1-nyy)}}; \text{ erit } xx = \frac{y}{1-nyy},$$

atque ſeries propoſita transmutabitur in hanc:

$$\frac{Sy}{\sqrt{(1-nyy)}} =$$

$$\begin{array}{l} ay^2 + nay^4 + n^2ay^6 + n^3ay^8 + n^4ay^{10} + \&c. \\ \quad + by^4 + 2nby^6 + 3n^2by^8 + 4n^3by^{10} + \&c. \\ \quad\quad + cy^6 + 3ncy^8 + 6n^2cy^{10} + \&c. \\ \quad\quad\quad + dy^8 + 4ndy^{10} + \&c. \\ \quad\quad\quad\quad + ey^{10} + \&c. \end{array}$$

Si igitur ſumma S ex priori ſerie fuerit cognita, habebitur ſimul ſumma ſequentis ſeriei:

$$\frac{S}{\sqrt{(1-nyy)}} =$$

$$ay + (na+b)y^3 + (n^2a + 2nb + c)y^5 + (n^3a + 3n^2b + 3nc + d)y^7 + \&c.$$

16. Si ſumatur $n = -1$; erunt coefficientes huius ſeriei differentiae continuae ipſius a, ex ſerie a, b, c, d, &c. ſin autem ſigna in ſerie propoſita alternentur, tum poſito $n = 1$, coefficientes erunt iſtae differentiae. Denotent ergo Δa, $\Delta^2 a$, $\Delta^3 a$, $\Delta^4 a$, &c.

 diffe-

differentias primas, ſecundas, tertias, &c. ipſius a ex ſerie numerorum $a,\ b,\ c,\ d,\ e,\ f,$ &c.

Ac ſi fuerit:

$$S = ax + bx^3 + cx^5 + dx^7 + ex^9 + \&c.$$

poſito $x = \frac{y}{\sqrt{(1+yy)}}$; erit

$$\frac{S}{\sqrt{(1+yy)}} = ay + \Delta a.y^3 + \Delta^2 a.y^5 + \Delta^3 a.y^7 + \&c.$$

Sin autem fuerit:

$$S = ax - bx^3 + cx^5 - dx^7 + ex^9 - \&c.$$

ponaturque $x = \frac{y}{\sqrt{(1-yy)}}$; erit

$$\frac{S}{\sqrt{(1-yy)}} = ay - \Delta a.y^3 + \Delta^2 a.y^5 - \Delta^3 a.y^7 + \&c.$$

Quodſi ergo ſeries $a,\ b,\ c,\ d,\ e,$ &c. tandem ad differentias conſtantes deducat, tum vtraque ſeries abſolute ſummari poterit; quae ſummatio autem quoque ex ſuperioribus ſequitur.

17. Ponamus coefficientes $a,\ b,\ c,\ d,$ &c. conſtituere hanc ſeriem $1,\ \frac{1}{3},\ \frac{1}{5},\ \frac{1}{7},\ \frac{1}{9},$ &c. eritque, vti ſupra iam vidimus:

$$a = 1$$

$$\Delta a = -\frac{2}{3}$$

$$\Delta^3 a = \frac{2.4}{3.5}$$

$$\Delta^3 a = -\frac{2.4.6}{3.5.7} \quad \&c.$$

vnde

vnde ſequentes duas ſeries ſummabimus:

I. Sit $S = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \&c.$

Erit $S = \frac{1}{2}l\frac{1+x}{1-x}$. Poſito iam $x = \frac{y}{\sqrt{(1+yy)}}$, fiet

$$S = \frac{1}{2}l\frac{\sqrt{(1+yy)}+y}{\sqrt{(1+yy)}-y} = l(\sqrt{(1+yy)}+y):$$

vnde erit

$$\frac{l(\sqrt{(1+yy)}+y)}{\sqrt{(1+yy)}} = y - \frac{2}{3}y^3 + \frac{2.4}{3.5}y^5 - \frac{2.4.6}{3.5.7}y^7 + \&c.$$

II. Sit $S = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \&c.$

Erit $S = A\ \mathrm{tang}\ x$. Poſito iam $x = \frac{y}{\sqrt{(1-yy)}}$, fiet

$$S = A\ \mathrm{tang}\ \frac{y}{\sqrt{(1-yy)}} = A\ \mathrm{ſin}\ y = A\ \mathrm{coſ}\sqrt{(1-yy)}.$$

Hancobrem obtinebitur iſta ſummatio:

$$\frac{A\ \mathrm{ſin}\ y}{\sqrt{(1-yy)}} = y + \frac{2}{3}y^3 + \frac{2.4}{3.5}y^5 + \frac{2.4.6}{3.5.7}y^7 + \&c.$$

18. Poſſunt quoque loco x functiones transcendentes ipſius y ſubſtitui, ſicque ſummationes aliae inuentu difficiliores erui; verumtamen ne ſeries nouae fiant nimis perplexae, eiusmodi functiones eligi debent, quarum poteſtates facile exhiberi queant; quales ſunt quantitates exponentiales e^y. Propoſita igitur hac ſerie:

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + fx^6 + \&c.$$

 po-

ponatur $x = e^{ny}y$, denotante e numerum, cuius logarithmus hyperbolicus $= 1$,

erit $x^2 = e^{2ny}y^2$; $x^3 = e^{3ny}y^3$; &c.

Generaliter vero eſt, vti conſtat:

$$e^z = 1 + z + \frac{z^2}{1.2} + \frac{z^3}{1.2.3} + \frac{z^4}{1.2.3.4} + \&c.$$

Quare ſeries propoſita in hanc transmutabitur:

$$\begin{aligned} S = ay &+ 1nay^2 + \tfrac{1}{2}n^2ay^3 + \tfrac{1}{6}n^3ay^4 + \tfrac{1}{24}n^4ay^5 + \&c. \\ &+ by^2 + \tfrac{2}{1}nby^3 + \tfrac{4}{2}n^2by^4 + \tfrac{8}{6}n^3by^5 + \&c. \\ &+ cy^3 + \tfrac{3}{1}ncy^4 + \tfrac{9}{2}n^2cy^5 + \&c. \\ &+ dy^4 + \tfrac{4}{1}ndy^5 + \&c. \\ &+ ey^5 + \&c. \end{aligned}$$

I. Sit ſeries propoſita geometrica:

$S = x + x^2 + x^3 + x^4 + x^5 + \&c.$ erit $S = \frac{x}{1 - x}$.

Ponatur iam

$n = -1$; vt ſit $x = e^{-y}y$; & $S = \frac{e^{-y}y}{1 - e^{-y}y} = \frac{y}{e^y - y}$;

reperietur ſumma haec:

$$\frac{y}{e^y - y} = y - \frac{1}{2}y^3 - \frac{1}{6}y^4 + \frac{5}{24}y^5 + \frac{19}{120}y^6 - \&c.$$

cuius autem ſeriei lex non perſpicitur.

II. Sint in altera ſerie omnes termini praeter primum $= 0$; erit $b = -na$; $c = \frac{3}{2}n^2a$; $d = -\frac{8}{3}n^3a$;

$$e = \frac{125}{24}n^4a;\ f = -\frac{29}{30}n^5a;\ \&c.$$

Cum

Cum ergo ſit ſumma $S = ay$; & $x = e^{ny}$; fiet:

$$y = x - nx^2 + \frac{3}{2}n^2x^3 - \frac{8}{3}n^3x^4 + \frac{125}{24}n^4x^5$$
$$- \frac{29}{30}n^5x^6 + \&c.$$

Quoniam vero in his ſeriebus lex progreſſionis non eſt manifeſta, ſummationes ex hac ſubſtitutione deductae parum habent vtilitatis. Praecipue autem notari merentur transformationes ex ſubſtitutione $x = \frac{y}{1 \pm y}$ deriuatae, quippe quae non ſolum eximias ſummationes, ſed etiam idoneos modos ad ſummas ſerierum appropinquandi ſuppeditant. His ergo, quae ſine calculi differentialis ope ſunt expedita, praemiſſis, ad ipſum huius calculi vſum in doctrina ſerierum oſtendendum progrediamur.

CAPUT

CAPUT II.

DE INUESTIGATIONE SERIERUM SUMMABILIUM.

19.

Si Seriei, in cuius terminis ineſt quantitas indeterminata x, ſumma fuerit cognita, quae vtique erit functio ipſius x; tum quicunque valor ipſi x tribuatur, ſeriei ſumma perpetuo aſſignari poterit. Quare ſi loco x ponatur $x + dx$, ſeriei reſultantis ſumma erit aequalis ſummae prioris, vna cum ipſius differentiali: vnde ſequitur fore differentiale ſummae $=$ differentiali ſeriei. Quia vero hoc modo tam ſumma, quam ſinguli ſeriei termini multiplicati erunt per dx, ſi vbique per dx diuidatur, habebitur noua ſeries, cuius ſumma erit cognita. Simili modo ſi haec ſeries cum ſua ſumma denuo differentietur, & vbique per dx diuidatur, noua exoritur ſeries cum ſua ſumma: ſicque ex vna ſerie ſummabili quantitatem indeterminatam x inuoluente, per continuam differentiationem innumerae nouae ſeries pariter ſummabiles elicientur.

20. Quo haec clarius perſpiciantur, propoſita ſit progreſſio geometrica indeterminata, quippe cuius ſumma eſt cognita, haec:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \&c.$$

Si

Si nunc differentiatio inſtituatur, erit:

$$\frac{dx}{(1-x)^2} = dx + 2xdx + 3x^2dx + 4x^3dx + 5x^4dx + \&c.$$

atque diuiſione per dx inſtituta, habebitur:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \&c.$$

Si denuo differentietur, atque per dx diuidatur, prodibit:

$$\frac{2}{(1-x)^3} = 2 + 2.3x + 3.4x^2 + 4.5x^3 + 5.6x^4 + \&c.$$

ſeu

$$\frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + \&c.$$

vbi coefficientes ſunt numeri trigonales. Si haec porro differentietur, atque per $3dx$ diuidatur, obtinebitur:

$$\frac{1}{(1-x)^4} = 1 + 4x + 10x^2 + 20x^3 + 35x^4 + \&c.$$

cuius coefficientes ſunt numeri pyramidales primi. Sicque vlterius procedendo oriuntur eaedem ſeries quas ex euolutione fractionis $\frac{1}{(1-x)^n}$ naſci conſtat.

21. Latius autem patebit haec ſerierum inueſtigatio, ſi antequam quaeuis differentiatio ſuſcipiatur, ipſa ſeries vna cum ſumma per quamuis ipſius x poteſtatem ſeu functionem multiplicetur. Sic cum ſit

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \&c.$$

multiplicetur vbique per x^m, eritque

$$\frac{x^m}{1-x} = x^m + x^{m+1} + x^{m+2} + x^{m+3} + x^{m+4} + \&c.$$

nunc differentietur haec ſeries, fietque per dx diuiſo:

$$\frac{mx^{m-1}-(m-1)x^m}{(1-x)^2} = mx^{m-1} + (m+1)x^m + (m+2)x^{m+1} + (m+3)x^{m+2} + \&c.$$

Diuidatur nunc per x^{m-1}, habebitur:

$$\frac{m-(m-1)x}{(1-x)^2} = \frac{m}{1-x} + \frac{x}{(1-x)^2} = m + (m+1)x + (m+2)x^2 + \&c.$$

multiplicetur haec antequam noua differentiatio ſuſcipiatur per x^n, vt ſit

$$\frac{mx^n}{1-x} + \frac{x^{n+1}}{(1-x)^2} = mx^n + (m+1)x^{n+1} + (m+2)x^{n+2} + \&c.$$

Nunc inſtituatur differentiatio, & diuiſo per dx erit:

$$\frac{mnx^{n-1}}{1-x} + \frac{(m+n+1)x^n}{(1-x)^2} + \frac{2x^{n+1}}{(1-x)^3} = mnx^{n-1} + (m+1)(n+1)x^n + (m+2)(n+2)x^{n+1} + \&c.$$

Diuiſione autem per x^{n-1} inſtituta, fiet:

$$\frac{mn}{1-x} + \frac{(m+n+1)x}{(1-x)^2} + \frac{2xx}{(1-x)^3} = mn + (m+1)(n+1)x + (m+2)(n+2)x^2 + \&c.$$

ſicque vlterius progredi licebit: inuenientur autem perpetuo eaedem ſeries, quae ex euolutione fractionum ſummam conſtituentium naſcuntur.

22. Quoniam progreſſionis geometricae primum aſſumtae ſumma ad quemuis terminum vsque aſſignari po-

poteſt, hoc modo quoque ſeries definito terminorum numero conſtantes ſummabuntur. Ita cum ſit

$$\frac{1-x^{n+1}}{1-x} = x + x^2 + x^3 + x^4 + \quad . \quad . \quad . \quad . \quad . \quad + x^n$$

erit differentiatione inſtituta & terminis per dx diuiſis:

$$\frac{1}{(1-x)^2} - \frac{(n+1)x^n + nx^{n+1}}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3$$
$$+ \quad . \quad . \quad . \quad . \quad . \quad + nx^{n-1}$$

Hinc ſummae poteſtatum numerorum naturalium ad quemuis terminum inueniri poterunt. Multiplicetur enim haec ſeries per x, vt fiat:

$$\frac{x-(n+1)x^{n+1} + nx^{n+2}}{(1-x)^2} = x + 2x^2 + 3x^3 + \quad . \quad . \quad . \quad + nx^n$$

quae denuo differentiata ac per dx diuiſa dabit:

$$\frac{1 + x - (n+1)^2 x^n + (2nn + 2n - 1)x^{n+1} - nnx^{n+2}}{(1-x)^3}$$
$$= 1 + 4x + 9x^2 + \quad . \quad . \quad . \quad . \quad . \quad . \quad + n^2 x^{n-1}$$

quae per x multiplicata dabit:

$$\frac{x + x^2 - (n+1)^2 x^{n+1} + (2nn + 2n - 1)x^{n+2} - nnx^{n+3}}{(1-x)^3}$$
$$= x + 4x^2 + 9x^3 + \quad . \quad . \quad . \quad . \quad . \quad . \quad + n^2 x^n$$

quae differentiata, per dx diuiſa ac per x multiplicata producet ſeriem hanc:

$$x + 8x^2 + 27x^3 + \quad . \quad . \quad . \quad . \quad . \quad . \quad + n^3 x^n$$

cuius ſumma propterea inuenietur. Ex hacque ſimili modo ſumma biquadratorum altiorumque poteſtatum indefinita eruetur.

23. Methodus igitur haec ad omnes ſeries quantitatem indeterminatam continentes accommodari poteſt, quarum quidem ſummae conſtant. Cum igitur praeter geometricas ſeries recurrentes omnes eadem praerogatiua gaudeant, vt non ſolum in infinitum, ſed etiam ad quemuis terminum ſummari queant; ex iis quoque hac methodo innumerae aliae ſeries ſummabiles inueniri poterunt. Quod cum opus foret maxime diffuſum, ſi id perſequi vellemus, vnicum caſum perpendamus.

Sit ſcilicet propoſita haec ſeries:

$$\frac{x}{1-x-xx} = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7 + \&c.$$

quae differentiata ac per dx diuiſa dabit:

$$\frac{1+xx}{(1-x-xx)^2} = 1 + 2x + 6x^2 + 12x^3 + 25x^4 + 48x^5 + 91x^6 + \&c.$$

Facile autem patet omnes has ſeries hoc modo reſultantes fore quoque recurrentes, quarum adeo ſummae ex ipſarum natura inueniri poterunt.

24. In genere igitur ſi ſeriei cuiuſpiam in hac forma contentae: $ax + bx^2 + cx^3 + dx^4 + \&c.$ ſumma fuerit cognita, quam ponamus $= S$, eiusdem ſeriei, ſi ſinguli termini ſingulatim per terminos progreſſionis arithmeticae multiplicentur, ſumma inueniri poterit.

Sit enim

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \&c.$$

multiplicetur per x^m, erit

$$Sx^m = ax^{m+1} + bx^{m+2} + cx^{m+3} + dx^{m+4} + \&c.$$

dif-

differentietur haec aequatio, & diuidatur per dx:

$$mSx^{m-1} + x^m \frac{dS}{dx} = (m+1)ax^m + (m+2)bx^{m+1} + (m+3)cx^{m+2} + \&c.$$

diuidatur per x^{m-1}, eritque

$$mS + \frac{xdS}{dx} = (m+1)ax + (m+2)bx^2 + (m+3)cx^3 + \&c.$$

Quodſi ergo huius ſequentis ſeriei ſumma deſideretur:

$$\alpha ax + (\alpha+\beta)bx^2 + (\alpha+2\beta)cx^3 + (\alpha+3\beta)dx^4 + \&c.$$

multiplicetur ſuperior per β ac ſtatuatur $m\beta + \beta = \alpha$ vt ſit $m = \frac{\alpha - \beta}{\beta}$; eritque huius ſeriei ſumma

$$= (\alpha - \beta)S + \frac{\beta xdS}{dx}.$$

25. Poterit etiam huius ſeriei propoſitae ſumma inueniri, ſi ſinguli eius termini multiplicentur per terminos ſeriei ſecundi ordinis ſingulatim, cuius ſcilicet differentiae demum ſecundae ſint conſtantes. Quoniam enim iam inuenimus.

$$mS + \frac{xdS}{dx} = (m+1)ax + (m+2)bx^2 + (m+3)cx^3 + \&c.$$

multiplicetur per x^n, vt ſit

$$mSx^n + \frac{x^{n+1}dS}{dx} = (m+1)ax^{n+1} + (m+2)bx^{n+2} + \&c.$$

differentietur poſito dx conſtante, & per dx diuidatur:

$$mnSx^{n-1} + \frac{(m+n+1)x^n dS}{dx} + \frac{x^{n+1}ddS}{dx^2} =$$

$$(m+1)(n+1)ax^n + (m+2)(n+2)bx^{n+1} + \&c.$$

 Di-

Diuidatur per x^{n-1}, ac multiplicatur per k, vt ſit:

$$mnkS + \frac{(m+n+1)kxdS}{dx} + \frac{kx^2ddS}{dx^2} =$$

$$(m+1)(n+1)kax + (m+2)(n+2)kbx^2 + (m+3)(n+3)cx^3 + \&c.$$

Comparetur nunc haec ſeries cum iſta:

erit:	Diff. 1.	Diff. 2.
$kmn + km + kn + k = \alpha$	$km + kn + 3k = \beta$	$2k = \gamma$
$kmn + 2km + 2kn + 4k = \alpha + \beta$	$km + kn + 5k = \beta + \gamma$	
$kmn + 3km + 3kn + 9k = \alpha + 2\beta + \gamma$		

Ergo $k = \frac{1}{2}\gamma$; & $m + n = \frac{2\beta}{\gamma} - 3$; atque

$$mn = \frac{\alpha}{k} - m - n - 1 = \frac{2\alpha}{\gamma} - \frac{2\beta}{\gamma} + 2 = \frac{2(\alpha - \beta + \gamma)}{\gamma}.$$

Hinc ſumma ſeriei quaeſita erit:

$$(\alpha - \beta + \gamma)S + \frac{(\beta - \gamma)xdS}{dx} + \frac{\gamma x^2 ddS}{2dx^2}.$$

26. Simili modo ſumma reperiri poterit ſeriei huius

$$Aa + Bbx + Ccx^2 + Ddx^3 + Eex^4 + \&c.$$

ſi quidem cognita fuerit ſumma S ſeriei huius:

$$S = a + bx + cx^2 + dx^3 + ex^4 + fx^5 + \&c.$$

atque A, B, C, D, &c. conſtituant ſeriem, quae ad differentias conſtantes deducitur. Fingatur enim ſumma, quoniam eius forma ex antecedentibus colligitur, haec

$$\alpha S + \frac{\beta xdS}{dx} + \frac{\gamma x^2 ddS}{2dx^2} + \frac{\delta x^3 d^3S}{6dx^3} + \frac{\varepsilon x^4 d^4S}{24dx^4} + \&c.$$

Nunc

Nunc ad litteras α, β, γ, δ, &c. inueniendas euoluantur ſingulae ſeries, eritque:

$$\alpha S = \alpha a + \alpha bx + \alpha cx^2 + \alpha dx^3 + \alpha ex^4 + \&c.$$

$$\frac{\beta x dS}{dx} = \beta bx + 2\beta cx^2 + 3\beta dx^3 + 4\beta ex^4 + \&c.$$

$$\frac{\gamma x^2 ddS}{2dx^2} = \gamma cx^2 + 3\gamma dx^3 + 6\gamma ex^4 + \&c.$$

$$\frac{\delta x^3 d^3 S}{6dx^2} = \delta dx^3 + 4\delta ex^4 + \&c.$$

$$\frac{\varepsilon x^4 d^4 S}{24dx^3} = \varepsilon ex^4 + \&c.$$

quae ſimul ſumtae comparentur cum propoſita:

$$Z = Aa + Bbx + Ccx^2 + Ddx^3 + Eex^4 + \&c.$$

fietque comparatione ſingulorum terminorum inſtituta:

$$\alpha = A$$
$$\beta = B - \alpha = B - A$$
$$\gamma = C - 2\beta - \alpha = C - 2B + A$$
$$\delta = D - 3\gamma - 3\beta - \alpha = D - 3C + 3B - A$$
&c.

His igitur valoribus inuentis erit ſumma quaeſita:

$$Z = AS + \frac{(B-A)\,xdS}{1\,dx} + \frac{(C-2B+A)\,x^2ddS}{1.2\,dx^2} + \frac{(D-3C+3B-A)\,x^3d^3S}{1.2.3\,dx^3} + \&c.$$

ſeu ſi ſeriei A, B, C, D, E, &c. differentiae continuae more conſueto indicentur, erit

$Z =$

$$Z = AS + \frac{\Delta A.x dS}{1\,dx} + \frac{\Delta^2 A.x^2 d^2S}{1.2\,dx^2} + \frac{\Delta^3 A.x^3 d^3S}{1.2.3\,dx^3} + \&c.$$

ſi quidem fuerit vti aſſumſimus:

$$S = a + bx + cx^2 + dx^3 + ex^4 + fx^5 + \&c.$$

Si ergo ſeries A, B, C, D, &c. tandem habeat differentias conſtantes, ſumma ſeriei Z finite exprimi poterit.

27. Quia ſumto e pro numero, cuius logarithmus hyperbolicus eſt $= 1$, eſt:

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \frac{x^5}{1.2.3.4.5} + \&c.$$

ſumatur haec ſeries pro priori, & cum ſit

$$S = e^x \quad \text{erit}$$

$$\frac{dS}{dx} = e^x$$

$$\frac{ddS}{dx^2} = e^x \quad \&c.$$

Quare huius ſeriei, quae ex illa & hac A, B, C, D, &c. componitur:

$$A + \frac{Bx}{1} + \frac{Cx^2}{1.2} + \frac{Dx^3}{1.2.3} + \frac{Ex^4}{1.2.3.4} + \&c.$$

ſumma hoc modo exprimetur:

$$e^x \left(A + \frac{x\Delta A}{1} + \frac{xx\Delta^2 A}{1.2} + \frac{x^3\Delta^3 A}{1.2.3} + \frac{x^4\Delta^4 A}{1.2.3.4} + \&c.\right)$$

Sic ſi proponatur haec ſeries:

$$2 + \frac{5x}{1} + \frac{10x^2}{1.2} + \frac{17x^3}{1.2.3} + \frac{26x^4}{1.2.3.4} + \frac{37x^5}{1.2.3.4.5} + \&c.$$

Ob

Ob ſeriem A, B, C, D, E, &c.

erit $A = 2, \; 5, \; 10, \; 17, \; 26$ &c.

$\Delta A = 3, \; 5, \; 7, \; 9,$ &c.

$\Delta^2 A = 2, \; 2, \; 2$ &c.

erit huius ſerei:

$$2 + 5x + \frac{10x^2}{2} + \frac{17x^3}{6} + \frac{26x^4}{24} + \&c.$$

ſumma $= e^x(2 + 3x + xx) = e^x(1+x)(2+x)$: quod quidem ſponte patet. Eſt enim

$$2e^x = 2 + \frac{2x}{1} + \frac{2x^2}{2} + \frac{2x^3}{6} + \frac{2x^4}{24} + \&c.$$

$$3xe^x = 3x + \frac{3x^2}{1} + \frac{3x^3}{2} + \frac{3x^4}{6} + \&c.$$

$$xxe^x = xx + \frac{x^3}{1} + \frac{x^4}{2} + \&c.$$

$$e^x(2+3x+xx) = 2+5x+\frac{10xx}{2} + \frac{17x^3}{6} + \frac{26x^4}{24} + \&c.$$

28. Quae hactenus ſunt tradita non ſolum ad ſeries in infinitum excurrentes ſpectant, ſed etiam ad ſummas quotcunque terminorum: coefficientes enim a, b, c, d, &c. vel in infinitum progredi, vel vbicunque libuerit abrumpi poſſunt. Verum cum hoc non egeat vberiori explicatione, quae ex hactenus allatis ſequuntur, accuratius perpendamus. Propoſita ergo quacunque ſerie, cuius ſinguli termini duobus conſtent factoribus, quorum alteri ſeriem ad differentias conſtantes deducen-

tem conſtituant, huius ſeriei ſumma poterit aſſignari; dummodo omiſſis iſtis factoribus, ſeries fuerit ſummabilis. Scilicet ſi propoſita ſit iſta ſeries

$$Z = Aa + Bbx + Ccx^2 + Ddx^3 + Eex^4 + \&c.$$

in qua quantitates A, B, C, D, E, &c. eiusmodi ſeriem conſtituant, quae tandem ad differentias conſtantes perducatur; tum iſtius ſeriei ſumma exhiberi poterit, dummodo habeatur ſumma S huius ſeriei

$$S = a + bx + cx^2 + dx^3 + ex^4 + \&c.$$

Sumtis enim ex progreſſione A, B, C, D, E, &c. differentiis continuis, vti initio huius libri oſtendimus:

$$\begin{array}{c} A, \quad B, \quad C, \quad D, \quad E, \quad F \quad \&c. \\ \Delta A, \quad \Delta B, \quad \Delta C, \quad \Delta D, \quad \Delta E \quad \&c. \\ \Delta^2 A, \quad \Delta^2 B, \quad \Delta^2 C, \quad \Delta^2 D \quad \&c. \\ \Delta^3 A, \quad \Delta^3 B, \quad \Delta^3 C \quad \&c. \\ \Delta^4 A, \quad \Delta^4 B, \quad \&c. \\ \Delta^5 A, \quad \&c. \\ \&c. \end{array}$$

erit ſeriei propoſitae ſumma:

$$Z = SA + \frac{xdS}{1\,dx}\Delta A + \frac{x^2 ddS}{1.2\,dx^2}\Delta^2 A + \frac{x^3 d^3 S}{1.2.3\,dx^3}\Delta^3 A + \&c.$$

poſito in altioribus ipſius S differentialibus dx conſtante.

29. Si igitur ſeries A, B, C, D, &c. nunquam ad differentias conſtantes deducat, ſumma ſeriei Z per nouam ſeriem infinitam exprimetur, quae interdum magis

gis conuerget quam propoſita; ſicque iſta ſeries in aliam ſibi aequalem transformabitur. Sit ad hoc declarandum propoſita haec ſeries:

$$Y = y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} + \frac{y^5}{5} + \frac{y^6}{6} + \&c.$$

quam conſtat exprimere $l\frac{1}{1-y}$, ita vt ſit $Y = -l(1-y)$. Diuidatur haec ſeries per y, & ſtatuatur $y = x$, &

$$Y = yZ, \text{ vt ſit } Z = -\frac{1}{y}l(1-y) = -\frac{1}{x}l(1-x),$$

erit

$$Z = 1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \frac{x^5}{6} + \&c.$$

quae comparata cum iſta:

$$S = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \&c. = \frac{1}{1-x},$$

dabit pro ſerie A, B, C, D, E, &c. hos valores:

$$1,\quad \frac{1}{2},\quad \frac{1}{3},\quad \frac{1}{4},\quad \frac{1}{5}$$

$$-\frac{1}{1.2},\ -\frac{1}{2.3},\ -\frac{1}{3.4},\ -\frac{1}{4.5}$$

$$\frac{1.2}{1.2.3},\quad \frac{1.2}{2.3.4},\quad \frac{1.2}{3.4.5}$$

$$-\frac{1.2.3}{1.2.3.4},\ -\frac{1.2.3}{2.3.4.5}$$

&c.

Erit ergo $A = 1$; $\Delta A = -\frac{1}{2}$; $\Delta^2 A = \frac{1}{3}$; $\Delta^3 A = -\frac{1}{4}$ &c.

 Porro

Porro cum ſit $S = \frac{1}{1-x}$, erit

$$\frac{dS}{dx} = \frac{1}{(1-x)^2}$$

$$\frac{ddS}{1.2\,dx^2} = \frac{1}{(1-x)^3}$$

$$\frac{d^3S}{1.2.3\,dx^3} = \frac{1}{(1-x)^4}$$

&c.

Quibus valoribus ſubſtitutis orietur ſumma: $Z =$

$$\frac{1}{1-x} - \frac{x}{2(1-x)^2} + \frac{x^2}{3(1-x)^3} - \frac{x^3}{4(1-x)^4} + \frac{x^4}{5(1-x)^5} + \&c.$$

Cum ergo ſit $x = y$, & $Y = -l(1-y) = yZ$;

erit

$$-l(1-y) = \frac{y}{1-y} - \frac{y^2}{2(1-y)^2} + \frac{y^3}{3(1-y)^3} - \frac{y^4}{4(1-y)^4} + \&c.$$

quae ſeries vtique exprimit

$$l\left(1 + \frac{y}{1-y}\right) = l\,\frac{1}{1-y} = -l(1-y),$$

cuius adeo veritas per ante demonſtrata conſtat.

30. Propoſita nunc ſit iſta ſeries, vt etiam vſus pateat, ſi poteſtates tantum impares occurrant, & ſigna alternentur.

$$Y = y - \frac{y^3}{3} + \frac{y^5}{5} - \frac{y^7}{7} + \frac{y^9}{9} - \frac{y^{11}}{11} + \&c.$$

ex qua conſtat eſſe $Y = A\,\mathrm{tang}\,y$.

Di-

Diuidatur haec ſeriei per y, & ponatur $\frac{Y}{y} = Z$, & $yy = x$; erit:

$$Z = 1 - \frac{x}{3} + \frac{xx}{5} - \frac{x^3}{7} + \frac{x^4}{9} - \frac{x^5}{11} + \&c.$$

quae ſi comparetur cum iſta:

$S = 1 - x + xx - x^3 + x^4 - \&c.$ fiet $S = \frac{1}{1+x}$, & ſeries coefficientium A, B, C, D, &c. fiet:

$$A = 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9} \; \&c.$$

$$\Delta A = -\frac{2}{3}; -\frac{2}{3.5}; -\frac{2}{5.7}; -\frac{2}{7.9}$$

$$\Delta^2 A = \frac{2.4}{3.5}; \frac{2.4}{3.5.7}; \frac{2.4}{5.7.9}$$

$$\Delta^3 A = -\frac{2.4.6}{3.5.7}; -\frac{2.4.6}{3.5.7.9}$$

$$\Delta^4 A = \frac{2.4.6.8}{3.5.7.9}$$

&c.

At cum ſit $S = \frac{1}{1+x}$; erit

$$\frac{dS}{1dx} = -\frac{1}{(1+x)^2}$$

$$\frac{ddS}{1.2\,dx^2} = \frac{1}{(1+x)^3}$$

$$\frac{d^3S}{1.2.3dx^3} = -\frac{1}{(1+x)^4} \; \&c.$$

 Qua-

Quare ſubſtitutis his valoribus fiet forma $Z =$

$$\frac{1}{1+x} + \frac{2x}{3(1+x)^2} + \frac{2.4x^2}{3.5(1+x)^3} + \frac{2.4.6x^3}{3.5.7(1+x)^4} + \&c.$$

Reſtituto ergo $x = yy$; & per y multiplicato fiet:

$$Y = A \text{ tang } y =$$

$$\frac{y}{1+yy} + \frac{2y^3}{3(1+yy)^2} + \frac{2.4y^5}{3.5(1+yy)^3} + \frac{2.4.6y^7}{3.5.7(1+yy)^4} + \&c.$$

31. Poteſt quoque ſuperior ſeries, qua arcus circuli per tangentem exprimitur, alio modo transmutari, eam comparando cum ſerie logarithmica.

Conſideremus nempe ſeriem

$$Z = 1 - \frac{x}{3} + \frac{xx}{5} - \frac{x^3}{7} + \frac{x^4}{9} - \frac{x^5}{11} + \&c.$$

quam comparemus cum hac:

$$S = \frac{1}{0} - \frac{x}{2} + \frac{xx}{4} - \frac{x^3}{6} + \frac{x^4}{8} - \&c.$$

$$= \frac{1}{0} - \frac{1}{2} l(1+x),$$

atque valores litterarum A, B, C, D, &c. erunt

$$A = \frac{0}{1};\ \frac{2}{3};\ \frac{4}{5};\ \frac{6}{7};\ \frac{8}{9};\ \&c.$$

$$\Delta A = \frac{2}{3};\ \frac{+2}{3.5};\ \frac{+2}{5.7};\ \frac{+2}{7.9};\ \&c.$$

$$\Delta^2 A = \frac{-2.4}{3.5};\ \frac{-2.4}{3.5.7};\ \frac{-2.4}{5.7.9};\ \&c.$$

$$\Delta^3 A = \frac{2.4.6}{3.5.7};\ \&c.$$

De

Deinde cum ſit $S = \frac{1}{0} - \frac{1}{2} l(1+x)$; erit

$$\frac{dS}{1.dx} = - \frac{1}{2(1+x)}$$

$$\frac{ddS}{1.2dx^2} = + \frac{1}{4(1+x)^2}$$

$$\frac{d^3S}{1.2.3\,dx^3} = - \frac{1}{6(1+x)^3}$$

$$\frac{d^4S}{1.2.3.4\,dx^3} = \frac{1}{8(1+x)^4}$$

&c.

Erit igitur $SA = S.\frac{0}{1} = 1$: & ex reliquis fiet:

$$Z = 1 - \frac{x}{3(1+x)} - \frac{2xx}{3.5(1+x)^2} - \frac{2.4x^3}{3.5.7(1+x)^3} - \&c.$$

Ponatur nunc $x = yy$, & multiplicetur per y fiet:

$$Y = A \text{ tang } y =$$

$$y - \frac{y^3}{3(1+yy)} - \frac{2y^5}{3.5(1+yy)^2} - \frac{2.4y^7}{3.5.7(1+yy)^3} - \&c.$$

Haec ergo transmutatio non impediebatur termino infinito $\frac{1}{0}$, qui in ſeriem S ingrediebatur. Sin autem cui ſuperſit dubium, is tantum ſingulos terminos praeter primum ſecundum poteſtates ipſius y in ſeries reſoluat, atque deprehendet actu ſeriem primum propoſitam reſultare.

32.

32. Hactenus eiusmodi tantum ſeries ſumus contemplati, in quibus omnes poteſtates variabilis occurrunt. Nunc igitur ad alias ſeries progrediamur, quae in ſingulis terminis eandem poteſtatem ipſius variabilis complectantur, cuiusmodi eſt haec:

$$S = \frac{1}{a+x} + \frac{1}{b+x} + \frac{1}{c+x} + \frac{1}{d+x} + \&c.$$

Huius enim ſeriei ſi ſumma S fuerit, cognita ac per functionem quampiam ipſius x exprimatur, erit differentiando ac per $-dx$ diuidendo:

$$\frac{-dS}{dx} = \frac{1}{(a+x)^2} + \frac{1}{(b+x)^2} + \frac{1}{(c+x)^2} + \frac{1}{(d+x)^2} + \&c.$$

Si haec vlterius differentietur, atque per $-2\,dx$ diuidatur, cognoſcetur ſeries cuborum:

$$\frac{ddS}{2dx^2} = \frac{1}{(a+x)^3} + \frac{1}{(b+x)^3} + \frac{1}{(c+x)^3} + \frac{1}{(d+x)^3} + \&c.$$

haecque denuo differentiata, atque per $-3\,dx$ diuiſa dabit:

$$\frac{-d^3S}{6\,dx^3} = \frac{1}{(a+x)^4} + \frac{1}{(b+x)^4} + \frac{1}{(c+x)^4} + \frac{1}{(d+x)^4} + \&c.$$

Similique modo omnium ſequentium poteſtatum ſummae reperientur, dummodo ſumma ſeriei primae fuerit cognita.

33. Huiusmodi autem ſeries fractionum quantitatem indeterminatam inuoluentes ſupra in introductione elicuimus, vbi oſtendimus, ſi circuli, cuius radius $= 1$, ſemiperipheria ſtatuatur $= \pi$; fore

π

$$\frac{\pi}{n \operatorname{ſin} \frac{m}{n} \pi} =$$

$$\frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \&c.$$

$$\frac{\pi \operatorname{coſ} \frac{m}{n} \pi}{n \operatorname{ſin} \frac{m}{n} \pi} =$$

$$\frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \&c.$$

Cum igitur pro m & n numeros quoscunque aſſumere liceat, ſtatuamus $n = 1$, & $m = x$; vt obtineamus ſeries illi quam in §. praec. propoſueramus ſimiles; hoc facto erit:

$$\frac{\pi}{\operatorname{ſin} \pi x} =$$

$$\frac{1}{x} + \frac{1}{1-x} - \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} + \frac{1}{3-x} - \&c.$$

$$\frac{\pi \operatorname{coſ} \pi x}{\operatorname{ſin} \pi x} =$$

$$\frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \frac{1}{3-x} + \&c.$$

Per differentiationes ergo ſummae quarumuis poteſtatum ex his fractionibus oriundarum exhiberi poterunt.

34. Conſideremus ſeriem priorem, ſitque breuitatis gratia $\frac{\pi}{\operatorname{ſin} \pi x} = S$, cuius differentialia altiora capiantur poſito dx conſtante: eritque

$$S = \frac{1}{x} + \frac{1}{1-x} - \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} + \frac{1}{3-x} - \&c.$$

$$\frac{-dS}{dx} = \frac{1}{xx} - \frac{1}{(1-x)^2} - \frac{1}{(1+x)^2} + \frac{1}{(2-x)^2} + \frac{1}{(2+x)^2} - \frac{1}{(3-x)^2} - \&c.$$

$$\frac{ddS}{2dx^2} = \frac{1}{x^3} + \frac{1}{(1-x)^3} - \frac{1}{(1+x)^3} - \frac{1}{(2-x)^3} + \frac{1}{(2+x)^3} + \frac{1}{(3-x)^3} - \&c.$$

$$\frac{-d^3S}{6dx^3} = \frac{1}{x^4} - \frac{1}{(1-x)^4} - \frac{1}{(1+x)^4} + \frac{1}{(2-x)^4} + \frac{1}{(2+x)^4} - \frac{1}{(3-x)^4} - \&c.$$

$$\frac{d^4S}{24dx^4} = \frac{1}{x^5} + \frac{1}{(1-x)^5} - \frac{1}{(1+x)^5} - \frac{1}{(2-x)^5} + \frac{1}{(2+x)^5} + \frac{1}{(3-x)^5} - \&c.$$

$$\frac{-d^5S}{120dx^5} = \frac{1}{x^6} - \frac{1}{(1-x)^6} - \frac{1}{(1+x)^6} + \frac{1}{(2-x)^6} + \frac{1}{(2+x)^6} - \frac{1}{(3-x)^6} - \&c.$$

&c.

vbi notandum eſt in poteſtatibus paribus ſigna eandem ſequi legem, pariterque in imparibus eandem legem ſignorum obſeruari. Omnium ergo iſtarum ſerierum ſummae inuenientur ex differentialibus expreſſionis

$$S = \frac{\pi}{\text{ſin}\,\pi x}.$$

35. Ad Differentialia haec ſimplicius exprimenda ponamus $\text{ſin}\,\pi x = p$ & $\text{coſ}\,\pi x = q$, erit $dp = \pi dx\,\text{coſ}\,\pi x = \pi q dx$, & $dq = -\pi p dx$. Cum ergo ſit

S =

$$S = \frac{\pi}{p} \qquad \text{erit}$$

$$\frac{-dS}{dx} = \frac{\pi^2 q}{pp}$$

$$\frac{ddS}{dx^2} = \frac{\pi^3(pp+2qq)}{p^3} = \frac{\pi^3(qq+1)}{p^3} \quad \text{ob} \quad pp+qq=1$$

$$\frac{-d^3S}{dx^3} = \pi^4\left(\frac{5q}{pp} + \frac{6q^3}{p^4}\right) = \frac{\pi^4(q^3+5q)}{p^4}$$

$$\frac{d^4S}{dx^4} = \pi^5\left(\frac{24q^4}{p^5} + \frac{28q^2}{p^3} + \frac{5}{p}\right)$$

$$\text{vel} = \frac{\pi^5(q^4+18q^2+5)}{p^5}$$

$$\frac{-d^5S}{dx^5} = \pi^6\left(\frac{120q^5}{p^6} + \frac{180q^3}{p^4} + \frac{61q}{pp}\right)$$

$$\text{vel} = \frac{\pi^6(q^5+58q^3+61q)}{p^6}$$

$$\frac{d^6S}{dx^6} = \pi^7\left(\frac{720q^6}{p^7} + \frac{1320q^4}{p^5} + \frac{662q^2}{p^3} + \frac{61}{p}\right)$$

$$\text{vel} = \frac{\pi^7(q^6+179q^4+479q^2+61)}{p^7}$$

$$\frac{-d^7S}{dx^7} = \pi^8\left(\frac{5040q^7}{p^8} + \frac{10920q^5}{p^6} + \frac{7266q^3}{p^4} + \frac{1385q}{p^2}\right)$$

$$\text{vel} = \frac{\pi^8}{p^8}(q^7+543q^5+3111q^3+1385q)$$

$$\frac{d^8S}{dx^8} = \pi^9\left(\frac{40320q^8}{p^9} + \frac{100800q^6}{p^7} + \frac{83664q^2}{p^5} + \frac{24568q^2}{p^3} + \frac{1385}{p}\right)$$

&c.

Quae expreſſiones facile vlterius quousque libuerit continuari poſſunt, ſi enim fuerit:

$$\pm\frac{d^nS}{dx^n}=\pi^{n+1}\left(\frac{\alpha q^n}{p^{n+1}}+\frac{\beta q^{n-2}}{p^{n-1}}+\frac{\gamma q^{n-4}}{p^{n-3}}+\frac{\delta q^{n-6}}{p^{n-5}}+\&c.\right)$$

erit differentiale ſequens ſignis mutatis:

$$\mp\frac{d^{n+1}S}{dx^{n+1}}=\pi^{n+2}\left(\frac{(n+1)\alpha q^{n+1}}{p^{n+2}}\begin{matrix}+n\alpha\\+(n-1)\beta\end{matrix}\Big\}\frac{q^{n-1}}{p^n}\begin{matrix}+(n-2)\beta\\+(n-3)\gamma\end{matrix}\Big\}\frac{q^{n-3}}{p^{n-2}}\begin{matrix}+(n-4)\gamma\\+(n-5)\delta\end{matrix}\Big\}\frac{q^{n-5}}{p^{n-4}}\ \&c.\right)$$

36. Ex his ergo obtinebuntur ſummae ſerierum ſuperiorum §. 34. exhibitarum ſequentes:

$$S=\pi.\frac{1}{p}$$

$$\frac{-dS}{dx}=\frac{\pi^2}{1}.\frac{q}{p^2}$$

$$\frac{ddS}{2dx^2}=\frac{\pi^3}{2}\left(\frac{2q^2}{p^3}+\frac{1}{p}\right)$$

$$\frac{-d^3S}{6dx^3}=\frac{\pi^4}{6}\left(\frac{6q^3}{p^4}+\frac{5q}{p^2}\right)$$

$$\frac{d^4S}{24dx^4}=\frac{\pi^5}{24}\left(\frac{24q^4}{p^5}+\frac{28q^2}{p^3}+\frac{5}{p}\right)$$

$$\frac{-d^5S}{120dx^5}=\frac{\pi 6}{120}\left(\frac{120q^5}{p^6}+\frac{180q^3}{p^4}+\frac{61q}{p^2}\right)$$

$$\frac{d^6S}{720dx^6}=\frac{\pi^7}{720}\left(\frac{720q^6}{p^7}+\frac{1320q^4}{p^6}+\frac{662q^4}{p^3}+\frac{61}{p}\right)$$

$$\frac{-d^7S}{5040dx^7}=\frac{\pi^8}{5040}\left(\frac{5040q^7}{p^8}+\frac{10920q^5}{p^6}+\frac{7266q^3}{p^4}+\frac{1385q}{p^2}\right)$$

$$\frac{d^8S}{40320dx^8}=\frac{\pi^9}{40320}\left(\frac{40320q^8}{p^9}+\frac{100800q^6}{p^7}+\frac{83664q}{p^5}+\frac{24568q^2}{p^3}+\frac{1385}{p}\right)$$

&c.

37.

37. Tractemus ſimili modo alteram ſeriem ſupra inuentam:

$$\frac{\pi \cos \pi x}{\sin \pi x} = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \frac{1}{3-x} + \&c.$$

atque poſito breuitatis ergo $\frac{\pi \cos \pi x}{\sin \pi x} = T$, orientur ſequentes ſummationes:

$$T = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \&c.$$

$$\frac{-dT}{dx} = \frac{1}{x^2} + \frac{1}{(1-x)^2} + \frac{1}{(1+x)^2} + \frac{1}{(2-x)^2} + \frac{1}{(2+x)^2} + \&c.$$

$$\frac{ddT}{2dx^2} = \frac{1}{x^3} - \frac{1}{(1-x)^3} + \frac{1}{(1+x)^3} - \frac{1}{(2-x)^3} + \frac{1}{(2+x)^3} - \&c.$$

$$\frac{-d^3T}{6dx^3} = \frac{1}{x^4} + \frac{1}{(1-x)^4} + \frac{1}{(1+x)^4} + \frac{1}{(2-x)^4} + \frac{1}{(2+x)^4} + \&c.$$

$$\frac{d^4T}{24dx^4} = \frac{1}{x^5} - \frac{1}{(1-x)^5} + \frac{1}{(1+x)^5} - \frac{1}{(2-x)^5} + \frac{1}{(2+x)^5} - \&c.$$

$$\frac{-d^5T}{120dx^5} = \frac{1}{x^6} + \frac{1}{(1-x)^6} + \frac{1}{(1+x)^6} + \frac{1}{(2-x)^6} + \frac{1}{(2+x)^6} + \&c.$$

&c.

vbi in poteſtatibus paribus omnes termini ſunt affirmatiui, in imparibus autem ſigna + & — alternatim ſe excipiunt.

38. Quo differentialium horum valores innoteſcant, ponamus vt ante $\sin \pi x = p$ & $\cos \pi x = q$, vt ſit $pp + qq = 1$; erit $dp = \pi q dx$ & $dq = -\pi p dx$.

Quibus valoribus adhibitis erit:

$$T = \pi.\frac{q}{p}$$

$$\frac{-dT}{dx} = \pi^2\left(\frac{qq}{pp}+1\right) = \frac{\pi^2}{pp}$$

$$\frac{ddT}{dx^2} = \pi^3\left(\frac{2q^3}{p^3}+\frac{2q}{p}\right) = \frac{2\pi^3 q}{p^3}$$

$$\frac{-d^3T}{dx^3} = \pi^4\left(\frac{6q^4}{p^4}+\frac{8qq}{pp}+2\right) = \pi^4\left(\frac{6qq}{p^4}+\frac{2}{pp}\right)$$

$$\frac{d^4T}{dx^4} = \pi^5\left(\frac{24q^3}{p^5}+\frac{16q}{p^3}\right)$$

$$\frac{-d^5T}{dx^5} = \pi^6\left(\frac{120q^4}{p^6}+\frac{120qq}{p^4}+\frac{16}{pp}\right)$$

$$\frac{d^6T}{dx^6} = \pi^7\left(\frac{720q^5}{p^7}+\frac{960q^3}{p^5}+\frac{272q}{p^3}\right)$$

$$\frac{-d^7T}{dx^7} = \pi^8\left(\frac{5040q^6}{p^8}+\frac{8400q^4}{p^6}+\frac{3696q^2}{p^4}+\frac{272}{p^2}\right)$$

$$\frac{d^8T}{dx^8} = \pi^9\left(\frac{40320q^7}{p^9}+\frac{80640q^5}{p^7}+\frac{48384q^3}{p^5}+\frac{7936q}{p^3}\right)$$

&c.

Quae formulae facile vlterius quousque libuerit continuari poſſunt. Si enim ſit

$$\pm\frac{d^nT}{dx^n} = \pi^{n+1}\left(\frac{\alpha q^{n-1}}{p^{n+1}}+\frac{\beta q^{n-3}}{p^{n-1}}+\frac{\gamma q^{n-5}}{p^{n-3}}+\frac{\delta q^{n-7}}{p^{n-5}}+\&c.\right)$$

erit expreſſio ſequens:

$$\mp\frac{d^{n+1}T}{dx^{n+1}} = \pi^{n+2}\left(\frac{(n+1)\alpha q^n}{p^{n+2}}+\frac{(n-1)(\alpha+\beta)q^{n-2}}{p^n}+\frac{(n-3)(\beta+\gamma)q^{n-4}}{p^{n-2}}+\&c.\right)$$

39.

39. Series ergo poteſtatum §. 37. datae ſequentes habebunt ſummas poſito $\sin \pi x = p$ & $\cos \pi x = q$.

$$T = \pi . \frac{q}{p}$$

$$\frac{-dT}{dx} = \pi^2 \frac{1}{pp}$$

$$\frac{ddT}{2dx^2} = \pi^3 \frac{q}{p^3}$$

$$\frac{-d^3T}{6dx^3} = \pi^4 \left(\frac{qq}{p^4} + \frac{1}{3pp}\right)$$

$$\frac{d^4T}{24dx^4} = \pi^5 \left(\frac{q^3}{p^5} + \frac{2q}{3p^3}\right)$$

$$\frac{-d^5T}{120dx^5} = \pi^6 \left(\frac{q^4}{p^6} + \frac{3qq}{3p^4} + \frac{2}{15pp}\right)$$

$$\frac{d^6T}{720dx^6} = \pi^7 \left(\frac{q^5}{p^7} + \frac{4q^3}{3p^5} + \frac{17q}{45p^3}\right)$$

$$\frac{-d^7T}{5040dx^7} = \pi^8 \left(\frac{q^6}{p^8} + \frac{5q^4}{3p^6} + \frac{11q^2}{15p^4} + \frac{17}{315pp}\right)$$

$$\frac{d^8T}{40320dx^8} = \pi^9 \left(\frac{q^7}{p^9} + \frac{6q^5}{3p^7} + \frac{6q^3}{5p^5} + \frac{62q}{315p^3}\right)$$

&c.

40. Praeter has ſeries inuenimus in introductione nonnullas alias, ex quibus ſimili modo per differentiationes nouae elici poſſunt. Oſtendimus enim eſſe:

$$\frac{1}{2x} - \frac{\pi \sqrt{x}}{2x \tang \pi \sqrt{x}} =$$

$$\frac{1}{1-x} + \frac{1}{4-x} + \frac{1}{9-x} + \frac{1}{16-x} + \frac{1}{25-x} + \&c.$$

Po-

Ponamus ſummam huius ſeriei eſſe $=$ S,

vt ſit $\quad S = \frac{1}{2x} - \frac{\pi}{2\sqrt{x}} \cdot \frac{\cos\pi\sqrt{x}}{\sin\pi\sqrt{x}}$, $\quad$ erit

$$\frac{dS}{dx} = -\frac{1}{2xx} + \frac{\pi}{4x\sqrt{x}} \cdot \frac{\cos\pi\sqrt{x}}{\sin\pi\sqrt{x}} + \frac{\pi\pi}{4x(\sin\pi\sqrt{x})^2},$$

quae ergo expreſſio praebet ſummam huius ſeriei:

$$\frac{1}{(1-x)^2} + \frac{1}{(4-x)^2} + \frac{1}{(9-x)^2} + \frac{1}{(16-x)^2} + \frac{1}{(25-x)^2} + \&c.$$

Deinde quoque oſtendimus eſſe:

$$\frac{\pi}{2\sqrt{x}} \cdot \frac{e^{2\pi\sqrt{x}}+1}{e^{2\pi\sqrt{x}}-1} - \frac{1}{2x} =$$

$$\frac{1}{1+x} + \frac{1}{4+x} + \frac{1}{9+x} + \frac{1}{16+x} + \&c.$$

Quodſi ergo haec ſumma ponatur $=$ S, erit:

$$\frac{-dS}{dx} = \frac{1}{(1+x)^2} + \frac{1}{(4+x)^2} + \frac{1}{(9+x)^2} + \frac{1}{(16+x)^2} + \&c.$$

At eſt

$$\frac{dS}{dx} = \frac{-\pi}{4x\sqrt{x}} \cdot \frac{e^{2\pi\sqrt{x}}+1}{e^{2\pi\sqrt{x}}-1} - \frac{\pi\pi}{x} \cdot \frac{e^{2\pi\sqrt{x}}}{(e^{2\pi\sqrt{x}}-1)^2} + \frac{1}{2xx}.$$

Ergo ſumma huius ſeriei erit:

$$\frac{-dS}{dx} = \frac{\pi}{4x\sqrt{x}} \cdot \frac{e^{2\pi\sqrt{x}}+1}{e^{2\pi\sqrt{x}}-1} + \frac{\pi\pi}{x} \cdot \frac{e^{2\pi\sqrt{x}}}{(e^{2\pi\sqrt{x}}-1)^2} - \frac{1}{2xx}.$$

Similique modo vlterioribus differentiationibus ſummae ſequentium poteſtatum inuenientur.

41. Si cognitus fuerit valor producti cuiuſpiam ex factoribus indeterminatam litteram inuoluentibus compo-

ſiti

ſiti, ex eo per eandem methodum innumerabiles ſeries ſummabiles inueniri poterunt. Sit enim huius producti

$$(1+\alpha x)(1+\beta x)(1+\gamma x)(1+\delta x)(1+\epsilon x) \text{ \&c.}$$

valor $= S$, functioni ſcilicet cuiuspiam ipſius x, erit logarithmis ſumendis:

$$lS = l(1+\alpha x) + l(1+\beta x) + l(1+\gamma x) + l(1+\delta x) + \text{\&c.}$$

Sumantur iam differentialia, erit diuiſione per dx inſtituta:

$$\frac{dS}{Sdx} = \frac{\alpha}{1+\alpha x} + \frac{\beta}{1+\beta x} + \frac{\gamma}{1+\gamma x} + \frac{\delta}{1+\delta x} + \text{\&c.}$$

ex cuius vlteriori differentiatione ſummae poteſtatum quarumuis iſtarum fractionum reperietur; plane vti in exemplis praecedentibus fuſius expoſuimus.

42. Exhibuimus autem in introductione nonnullas iſtiusmodi expreſſiones, ad quas hanc methodum accommodemus. Scilicet ſi ſit π arcus 180° circuli, cuius radius $= 1$, oſtendimus eſſe:

$$\text{ſin}\frac{m\pi}{2n} = \frac{m\pi}{2n} \cdot \frac{4nn - mm}{4nn} \cdot \frac{16nn - mm}{16nn} \cdot \frac{36nn - mm}{36nn} \text{ \&c.}$$

$$\text{coſ}\frac{m\pi}{2n} = \frac{nn - mm}{nn} \cdot \frac{9nn - mm}{9nn} \cdot \frac{25nn - mm}{25nn} \cdot \frac{49nn - mm}{49nn} \text{ \&c.}$$

Ponamus $n = 1$ & $m = 2x$; vt ſit

$$\text{ſin}\,\pi x = \pi x \cdot \frac{1 - xx}{1} \cdot \frac{4 - xx}{4} \cdot \frac{9 - xx}{9} \cdot \frac{16 - xx}{16} \cdot \text{\&c. vel}$$

$$\text{ſin}\,\pi x = \pi x \cdot \frac{1-x}{1} \cdot \frac{1+x}{1} \cdot \frac{2-x}{2} \cdot \frac{2+x}{2} \cdot \frac{3-x}{3} \cdot \frac{3+x}{3} \cdot \frac{4-x}{4} \cdot \text{\&c.} \quad \&$$

$$\operatorname{coſ}\pi x = \frac{1-4xx}{1}\cdot\frac{9-4xx}{9}\cdot\frac{25-4xx}{25}\cdot\frac{49-4xx}{49}\cdot \&c. \quad \text{ſeu}$$

$$\operatorname{coſ}\pi x = \frac{1-2x}{1}\cdot\frac{1+2x}{1}\cdot\frac{3-2x}{3}\cdot\frac{3+2x}{3}\cdot\frac{5-2x}{5}\cdot\frac{5+2x}{5}\cdot \&c.$$

Ex his ergo expreſſionibus, ſi logarithmi ſumantur, erit:

$$l\operatorname{ſin}\pi x = l\pi x + l\frac{1-x}{1} + l\frac{1+x}{1} + l\frac{2-x}{2} + l\frac{2+x}{2} + l\frac{3-x}{3} + \&c.$$

$$l\operatorname{coſ}\pi x = l\frac{1-2x}{1} + l\frac{1+2x}{1} + l\frac{3-2x}{3} + l\frac{3+2x}{3} + l\frac{5-2x}{5} + \&c.$$

43. Sumamus nunc harum ſerierum logarithmicarum differentialia, & diuiſione vbique per dx facta prior ſeries dabit:

$$\frac{\pi\operatorname{coſ}\pi x}{\operatorname{ſin}\pi x} = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \frac{1}{3-x} + \&c.$$

quae eſt ea ipſa ſeries, quam §. 37. tractauimus. Altera vero ſeries dabit:

$$\frac{-\pi\operatorname{ſin}\pi x}{\operatorname{coſ}\pi x} = \frac{-2}{1-2x} + \frac{2}{1+2x} - \frac{2}{3-2x} + \frac{2}{3+2x} - \frac{2}{5-2x} + \&c.$$

Ponamus $2x = z$, vt ſit $x = \frac{z}{2}$, & diuidamus per -2 erit:

$$\frac{\pi\operatorname{ſin}\frac{1}{2}\pi z}{2\operatorname{coſ}\frac{1}{2}\pi z} = \frac{1}{1-z} - \frac{1}{1+z} + \frac{1}{3-z} - \frac{1}{3+z} + \frac{1}{5-z} - \&c.$$

Cum autem ſit

$$\operatorname{ſin}\tfrac{1}{2}\pi z = \sqrt{\frac{1-\operatorname{coſ}\pi z}{2}} \quad \& \quad \operatorname{coſ}\tfrac{1}{2}\pi z = \sqrt{\frac{1+\operatorname{coſ}\pi z}{2}}$$

erit:

$$\frac{\pi\sqrt{(1-\operatorname{coſ}\pi z)}}{\sqrt{(1+\operatorname{coſ}\pi z)}} = \frac{2}{1-z} - \frac{2}{1+z} + \frac{2}{3-z} - \frac{2}{3+z} + \frac{2}{5-z} - \&c.$$

ſeu

ſeu loco z ſcribendo x erit:

$$\frac{\pi V(1-\operatorname{coſ}\pi x)}{V(1+\operatorname{coſ}\pi x)} = \frac{2}{1-x} - \frac{2}{1+x} + \frac{2}{3-x} - \frac{2}{3+x} + \frac{2}{5-x} - \&c.$$

Addatur haec ſeries ad primum inuentam:

$$\frac{\pi \operatorname{coſ}\pi x}{\operatorname{ſin}\pi x} = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \frac{1}{3-x} + \&c.$$

Atque reperietur huius ſeriei:

$$\frac{1}{x} + \frac{1}{1-x} - \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} + \frac{1}{3-x} - \frac{1}{3+x} - \&c.$$

Summa $= \frac{\pi V(1-\operatorname{coſ}\pi x)}{V(1+\operatorname{coſ}\pi x)} + \frac{\pi \operatorname{coſ}\pi x}{\operatorname{ſin}\pi x}$. At fractio haec $\frac{V(1-\operatorname{coſ}\pi x)}{V(1+\operatorname{coſ}\pi x)}$, ſi numerator & denominator multiplicetur per $V(1-\operatorname{coſ}\pi x)$ abit in $\frac{1-\operatorname{coſ}\pi x}{\operatorname{ſin}\pi x}$. Quocirca ſumma ſeriei erit $= \frac{\pi}{\operatorname{ſin}\pi x}$, quae eſt ea ipſa, quam §. 34. habuimus: vnde eam vlterius non perſequemur.

CAPUT III.

DE INUENTIONE DIFFERENTIARUM FINITARUM.

44.

Quemadmodum ex functionum differentiis finitis earum differentialia facile inueniri queant, in initio fuſius expoſuimus, atque adeo ex hoc fonte principium differentialium deriuauimus. Si enim differentiae, quae aſſumtae erant finitae, euaneſcant, in nihilumque abeant, oriuntur differentialia; & quia hoc caſu plures & ſaepe innumeri termini, qui differentiam finitam conſtituunt, reiiciuntur, differentialia multo facilius inueniri, atque commodius ſuccinctiusque exprimi poſſunt, quam differentiae finitae. Neque igitur hinc viciſſim via patere videtur, a differentialibus ad differentias finitas aſcendendi. Interim tamen eo modo, quo hic vtemur, ex differentialibus omnium ordinum cuiuscunque functionis, eiusdem differentiae finitae omnes definiri poterunt.

45. Sit y functio quaecunque ipſius x, quae cum poſito $x+dx$ loco x abeat in $y+dy$, ſi denuo loco x ponatur $x+dx$, valor $y+dy$ ſuo differentiali $dy+ddy$ augebitur, fietque $=y+2dy+ddy$, qui ergo valor reſpondebit ipſius x valori $x+2dx$. Simili modo ſi ponamus quantitatem x continuo ſuo differentiali dx augeri, vt ſucceſſiue valores

$x+$

$x + dx$; $x + 2dx$; $x + 3dx$; $x + 4dx$; &c. induat, valores ipſius y reſpondentes erunt, quos haec tabella indicat:

Valores ipſius x	Valores reſpondentes functionis y
$x + dx$	$y + dy$
$x + 2dx$	$y + 2dy + ddy$
$x + 3dx$	$y + 3dy + 3ddy + dy$
$x + 5dx$	$y + 4dy + 6ddy + 4d^3y + d^4y$
$x + 5dx$	$y + 5dy + 10ddy + 10d^3y + 5d^4y + d^5y$
$x + 6dx$	$y + 6dy + 15ddy + 20d^3y + 15d^4y + 6d^5y + d^6y$
&c.	&c.

46. Generaliter ergo ſi x abeat in $x + ndx$, functio y recipiet hanc formam:

$$y + \frac{n}{1} dy + \frac{n(n-1)}{1.\ 2} ddy + \frac{n(n-1)(n-2)}{1.\ 2.\ 3} d^3y + \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4} d^4y + \&c.$$

in qua expreſſione, etſi quilibet terminus infinities minor eſt quam praecedens, tamen nullum praetermiſimus, quo iſta formula ad praeſens negotium apta redderetur. Statuemus enim pro n numerum infinite magnum, & quoniam notauimus, fieri poſſe vt productum ex quantitate infinite magna in infinite paruam aequetur quantitati finitae, terminus ſecundus vtique homo-

geneus fieri poterit primo, ſeu ndy quantitatem finitam repraeſentare poterit. Ob eandemque rationem terminus tertius $\frac{n(n-1)}{1.\,2}ddy$, etſi ddy infinities minus eſt quam dy, tamen quia alter factor $\frac{n(n-1)}{1.\,2}$ infinities maior eſt quam n, terminus quoque tertius quantitatem finitam exprimere poterit: ſicque poſito n numero infinito nullum illius expreſſionis terminum reiicere licebit.

47. Poſito autem n numero infinito quocunque is numero finito ſiue augeatur ſiue diminuatur, numerus reſultans ad n habebit rationem aequalitatis, hincque pro ſingulis factoribus $n-1$, $n-2$, $n-3$, $n-4$, &c. vbique ſcribi poterit n. Cum enim ſit

$$\frac{n(n-1)}{1.\,2}ddy = \tfrac{1}{2}nnddy - \tfrac{1}{2}nddy$$

prior terminus $\frac{1}{2}nnddy$ ad poſteriorem $\frac{1}{2}nddy$ rationem tenebit vt n ad 1, ſicque hic reſpectu illius euaneſcet; loco $\frac{n(n-1)}{1.\,2}$ ergo ſcribi poterit $\frac{1}{2}nn$. Simili modo quarti termini coefficiens $\frac{n(n-1)(n-2)}{1.\,2.\,3}$ contrahi poterit in $\frac{n^3}{6}$ pariterque in ſequentibus numeri, quibus n in factoribus diminuitur, negligi poterunt. Hoc vero facto functio y, ſi loco x ponatur $x+ndx$, exiſtente numero n infinito, ſequentem valorem accipiet:

$y+$

$$y + \frac{ndy}{1} + \frac{nnddy}{1.2} + \frac{n^3 d^3 y}{1.2.3} + \frac{n^4 d^4 y}{1.2.3.4} + \frac{n^5 d^5 y}{1.2.3.4.5} + \&c.$$

48. Cum igitur ſumto n numero infinite magno etiamſi dx ſit infinite paruum, productum ndx quantitatem finitam exprimere poſſit, ponamus $ndx = \omega$, vt ſit $n = \frac{\omega}{dx}$; erit vtique n numerus infinitus, cum ſit quotus ex diuiſione quantitatis finitae ω per infinite paruam dx reſultans. Valore autem hoc loco n adhibito cognoſcemus, ſi quantitas variabilis x augeatur quauis quantitate finita ω, ſeu ſi loco x ponatur $x + \omega$, tum quamuis ipſius functionem y abituram eſſe in hanc formam:

$$y + \frac{\omega dy}{1 dx} + \frac{\omega^2 ddy}{1.2 dx^2} + \frac{\omega^3 d^3 y}{1.2.3 dx^3} + \frac{\omega^4 d^4 y}{1.2.3.4 dx^4} + \&c.$$

cuius expreſſionis ſinguli termini per continuam ipſius y differentiationem inueniri poterunt. Cum enim y ſit functio ipſius x, oſtendimus ſupra, has functiones omnes $\frac{dy}{dx}$; $\frac{ddy}{dx^2}$; $\frac{d^3 y}{dx^3}$; &c. quantitates finitas exhibere.

49. Cum igitur, dum quantitas variabilis x quantitate finita ω augeri aſſumitur, functio eius quaecunque y augeatur ſua differentia prima, quam ſupra per Δy indicauimus, exiſtente $\omega = \Delta x$: differentia ipſius y per continuam differentiationem reperiri poterit; erit enim:

Δy

$$\Delta y = \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2\,dx^2} + \frac{\omega^3 d^3 y}{6\,dx^3} + \frac{\omega^4 d^4 y}{24\,dx^4} + \&c.$$

ſeu

$$\Delta y = \frac{\Delta x}{1} \cdot \frac{dy}{dx} + \frac{\Delta x^2}{2} \cdot \frac{ddy}{dx^2} + \frac{\Delta x^3}{6} \cdot \frac{d^3 y}{dx^3} + \frac{\Delta x^4}{24} \cdot \frac{d^4 y}{dx^4} + \&c.$$

Sicque differentia finita Δy exprimitur per progreſſionem, cuius ſinguli termini ſecundum poteſtates ipſius Δx procedunt. Atque hinc viciſſim patet, ſi quantitas x tantum quantitate infinite parua augeatur, vt Δx abeat in eius differentiale dx, omnes terminos prae primo euaneſcere, foreque $\Delta y = dy$; facto enim $\Delta x = dx$, differentia Δy abit per definitionem in differentiale dy.

50. Quoniam ſi loco x ponatur $x + \omega$, eius functio quaecuuque y induit ſequentem valorem:

$$y + \frac{\omega dy}{dx} + \frac{\omega^2 ddx}{2\,dx^2} + \frac{\omega^3 d^3 y}{6\,dx^3} + \frac{\omega^4 d^4 y}{24 dx^4} + \&c.$$

veritas huius expreſſionis comprobari poterit eiusmodi exemplis, quibus differentialia altiora ipſius y tandem euaneſcunt: his enim caſibus numerus terminorum ſuperioris expreſſionis fiet finitus:

EXEMPLUM I.

Quaeratur valor expreſſionis $xx - x$ *ſi loco* x *ponatur* $x + 1$.

Ponatur $y = xx - x$; & cum x in $x + 1$ abire ſtatuatur, fiet $\omega = 1$, ſumtis iam differentialibus erit:

$$\frac{dy}{dx} = 2x - 1;\quad \frac{ddy}{dx^2} = 2;\quad \frac{d^3 y}{dx^3} = 0;\quad \&c.$$

Hinc

Hinc functio $y = xx - x$ poſito $x + 1$ loco x abibit in : $xx - x + 1(2x - 1) + \frac{1}{2}.2 = xx + x$.

Quodſi autem in $xx - x$ loco x actu ponatur $x + 1$

abibit	xx	in	$xx + 2x + 1$
	x	in	$x + 1$
Ergo	$xx - x$	in	$xx + x$.

EXEMPLUM II.

Quaeratur valor expreſſionis $x^3 + xx + x$, *ſi loco* x *ponatur* $x + 2$.

Ponatur $y = x^3 + xx + x$, fietque $\omega = 2$; nunc cum ſit $y = x^3 + xx + x$

erit
$$\frac{dy}{dx} = 3xx + 2x + 1$$
$$\frac{ddy}{dx^2} = 6x + 2$$
$$\frac{d^3y}{dx^3} = 6$$
$$\frac{d^4y}{dx^4} = 0.$$

Ex his valor functionis $y = x^3 + xx + x$, ſi pro x ſtatuatur $x + 2$, erit ſequens:

$x^3 + xx + x + 2(3xx + 2x + 1) + \frac{4}{2}(6x + 2) + \frac{8}{6}.6$ $= x^3 + 7xx + 17x + 14$, qui idem prodit ſi actu loco x ſubſtituatur $x + 2$.

EXEMPLUM III.

Quaeratur valor expreſſionis $xx + 3x + 1$, *ſi loco* x *ponatur* $x - 3$.

 Fiet

Fiet ergo $\omega = -3$; & poſito

$$y = xx + 3x + 1, \text{ erit}$$

$$\frac{dy}{dx} = 2x + 3$$

$$\frac{ddy}{dx^2} = 2:$$

vnde poſito $x - 3$ loco x functio $x^2 + 3x + 1$ abibit in $x^2 + 3x + 1 - \frac{3}{1}(xx + 3) + \frac{9}{2}.2 = x^2 - 3x + 1$.

51. Si pro ω ſumatur numerus negatiuus, reperietur valor, quem functio quaecunque ipſius x induit, dum ipſa quantitas x diminuitur data quantitate ω. Scilicet ſi loco x ponatur $x - \omega$, functio ipſius x quaecunque y accipiet iſtum valorem:

$$y - \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} - \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} - \&c.$$

vnde omnes variationes, quas functio y ſubire poteſt, dum quantitas x vtrinque variatur, inueniri poterunt. Quodſi autem y fuerit functio rationalis integra ipſius x, quoniam tandem ad eius differentialia euaneſcentia deuenitur, valor variatus per expreſſionem finitam exprimetur; ſin autem y non fuerit huiusmodi functio, valor variatus per ſeriem infinitam exprimetur, cuius propterea ſumma, quoniam ſi ſubſtitutio actu inſtituatur, valor variatus facile aſſignatur, expreſſione finita exhiberi poterit.

52. Quemadmodum autem differentia prima eſt inuenta, ita quoque differentiae ſequentes ſimilibus expres-

preſſionibus exhiberi poſſunt. Induat enim x ſucceſſiue valores $x+\omega$, $x+2\omega$, $x+3\omega$, $x+4\omega$, &c. atque valores ipſius y reſpondentes indicentur per y^{I}, y^{II}, y^{III}, y^{IV}, &c. ſicuti in initio huius libri poſuimus. Quoniam ergo y^{I}, y^{II}, y^{III}, y^{IV}, &c. ſunt valores, quos y nanciſcitur, ſi loco x ſcribatur reſpectiue

$$x+\omega,\ x+2\omega,\ x+3\omega,\ x+4\omega,\ \&c.$$

per modo demonſtrata iſti ipſius y valores ita exprimentur:

$$y^{I}=y+\frac{\omega dy}{dx}+\frac{\omega^2 ddy}{2dx^2}+\frac{\omega^3 d^3y}{6dx^3}+\frac{\omega^4 d^4y}{24dx^4}+\&c.$$

$$y^{II}=y+\frac{2\omega dy}{dx}+\frac{4\omega^2 ddy}{2dx^2}+\frac{8\omega^3 d^3y}{6dx^3}+\frac{16\omega^4 d^4y}{24dx^4}+\&c.$$

$$y^{III}=y+\frac{3\omega dy}{dx}+\frac{9\omega^2 ddy}{2dx^2}+\frac{27\omega^3 d^3y}{6dx^3}+\frac{81\omega^4 d^4y}{24dx^4}+\&c.$$

$$y^{IV}=y+\frac{4\omega dy}{dx}+\frac{16\omega^2 ddy}{2dx^2}+\frac{64\omega^3 d^3y}{6dx^3}+\frac{256\omega^4 d^4y}{24dx^4}+\&c.$$

&c.

53. Cum igitur, ſi Δy, $\Delta^2 y$, $\Delta^3 y$, $\Delta^4 y$, &c. denotent differentias, primam, ſecundam, tertiam, quartam, &c. ſit:

$$\Delta y = y^{I} - y$$

$$\Delta^2 y = y^{II} - 2y^{I} + y$$

$$\Delta^3 y = y^{III} - 3y^{II} + 3y^{I} - y$$

$$\Delta^4 y = y^{IV} - 4y^{III} + 6y^{II} - 4y^{I} + y$$

&c.

 iſtae

iſtae differentiae per differentialia hoc modo exprimentur:

$$\Delta y = \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} + \&c.$$

$$\Delta^2 y = \frac{(2^2-2.1)\omega^2 ddy}{2dx^2} + \frac{(2^3-2.1)\omega^3 d^3y}{6dx^3} + \frac{(2^4-2.1)\omega^4 d^4y}{24dx4} + \&c.$$

$$\Delta^3 y = \frac{(3^3-3.2^3+3.1)\omega^3 d^3y}{6dx^3} + \frac{(3^4-3.2^4+3.1)\omega^4 d^4y}{24dx^4} + \&c.$$

$$\Delta^4 y = \frac{(4^4-4.3^4+6.2^4-4.1)\omega^4 d^4y}{24dx^4} + \frac{(4^5-4.3^5+6.2^5-4.1)\omega^5 d^5y}{120dx^5} + \&c.$$

&c.

54. Quantam vtilitatem afferant iſtae differentiarum expreſſionis in doctrina ſerierum & progreſſionum, cum ſponte patet, tum in ſequentibus vberius exponemus. Interim tamen in hoc capite vſum, qui hinc ad ſerierum notitiam immediate redundat, perpendamus. Quanquam vulgo indices terminorum ſeriei cuiuscunque progreſſionem arithmeticam, cuius differentia eſt vnitas, conſtituere aſſumuntur; tamen quo vſus latius pateat, atque applicatio facilius fieri poſſit, differentiam ſtatuamus $= \omega$, ita vt, ſi terminus generalis ſeu is qui indici x reſpondet, fuerit y; ſequentes conueniant indicibus

$$x+\omega, \quad x+2\omega, \quad x+3\omega, \quad \&c.$$

Quodſi ergo his indicibus reſpondeant ſequentes ſeriei termini:

$$x, \; x+\omega, \; x+2\omega, \; x+3\omega, \; x+4\omega, \; \&c.$$
$$y, \; P, \; Q, \; R, \; S, \; \&c.$$

ſin-

ſinguli ex y eiusque differentialibus definientur hoc modo:

$$P = y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2 dx^2} + \frac{\omega^3 d^3 y}{6 dx^3} + \frac{\omega^4 d^4 y}{24 dx^4} + \&c.$$

$$Q = y + \frac{2\omega dy}{dx} + \frac{4\omega^2 ddy}{2 dx^2} + \frac{8\omega^3 d^3 y}{6 dx^3} + \frac{16\omega^4 d^4 y}{24 dx^4} + \&c.$$

$$R = y + \frac{3\omega dy}{dx} + \frac{9\omega^2 ddy}{2 dx^2} + \frac{27\omega^3 d^3 y}{6 dx^3} + \frac{81\omega^4 d^4 y}{24 dx^4} + \&c.$$

$$S = y + \frac{4\omega dy}{dx} + \frac{16\omega^2 ddy}{2 dx^2} + \frac{64\omega^3 d^3 y}{6 dx^3} + \frac{256\omega^4 d^4 y}{24 dx^4} + \&c.$$

&c.

55. Si hae expreſſiones a ſe inuicem ſubtrahantur, in differentias non amplius ingredietur y, eritque

$$P - y = \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2 dx^2} + \frac{\omega^3 d^3}{6 dx^3} + \frac{\omega^4 d^4 y}{24 dx^4} + \&c.$$

$$Q - P = \frac{\omega dy}{dx} + \frac{3\omega^2 ddy}{2 dx^2} + \frac{7\omega^3 d^3 y}{6 dx^3} + \frac{15\omega^4 d^4 y}{24 dx^4} + \&c.$$

$$R - Q = \frac{\omega dy}{dx} + \frac{5\omega^2 ddy}{2 dx^2} + \frac{19\omega^3 d^3 y}{6 dx^3} + \frac{65\omega^4 d^4 y}{20 dx^4} + \&c.$$

$$S - R = \frac{\omega dy}{dx} + \frac{7\omega^2 ddy}{2 dx^2} + \frac{37\omega^3 d^3 y}{6 dx^3} + \frac{175\omega^4 d^4 y}{24 dx^4} + \&c.$$

$$T - S = \frac{\omega dy}{dx} + \frac{9\omega^2 ddy}{2 dx^2} + \frac{61\omega^3 d^3 y}{6 dx^3} + \frac{369\omega^4 d^4 y}{24 dx^4} + \&c.$$

&c.

Si hae expreſſiones denuo a ſe inuicem ſubtrahantur, etiam differentialia prima ſe deſtruent; eritque

$$Q-2P+y=\frac{2\omega^2 ddy}{2dx^2}+\frac{6\omega^3 d^3y}{6dx^3}+\frac{14\omega^4 d^4y}{24dx^4}+\&c.$$

$$R-2Q+P=\frac{2\omega^2 ddy}{2dx^2}+\frac{12\omega^3 d^3y}{6dx^3}+\frac{50\omega^4 d^4y}{24dx^4}+\&c.$$

$$S-2R+Q=\frac{2\omega^2 ddy}{2dx^2}+\frac{18\omega^3 d^3y}{6dx^3}+\frac{110\omega^4 d^4y}{24dx^4}+\&c.$$

$$T-2S+R=\frac{2\omega^2 ddy}{2dx^2}+\frac{24\omega^3 d^3y}{6dx^3}+\frac{194\omega^4 d^4y}{24dx^4}+\&c.$$

&c.

His autem denuo a ſe inuicem ſubtractis differentialia quoque ſecunda ex computo egredientur:

$$R-3Q+3P-y=\frac{6\omega^3 d^3y}{6dx^3}+\frac{36\omega^4 d^4y}{24dx^4}+\&c.$$

$$S-3R+3Q-P=\frac{6\omega^3 d^3y}{6dx^3}+\frac{60\omega^4 d^4y}{24dx^4}+\&c.$$

$$T-3S+3R-Q=\frac{6\omega^3 d^3y}{6dx^3}+\frac{84\omega^4 d^4y}{24dx^4}+\&c.$$

ſubtractionem autem vlterius continuando fiet:

$$S-4R+6Q-4P+y=\frac{24\omega^4 d^4y}{24dx^4}+\&c.$$

$$T-4S+6R-4Q+P=\frac{24\omega^4 d^4y}{24dx^4}+\&c.$$

atque

$$T-5S+10R-10Q+5P-y=\frac{120\omega^5 d^5y}{120dx^5}+\&c.$$

56. Quodſi ergo y fuerit functio rationalis integra ipſius x, quia eius differentialia altiora tandem euaneſcent

cent, hoc modo procedendo tandem ad expreſſiones euaneſcentes peruenietur. Cum igitur iſtae expreſſiones ſint differentiae ipſius y, earum formas & coefficientes diligentius perpendamus:

$$y = y$$

$$\Delta y = \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} + \frac{\omega^5 d^5y}{120dx^5} + \&c.$$

$$\Delta^2 y = \frac{\omega^2 ddy}{dx^2} + \frac{3\omega^3 d^3y}{3dx^3} + \frac{7\omega^4 d^4y}{3.4\,dx^4} + \frac{15\omega^5 d^5y}{3.4.5\,dx^5} + \frac{31\omega^6 d^6y}{3.4.5.6\,dx^6} + \&c.$$

$$\Delta^3 y = \frac{\omega^3 d^3y}{dx^3} + \frac{6\omega^4 d^4y}{4dx^4} + \frac{25\omega^5 d^5y}{4.5\,dx^5} + \frac{90\omega^6 d^6y}{4.5.6\,dx^6} + \frac{301\omega^7 d^7y}{4.5.6.7\,dx^7} + \&c.$$

$$\Delta^4 y = \frac{\omega^4 d^4y}{dx^4} + \frac{10\omega^5 d^5y}{5dx^5} + \frac{65\omega^6 d^6y}{5.6\,dx^6} + \frac{350\omega^7 d^7y}{5.6.7.\,dx^7} + \&c.$$

$$\Delta^5 y = \frac{\omega^5 d^5y}{dx^5} + \frac{15\omega^6 d^6y}{6\,dx^6} + \frac{140\omega^7 d^7y}{6.7\,dx^7} + \frac{1050\omega^8 d^8y}{6.7.8\,dx^8} + \&c.$$

$$\Delta^6 y = \frac{\omega^6 d^6y}{dx^6} + \frac{21\omega^7 d^7y}{7\,dx^7} + \frac{266\omega^8 d^8y}{7.8\,dx^8} + \frac{2646\omega^9 d^9y}{7.8.9\,dx^9} + \&c.$$

&c.

In quibus ſeriebus quemadmodum denominatores procedant, clarum eſt; numeratorum autem coefficientes ita formantur, vt quiuis coefficiens numeratoris ſit aggregatum ex ſupra ſtante & praecedente per exponentem differentiae multiplicato. Sic in ſerie differentiam $\Delta^6 y$ exprimente, eſt $2646 = 1050 + 6.266$.

57. Conſideremus quoque ſeriem eandem ſimul retro continuatam, quae continet terminos indicibus $x - \omega$; $x - 2\omega$; $x - 3\omega$; &c. reſpondentes:

In

$x-4\omega;\ x-3\omega;\ x-2\omega;\ x-\omega;\ x;\ x+\omega;\ x+2\omega;\ x+3\omega;\ x+4\omega;$ &c.

$s,\quad r,\quad q,\quad p,\quad y,\quad P,\quad Q,\quad R,\quad S,$ &c.

Cum igitur ſit:

$$p = y - \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} - \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} - \&c.$$

$$q = y - \frac{2\omega dy}{dx} + \frac{4\omega^2 ddy}{2dx^2} - \frac{8\omega^3 d^3y}{6dx^3} + \frac{16\omega^4 d^4y}{24dx^4} - \&c.$$

$$r = y - \frac{3\omega dy}{dx} + \frac{9\omega^2 ddy}{2dx^2} - \frac{27\omega^3 d^3y}{6dx^3} + \frac{81\omega^4 d^4y}{24dx^4} - \&c.$$

$$s = y - \frac{4\omega dy}{dx} + \frac{16\omega^2 ddy}{2dx^2} - \frac{64\omega^3 d^3y}{6dx^3} + \frac{256\omega^4 dx^4}{24dx^4} - \&c.$$

&c.

erit his valoribus a ſuperioribus P, Q, R, S, &c. ſubtrahendis:

$$\frac{P-p}{2} = \frac{\omega dy}{dx} + \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^5 d^5y}{120dx^5} + \&c.$$

$$\frac{Q-q}{2} = \frac{2\omega dy}{dx} + \frac{8\omega^3 d^3y}{6dx^3} + \frac{32\omega^5 d^5y}{120dx^5} + \&c.$$

$$\frac{R-r}{2} = \frac{3\omega dy}{dx} + \frac{27\omega^3 d^3y}{6dx^3} + \frac{243\omega^5 d^5y}{120dx^5} + \&c.$$

$$\frac{S-s}{2} = \frac{4\omega dy}{dx} + \frac{64\omega^3 d^3y}{6dx^3} + \frac{1024\omega^5 d^5y}{120dx^5} + \&c.$$

&c.

ſin

ſin autem termini hi ad ſuperiores addantur, tum, quemadmodum hic differentialia parium ordinum deerant, differentialia imparia ex computo egredientur.

Erit enim

$$\frac{P+p}{2} = y + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^4 d^4y}{24dx^4} + \frac{\omega^6 d^6y}{720dx^6} + \&c.$$

$$\frac{Q+q}{2} = y + \frac{4\omega^2 ddy}{2dx^2} + \frac{16\omega^4 d^4y}{24dx^4} + \frac{64\omega^6 d^6y}{720dx^6} + \&c.$$

$$\frac{R+r}{2} = y + \frac{9\omega^2 ddy}{2dx^2} + \frac{81\omega^4 d^4y}{24dx^4} + \frac{729\omega^6 d^6y}{720dx^6} + \&c.$$

$$\frac{S+s}{2} = y + \frac{16\omega^2 ddy}{2dx^2} + \frac{256\omega^4 d^4y}{24dx^4} + \frac{4096\omega^6 d^6y}{720dx^6} + \&c.$$

&c.

58. Quoniam termini antecedentes omnes exprimi poſſunt, ſi ii in vnam ſummam colligantur, prodibit ſeriei propoſitae terminus ſummatorius. Reſpondeat ſcilicet terminus primus indici $x - n\omega$, eritque ipſe terminus

primus $=$

$$y - \frac{n\omega dy}{dx} + \frac{n^2\omega^2 ddy}{2dx^2} - \frac{n^3\omega^3 d^3y}{6dx^3} + \frac{n^4\omega^4 d^4y}{24dx^4} + \&c.$$

Cum igitur terminus indici x reſpondens ſit $= y$, terminorumque omnium numerus ſit $= n + 1$, erit ſumma omnium a primo ad vltimum y incluſiue ſumtorum

ſeu terminus ſummatorius $=$

$$(n+1)y - \frac{\omega dy}{dx}(1+2+3+ \ldots +n)$$
$$+ \frac{\omega^2 ddy}{2dx^2}(1+2^2+3^2+ \ldots +n^2)$$
$$- \frac{\omega^3 d^3y}{6dx^3}(1+2^3+3^3+ \ldots +n^3)$$
$$+ \frac{\omega^4 d^4y}{24dx^4}(1+2^4+3^4+ \ldots +n^4)$$
$$- \frac{\omega^5 d^5y}{120dx^5}(1+2^5+3^5+ \ldots +n^5)$$
&c.

59. Supra autem ſingularum harum ſerierum ſummas exhibuimus, quae ſi hic ſubſtituantur, erit ſumma ſeriei noſtrae propoſitae $=$

$$(n+1)y - \frac{\omega dy}{dx}(\tfrac{1}{2}nn+\tfrac{1}{2}n)$$
$$+ \frac{\omega^2 ddy}{2dx^2}(\tfrac{1}{3}n^3+\tfrac{1}{2}nn+\tfrac{1}{6}n)$$
$$- \frac{\omega^3 d^3y}{6dx^3}(\tfrac{1}{4}n^4+\tfrac{1}{2}n^3+\tfrac{1}{4}n^2)$$
$$+ \frac{\omega^4 d^4y}{24dx^4}(\tfrac{1}{5}n^5+\tfrac{1}{2}n^4+\tfrac{1}{3}n^3-\tfrac{1}{30}n)$$
$$- \frac{\omega^5 d^5y}{120dx^5}(\tfrac{1}{6}n^6+\tfrac{1}{2}n^5+\tfrac{5}{12}n^4-\tfrac{1}{12}n^2)$$
&c.

vbi n dabitur ex indice termini primi, a qua ſumma computatur. Ita ſi ponatur $\omega = 1$, & index termini pri-

primi ſit $=1$, ſecundi $=2$, & vltimi $=x$, ita vt haec ſeries ſit propoſita:

$$\begin{array}{l} 1,\ 2,\ 3,\ 4,\ \dots\dots\ x \\ a,\ b,\ c,\ d,\ \dots\dots\ y \end{array}$$

erit huius ſeriei ſumma (ob $x-n=1$ & $n=x-1$)

$$= xy - \frac{dy}{dx}\left(\tfrac{1}{2}xx - \tfrac{1}{2}x\right)$$
$$+ \frac{ddy}{2dx^2}\left(\tfrac{1}{3}x^3 - \tfrac{1}{2}xx + \tfrac{1}{6}x\right)$$
$$- \frac{d^3y}{6dx^3}\left(\tfrac{1}{4}x^4 - \tfrac{1}{2}x^3 + \tfrac{1}{4}xx\right)$$
$$+ \frac{d^4y}{24dx^4}\left(\tfrac{1}{5}x^5 - \tfrac{1}{2}x^4 + \tfrac{1}{3}x^3 - \tfrac{1}{30}x\right)$$
$$- \frac{d^5y}{120dx^5}\left(\tfrac{1}{6}x^6 - \tfrac{1}{2}x^5 + \tfrac{5}{12}x^4 - \tfrac{1}{12}x^2\right)$$
$$+ \frac{d^6y}{720dx^6}\left(\tfrac{1}{7}x^7 - \tfrac{1}{2}x^6 + \tfrac{1}{2}x^5 - \tfrac{1}{6}x^3 + \tfrac{1}{42}x\right)$$

&c.

60. Ex hac ſummae expreſſione, quia coefficientes vehementer augentur, ſi x fuerit numerus magnus, parum vtilitatis ad doctrinam ſerierum redundat; interim tamen iuvabit aliquas proprietates inde fluentes commemoraſſe. Sit terminus generalis $y=x^n$, atque terminus ſummatorius per Sy ſeu S.x^n indicetur. Qua deſignatione vbique adhibita erit:

$$\tfrac{1}{2}xx - \tfrac{1}{2}x = \text{S}.x - x$$
$$\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + \tfrac{1}{6}x = \text{S}.x^2 - x^2$$
$$\tfrac{1}{4}x^4 - \tfrac{1}{2}x^3 + \tfrac{1}{4}xx = \text{S}.x^3 - x^3 \quad \&c.$$

 Quam-

Quamobrem ex ſuperiori expreſſione obtinebitur:

$$S.x^n = x^{n+1} - nx^{n-1}S.x + nx^n$$

$$+ \frac{n(n-1)}{1.\ 2} x^{n-2} S.x^2 - \frac{n(n-1)}{1.\ 2} x^n$$

$$- \frac{n(n-1)(n-2)}{1.\ 2.\ 3} x^{n-3} S.x^3 + \frac{n(n-1)(n-2)}{1.\ 2.\ 3} x^n$$

&c.

At cum ſit

$$(1-1)^n = 0 = 1 - n + \frac{n(n-1)}{1.\ 2} - \frac{n(n-1)(n-2)}{1.\ 2.\ 3} + \&c.$$

erit $n - \frac{n(n-1)}{1.\ 2} + \frac{n(n-1)(n-2)}{1.\ 2.\ 3} - \&c.$

$= 1$, ideoque excepto caſu $n = 0$, quo iſta expreſſio fit $= 0$.

$$S.x^n = x^{n+1} + x^n - nx^{n-1}S.x$$

$$+ \frac{n(n-1)}{1.\ 2} x^{n-2} S.x^2$$

$$- \frac{n(n-1)(n-2)}{1.\ 2.\ 3} x^{n-3} S.x^3$$

$$+ \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4} x^{n-4} S.x^4$$

&c.

61. Quo tam veritas quam vis huius formulae clarius perſpiciatur, euoluamus ſingulos caſus, ſitque primo $n = 1$, eritque:

$S.x = x^2 + x - S.x$, ideoque $S.x = \frac{xx + x}{2}$, quemadmodum ſatis conſtat.

Pona-

Ponamus ergo $n = 2$, eſt erit:

$$S.x^2 = x^3 + xx - 2xS.x + S.x^2,$$

quae aequatio, cum vtrinque termini $S.x^2$ ſe tollant, idem dat, quod praecedens $S.x = \frac{xx+x}{2}$. Si ſit $n = 3$, erit

$$S.x^3 = x^4 + x^3 - 3x.^2 Sx + 3xS.x^2 - S.x^3,$$

ideoque

$$S.x^3 = \tfrac{3}{2}xS.x^2 - \tfrac{3}{2}x^2 Sx + \tfrac{1}{2}x^3(x+1),$$

ſi ponatur $n = 4$ prodibit:

$$S.x^4 = x^5 + x^4 - 4x^3 Sx + 6x^2 S.x^2 - 4xS.x^3 + S.x4,$$

vnde ob $S.x^4$ deſtructum erit:

$$S.x^3 = \tfrac{3}{2}xS.x^2 - x^2 S.x + \tfrac{1}{4}x^3(x+1)$$

a cuius triplo, ſi praecedentis duplum ſubtrahatur remanebit: $S.x^3 = \tfrac{3}{2}xS.x^2 - \tfrac{1}{4}x^3(x-1)$.

Si ponatur $n = 5$ fiet:

$$S.x^5 = x^6 + x^5 - 5x^4 Sx + 10x^3 S.x^2 - 10x^2 S.x^3 + 5xS.x^4 - S.x^5$$

ſeu

$$S.x^5 = \tfrac{5}{2}xS.x^4 - 5x.^2 Sx^3 + 5x.^3 Sx^2 - \tfrac{5}{2}x^4 Sx + \tfrac{1}{2}x^5(x+1)$$

atque ex $n = 6$ ſequitur:

$$S.x^6 = x^7 + x^6 - 6x^5 Sx + 15x^4 S.x^2 - 20x^3 S.x^3 + 15x^2 S.x^4 - 6xS.x^5 + S.x^6$$

ſeu

$$S.x^5 = \tfrac{5}{2}xS.x^4 - \tfrac{10}{3}x^2 S.x^3 + \tfrac{5}{2}x^3 S.x^2 - x^4 S.x + \tfrac{1}{6}x^5(x+1).$$

62. Ex his ergo generaliter concludimus, ſi fuerit $n = 2m + 1$ fore:

$$S.x^{2m+1} = \frac{2m+1}{2} x S x^{2m} - \frac{(2m+1)2m}{2.\ 1.\ 2} x^2 S.x^{2m-1} + \frac{(2m+1)2m(2m-1)}{2.\ 1.\ 2.\ 3} x^3 S.x^{2m-2}$$

$$- \ . \ . \ . \ . \ - \frac{(2m+1)}{2} x^{2m} S x + \tfrac{1}{2} x^{2m+1} (x+1).$$

Sin autem ſit $n = 2m + 2$, quia termini $S.x^{2m+2}$ ſe mutuo deſtruunt, reperietur:

$$S.x^{2m+1} = \frac{2m+1}{2} x S x^{2m} - \frac{(2m+1)2m}{2.\ 3} x^2 S x^{2m-1} + \frac{(2m+1)2m(2m-1)}{2.\ 3.\ 4} x^3 S.x^{2m-2}$$

$$- \ . \ . \ . \ . \ - x^{2m} S x + \frac{1}{2m+2} x^{2m+1} (x+1).$$

Duplici ergo modo ſummae poteſtatum imparium ex ſummis poteſtatum inferiorum determinari poſſunt: atque ex varia combinatione harum duarum formularum infinitae aliae formari poſſunt.

63. Multo facilius autem ſummae poteſtatum imparium ex antecedentibus definiri poſſunt: atque ad hoc quidem ſufficit ſolam ſummam poteſtatis paris antecedentis nouiſſe. Ex ſummis enim poteſtatum ſupra exhibitis conſtat, numerum terminorum ſummas conſtituentium, imparibus tantum poteſtatibus augeri, ita vt ſumma poteſtatis imparis totidem conſtet terminis, quot ſumma poteſtatis paris praecedentis. Sic ſi poteſtatis paris x^{2n} ſumma ſit:

$$S.x^{2n} = \alpha x^{2n+1} + \beta x^{2n} + \gamma x^{2n-1} - \delta x^{2n-3} + \varepsilon x^{2n-5} - \&c.$$

vidi-

vidimus enim, poſt terminum tertium alternos terminos deficere, ſimulque ſigna alternari; hinc ſumma ſequentis poteſtatis x^{2n+1} inuenietur, ſi ſinguli illius termini reſpectiue multiplicentur per hos numeros:

$$\frac{2n+1}{2n+2}x;\ \frac{2n+1}{2n+1}x;\ \frac{2n+1}{2n}x;\ \frac{2n+1}{2n-1}x;\ \frac{2n+1}{2n-2}x;\ \&c.$$

non omittendo terminos deficientes; eritque ergo

$$S.x^{2n+1} = \frac{2n+1}{2n+2}\alpha x^{2n+2} + \frac{2n+1}{2n+1}\beta x^{2n+1} + \frac{2n+1}{2n}\gamma x^{2n}$$

$$- \frac{2n+1}{2n-1}\delta x^{2n-2} + \frac{2n+1}{2n-4}\varepsilon x^{2n-4} - \frac{2n+1}{2n-6}\zeta x^{2n-6} + \&c.$$

Quodſi ergo conſtet ſumma poteſtatis x^{2n}, ex ea expedite ſumma ſequentis poteſtatis x^{2n+1} formari poterit.

64. Haec ſequentium ſummarum inueſtigatio etiam ad poteſtates pares extenditur; quoniam autem harum ſummae nouum terminum recipiunt, hic per iſtam methodum non inuenitur, ex natura tamen ipſius ſeriei, qua conſtat, ſi ponatur $x = 1$, ſummam quoque fieri debere $= 1$, ſemper erui poterit. Viciſſim autem ſemper ex ſumma cuiusuis poteſtatis cognita praecedentium poteſtatum ſummae inueniri poterunt. Si enim fuerit:

$$S.x^n = \alpha x^{n+1} + \beta x^n + \gamma x^{n-1} - \delta x^{n-3} + \varepsilon x^{n-5} - \zeta x^{n-7} + \&c.$$

erit pro poteſtate praecedente:

$$S.x^{n-1} = \frac{n+1}{n}\alpha x^n + \frac{n}{n}\beta x^{n-1} + \frac{(n-1)}{n}\gamma x^{n-2} - \frac{(n-3)}{n}\delta x^{n-4} + \&c.$$

hinc-

hincque vlterius regredi licet, quousque libuerit. Notandum autem eſt eſſe perpetuo $\alpha = \frac{1}{n+1}$ & $\beta = \frac{1}{2}$, vti ex formulis iam ſupra datis apparet.

65. Attendenti ſtatim patebit ſummam poteſtatum x^{n-1} prodire, ſi ſumma poteſtatum x^n differentietur, eiusque differentiale per ndx diuidatur; eritque adeo $d.Sx^n = ndx.Sx^{n-1}$ & quia eſt $d.x^n = nx^{n-1}dx$;

$$\text{erit} \quad d.Sx^n = S.nx^{n-1}dx = S.d.x^n;$$

ex quo intelligitur differentiale ſummae aequari ſummae differentialis: ita in genere ſi ſeriei cuiuſpiam terminus generalis fuerit $= y$, & Sy eius terminus ſummatorius; erit quoque $S.dy = d.Sy$: hoc eſt ſumma differentialium omnium terminorum aequatur differentiali ſummae ipſorum terminorum. Ratio autem huius aequalitatis facile perſpicitur ex iis, quae ſupra de ſerierum differentiatione attulimus. Cum enim ſit

$$S.x^n = x^n + (x-1)^n + (x-2)^n + (x-3)^n + (x-4)^n + \&c.$$

erit

$$\frac{d.Sx^n}{ndx} = x^{n-1} + (x-1)^{n-1} + (x-2)^{n-1} + (x-3)^{n-1} + \&c. = S.x^{n-1}$$

quae demonſtratio ad omnes alias ſeries patet.

66. Reuertamur autem, vnde digreſſi ſumus, ad differentias functionum, circa quas adhuc quaedam annotanda ſunt. Quoniam vidimus, ſi y fuerit functio quaecunque ipſius x, atque loco x vbique ponatur $x \pm \omega$, functionem y adepturam eſſe ſequentem valorem:

$y \pm$

$$y \pm \frac{\omega dy}{1dx} + \frac{\omega^2 ddy}{1.2dx^2} \pm \frac{\omega^3 d^3y}{1.2.3dx^3} + \frac{\omega^4 d^4y}{1.2.3.4dx^4} \pm \frac{\omega^5 d^5y}{1.2.3.4.5dx^5} + \&c.$$

haec expreſſio locum habebit, ſiue pro ω quantitas quaecunque conſtans accipiatur, ſiue etiam variabilis, ab ipſa x pendens. Inuentis enim per differentiationem valoribus fractionum $\frac{dy}{dx}$; $\frac{ddy}{dx^2}$; $\frac{d^3y}{dx^3}$; &c. in factoribus ω, ω^2, ω^3, &c. variabilitas non ſpectatur, hincque perinde eſt ſiue ω denotet quantitatem conſtantem, ſiue variabilem ab x pendentem.

67. Ponamus ergo eſſe $\omega = x$, atque in functione y loco x ſcribi $x - x = 0$. Quamobrem ſi in functione ipſius x quacunque y loco x vbique ſcribatur 0, valor functionis erit hic:

$$y - \frac{xdy}{1dx} + \frac{x^2ddy}{1.2dx^2} - \frac{x^3d^3y}{1.2.3dx^3} + \frac{x^4d^4y}{1.2.3.4dx^4} - \&c.$$

Haec ergo expreſſio ſemper indicat valorem, quem functio quaecunque y induit, ſi in ea ponatur $x = 0$, cuius veritatem ſequentia exempla illuſtrabunt:

EXEMPLUM I.

Sit $y = xx + ax + ab$, cuius valor, ſi ponatur $x = 0$, quaeratur, quem quidem conſtat fore $= ab$.

Cum ſit $y = xx + ax + ab$

erit $\frac{dy}{1dx} = 2x + a$

$\frac{ddy}{1.2dx^2} = 1$

ideoque prodibit valor quaeſitus

$$= xx + ax + ab - x(2x + a) + xx.\,1 = ab.$$

EXEMPLUM II.

Sit $y = x^3 - 2x + 3$, cuius valor, poſito $x = 0$, quaeratur, quem conſtat fore $= 3$.

Cum ſit $y = x^3 - 2x + 3$

erit $\frac{dy}{dx} = 3xx - 2$

$$\frac{ddy}{1.2\,dx^2} = 3x$$

$$\frac{d^3y}{1.2.3\,dx^3} = 1$$

obtinebitur valor quaeſitus

$$= x^3 - 2x + 3 - x(3xx - 2) + xx.\,3x - x^3.\,1 = 3.$$

EXEMPLUM III.

Sit $y = \frac{x}{1-x}$, cuius valor poſito $x = 0$, quaeritur, quem conſtat fore $= 0$.

Cum ſit $y = \frac{x}{1-x}$; erit $\frac{dy}{dx} = \frac{1}{(1-x)^2}$;

$\frac{ddy}{1.2\,dx^2} = \frac{1}{(1-x)^3}$; $\frac{d^3y}{1.2.3\,dx^3} = \frac{1}{(1-x)^4}$; &c.

Hinc erit valor quaeſitus

$$= \frac{x}{1-x} - \frac{x}{(1-x)^2} + \frac{xx}{(1-x)^3} - \frac{x^3}{(1-x)^4} + \frac{x^4}{(1-x)^5} - \text{\&c.}$$

huiusque ergo ſeriei valor eſt $= 0$.

Quod

Quod etiam hinc patet, quod haec ſeries primo termino truncata $\frac{x}{(1-x)^2} - \frac{xx}{(1-x)^3} + \frac{x^3}{(1-x)^4} -$ &c. ſit ſeries geometrica, eiusque ſumma $= \frac{x}{(1-x)^2 + x(1-x)} = \frac{x}{1-x}$, vnde valor inuentus erit $= \frac{x}{1-x} - \frac{x}{1-x} = 0$.

EXEMPLUM IV.

Sit $y = e^x$, denotante e numerum, cuius logarithmus hyperbolicus eſt vnitas, & quaeratur valor ipſius y ſi ponatur $x = 0$, quem quidem conſtat fore $= 1$.

Cum ſit $y = e^x$; erit $\frac{dy}{dx} = e^x$; $\frac{ddy}{dx^2} = e^x$; &c. ideoque valor quaeſitus erit

$$= e^x - \frac{e^x x}{1} + \frac{e^x xx}{1.2} - \frac{e^x x^3}{1.2.3} + \frac{e^x x^4}{1.2.3.4} - \&c.$$

$$= e^x \left(1 - \frac{x}{1} + \frac{xx}{1.2} - \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} - \&c.\right)$$

At ſupra vidimus ſeriem $1 - \frac{x}{1} + \frac{xx}{1.2} - \frac{x^3}{1.2.3} + \&c.$ exprimere valorem e^{-x}, erit ergo valor quaeſitus vtique $= e^x . e^{-x} = \frac{e^x}{e^x} = 1$.

EXEMPLUM V.

Sit $y = \sin x$, atque poſito $x = 0$, manifeſtum eſt fore $y = 0$, id quod etiam formula generalis indicabit.

 Cum

Cum enim ſit $y = \mathrm{ſin}\, x$; erit $\frac{dy}{dx} = \mathrm{coſ}\, x$;

$\frac{ddy}{dx^4} = - \mathrm{ſin}\, x$; $\frac{d^3y}{dx^3} = - \mathrm{coſ}\, x$; $\frac{d^4y}{dx^4} = \mathrm{ſin}\, x$; &c.

erit poſito $x = 0$ valor ipſius y hic:

$$\mathrm{ſin}\, x - \frac{x}{1} \mathrm{coſ}\, x - \frac{xx}{1.2} \mathrm{ſin}\, x + \frac{x^3}{1.2.3} \mathrm{coſ}\, x + \frac{x^4}{1.2.3.4} \mathrm{ſin}\, x - \&c.$$

qui eſt $= \mathrm{ſin}\, x \left(1 - \frac{xx}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3\ldots6} + \&c.\right)$

$$- \mathrm{coſ}\, x \left(\frac{x}{1} - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3\ldots.7} + \&c.\right)$$

harum autem ſerierum ſuperior exprimit $\mathrm{coſ}\, x$, inferior autem $\mathrm{ſin}\, x$, vnde valor quaeſites erit

$$= \mathrm{ſin}\, x \,.\, \mathrm{coſ}\, x - \mathrm{coſ}\, x \,.\, \mathrm{ſin}\, x = 0.$$

68. Hinc igitur viciſſim cognoſcimus, ſi y eiusmodi fuerit functio ipſius x, vt ipſa euaneſcat, poſito $x = 0$, tum fore

$$y - \frac{xdy}{1dx} + \frac{xxddy}{1.2\,dx^2} - \frac{x^3d^3y}{1.2.3dx^3} + \frac{x^4d^4y}{1.2.3.4dx^4} - \&c. = 0.$$

Vnde haec eſt aequatio generalis omnium omnino functionum ipſius x, quae dum fit $x = 0$, ſimul ipſae euaneſcunt. Et hancobrem iſta aequatio ita eſt comparata, vt, quaecunque functio ipſius x, dummodo ea euaneſcat euaneſcente x, loco y ſubſtituatur, aequationi perpetuo ſatisfiat. Quodſi vero y eiusmodi fuerit functio ipſius x, quae

quae poſito $x = 0$, recipiat valorem datum $= A$, tum erit:

$$y - \frac{xdy}{1dx} + \frac{x^2ddy}{1.2dx^2} - \frac{x^3d^3y}{1.2.3dx^3} + \frac{x^4d^4y}{1.2.3.4dx^4} - \&c. = A.$$

in qua aequatione omnes continentur functiones ipſius x, quae poſito $x = 0$, abeunt in A.

69. Si loco x ſcribatur $2x$, ſeu $x + x$, functio quaecunque ipſius x, quae deſignetur per y hunc induet valorem

$$y + \frac{xdy}{1dx} + \frac{x^2ddy}{1.2dx^2} + \frac{x^3d^3y}{1.2.3dx^3} + \frac{x^4d^4y}{1.2.3.4dx^4} + \&c.$$

Atque ſi loco x ſcribamus nx, hoc eſt $x + (n - 1)x$ functio y accipiet valorem ſequentem:

$$y + \frac{(n-1)xdy}{1dx} + \frac{(n-1)^2xxddy}{1.2dx^2} + \frac{(n-1)^3x^3d^3y}{1.2.3dx^3} + \&c.$$

Sin autem generaliter pro x ſcribamus t, functio quaecunque y ipſius x, transmutabitur ob $t = x + t - x$ in formam ſequentem:

$$y + \frac{(t-x)dy}{1dx} + \frac{(t-x)^2ddy}{1.2dx^2} + \frac{(t-x)^3d^3y}{1.2.3dx^3} + \&c.$$

Si igitur v fuerit talis functio ipſius t, qualis y eſt ipſius x, quia v ex y naſcitur, ponendo t loco x, erit:

$$v = y + \frac{(t-x)dy}{1dx} + \frac{(t-x)^2ddy}{1.2dx^2} + \frac{(t-x)^3d^3y}{1.2.3dx^3} + \&c.$$

cuius veritas quibuscunque exemplis comprobari poteſt.

EXEMPLUM.

Sit enim $y = xx - x$: manifeſtum eſt poſito t loco x fore $v = tt - t$, quod idem expreſſio inuenta declarabit.

Nam ob

$$y = xx - x; \text{ erit } \frac{dy}{dx} = 2x - 1; \quad \& \quad \frac{ddy}{2dx^2} = 1;$$

vnde fiet

$$v = xx - x + (t - x)(2x - 1) + (t - x)^2 =$$
$$xx - x + 2tx - 2xx - t + x + tt - 2tx + xx = tt - t.$$

Si itaque y fuerit eiusmodi functio ipſius x, quae poſito $x = a$ abeat in A; ob $t = a$ & $v = A$ fiet

$$A = y + \frac{(a-x)dy}{1dx} + \frac{(a-x)^2 ddy}{1.2dx^2} + \frac{(a-x)^3 d^3y}{1.2.3dx^3} + \&c.$$

huicque ergo aequationi omnes functiones ipſius x, quae facto $x = a$ abeunt in A, ſatisfaciunt.

CAPUT

CAPUT IV.

DE CONUERSIONE FUNCTIONUM IN SERIES.

70.

In Capite ſuperiori iam ex parte oſtenſus eſt vſus, quem expreſſiones generales ibi pro differentiis finitis inventae habent in inueſtigatione ſerierum, quae valorem cuiusque functionis ipſius x exhibeant. Si enim y fuerit functio data ipſius x, eius valor quem induit poſito $x = 0$, erit cognitus; hicque ſi ponatur $= A$, erit vti inuenimus:

$$y - \frac{xdy}{dx} + \frac{x^2 ddy}{1.2\,dx^2} - \frac{x^3 d^3y}{1.2.3\,dx^3} + \frac{x^4 d^4y}{1.2.3.4\,dx^4} - \&c. = A.$$

Hinc ergo non ſolum habemus ſeriem plerumque in infinitum excurrentem, cuius ſumma aequetur quantitati conſtanti A, etiamſi in ſingulis terminis inſit quantitas variabilis x, ſed etiam ipſam functionem y per ſeriem exprimere poterimus, erit enim:

$$y = A + \frac{xdy}{dx} - \frac{xxddy}{1.2\,dx^2} + \frac{x^3 d^3y}{1.2.3\,dx^3} - \frac{x^4 d^4y}{1.2.3.4\,dx^3} + \&c.$$

cuius exempla iam aliquot ſunt allata.

71. Quo autem haec inueſtigatio latius pateat, ponamus functionem y abire in z, ſi loco x vbique ſcribatur $x + \omega$, ita vt z talis ſit functio ipſius $x + \omega$, qualis y eſt ipſius x, atque oſtendimus fore:

$z =$

$$z = y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{1.2\, dx^2} + \frac{\omega^3 d^3 y}{1.2.3\, dx^3} + \frac{\omega^4 d^4 y}{1.2.3.4\, dx^4} + \&c.$$

Cum igitur huius ſeriei ſinguli termini per continuam ipſius y differentiationem ponendo dx conſtans inueniri, ſimulque valor ipſius z per ſubſtitutionem $x + \omega$ in locum ipſius x actu exhiberi queat; hoc modo perpetuo obtinebitur ſeries valori ipſius z aequalis, quae ſi ω fuerit quantitas vehementer parua, maxime conuergit, atque non admodum multis terminis capiendis valorem ipſius z proxime verum praebebit. Ex quo huius formulae in negotio approximationum vberrimus erit vſus.

72. Vt igitur in inſigni huius formulae vſu oſtendendo ordine procedamus, ſubſtituamus primo in locum ipſius y functiones ipſius x algebraicas. Ac primo quidem ſit $y = x^n$; eritque ſi $x + \omega$ loco x ponatur $z = (x + \omega)^n$. Cum igitur ſit:

$$\frac{dy}{dx} = nx^{n-1} \quad ; \quad \frac{ddy}{dx^2} = n(n-1)x^{n-2}$$

$$\frac{d^3y}{dx^3} = n(n-1)(n-2)x^{n-3} \quad ; \quad \frac{d^4y}{dx^4} = n(n-1)(n-2)(n-3)x^{n-4}$$

&c.

his valoribus ſubſtitutis fiet:

$$(x+\omega)^n = x^n + \frac{n}{1}x^{n-1}\omega + \frac{n(n-1)}{1.\,2}x^{n-2}\omega^2 + \frac{n(n-1)(n-2)}{1.\,2.\,3}x^{n-3}\omega^3 + \&c.$$

quae eſt notiſſima expreſſio Neutoniana, qua poteſtas binomii $(x + \omega)^n$ in ſeriem conuertitur. Huiusque

ſeri-

ſeriei terminorum numerus ſemper eſt finitus, ſi n fuerit numerus integer affirmatiuus.

73. Poterimus hinc quoque progreſſionem inuenire, quae valorem poteſtatis binomii ita exprimat, vt ea abrumpatur, quoties exponens poteſtatis fuerit numerus negatiuus. Statuamus enim

$$\omega = \frac{-ux}{x+u}\text{; erit } z = (x+\omega)^n = \left(\frac{xx}{x+u}\right)^n$$

ideoque habebitur:

$$\frac{x^{2n}}{(x+u)^n} = x^n - \frac{nx^n u}{1(x+u)} + \frac{n(n-1)x^n u^2}{1.2(x+u)^2} - \frac{n(n-1)(n-2)x^n u^3}{1.2.3(x+u)^3} + \&c.$$

diuidatur vbique per x^{2n}, eritque

$$(x+u)^{-n} = x^{-n} - \frac{nx^{-n}u}{1(x+u)} + \frac{n(n-1)x^{-n}u^2}{1.2(x+u)^2} - \frac{n(n-1)(n-2)x^{-n}u^3}{1.2.3(x+u)^3} + \&c.$$

Ponatur nunc $-n = m$; prodibitque

$$(x+u)^m = x^m + \frac{mx^m u}{1(x+u)} + \frac{m(m+1)x^m u^2}{1.2(x+u)^2} + \frac{m(m+1)(m+2)x^m u^3}{1.2.3(x+u)^3} + \&c.$$

quae ſeries, quoties m eſt numerus integer negatiuus, finito terminorum numero conſtabit. Haec igitur ſeries aequalis eſt primum inuentae, ſi pro ω & n ſcribantur u & m; erit enim inde

$$(x+u)^m = x^m + \frac{mx^{m-1}u}{1} + \frac{m(m-1)x^{m-2}u^2}{1.2} + \frac{m(m-1)(m-2)x^{m-3}u^3}{1.2.3} + \&c.$$

74. Haec eadem ſeries quoque deduci poteſt ex expreſſione initio §. 70. data. Cum enim, ſi poſito $x = 0$, abeat y in A ſit:

$$y - \frac{xdy}{dx} + \frac{xxddy}{1.2dx^2} - \frac{x^3d^3y}{1.2.3dx^3} + \frac{x^4d^4y}{1.2.3.4dx^4} - \&c. = A,$$

ponatur $y = (x+a)^n$; eritque $A = a^n$; & ob

$$\frac{dy}{dx} = n(x+a)^{n-1} \;;\; \frac{ddy}{dx^2} = n(n-1)(x+a)^{n-2} \;;$$

$$\frac{d^3y}{dx^3} = n(n-1)(n-2)(x+a)^{n-3} \;; \&c. \text{ fiet}$$

$$(x+a)^n - \frac{n}{1}x(x+a)^{n-1} + \frac{n(n-1)}{1.2}x^2(x+a)^{n-2} - \&c. = a^n$$

diuidatur per $a^n(x+a)^n$, atque prodibit:

$$(x+a)^{-n} = a^{-n} - \frac{na^{-n}x}{1(x+a)} + \frac{n(n-1)a^{-n}x^2}{1.2(x+a)^2} - \&c.$$

quae poſitis reſpectiue u, x & $-m$ pro x, a & n orietur ſeries ante inuenta.

75. Si pro m ſtatuantur numeri fracti, ambae ſeries in infinitum excurrent, interim tamen ſi u prae x fuerit quantitas valde parua, vehementer ad verum valorem conuergent. Sit igitur $m = \frac{\mu}{\nu}$; & $x = a^\nu$, erit ex ſeriem primum inuenta:

$$(a^\nu + u)^{\frac{\mu}{\nu}} = a^\mu\left(1 + \frac{\mu u}{\nu . a^\nu} + \frac{\mu(\mu-\nu)}{\nu . 2\nu} . \frac{uu}{a^{2\nu}} + \frac{\mu(\mu-\nu)(\mu-2\nu)}{\nu . 2\nu . 3\nu} . \frac{u^3}{a^{3\nu}} + \&c.\right)$$

Series autem poſterius inuenta dabit:

$$(a^\nu + u)^{\frac{\mu}{\nu}} = a^\mu\left(1 + \frac{\mu u}{\nu(a^\nu + u)} + \frac{\mu(\mu+\nu)u^2}{\nu . 2\nu(a^\nu + u)^2} + \frac{\mu(\mu+\nu)(\mu+2\nu)u^3}{\nu . 2\nu . 3\nu\,(a^\nu + u)^3} + \&c.\right)$$

Haec

Haec autem poſterior ſeries magis conuergit quam prior, cum eius termini etiam decreſcant, ſi fuerit $u > a^{\nu}$, quo caſu tamen prior ſeries diuergit.

Si igitur ſit $\mu = 1$, $\nu = 2$, erit

$$\sqrt{(a^2+u)} = a\left(1 + \frac{1\, u}{2(a^2+u)} + \frac{1.\,3\; u^2}{2.4(a^2+u)^2} + \frac{1.\,3.\,5\; u^3}{2.4.6(a^2+u)^3} + \&c.\right)$$

Simili modo pro ν ponendo numeros 3, 4, 5 &c. manente $\mu = 1$, erit:

$$\sqrt[3]{(a^3+u)} = a\left(1 + \frac{1\, u}{3(a^3+u)} + \frac{1.\,4.\; u^2}{3.6(a^3+u)^2} + \frac{1.\,4.\,7\; u^3}{3.6.9(a^4+u)^3} + \&c.\right)$$

$$\sqrt[4]{(a^4+u)} = a\left(1 + \frac{1\, u}{4(a^4+u)} + \frac{1.\,5.\; u^2}{4.8(a^4+u)^2} + \frac{1.\,5.\,9\; u^3}{4.8.12(a^4+u)^3} + \&c.\right)$$

$$\sqrt[5]{(a^5+u)} = a\left(1 + \frac{1\, u}{5(a^5+u)} + \frac{1.\,6\; u^2}{5.10(a^5+u)^2} + \frac{1.\,6.\,11\; u^3}{5.10.15(a^5+u)^3} + \&c.\right)$$

&c.

76. Ex his ergo formulis facile cuiusque numeri propoſiti radix cuiusuis poteſtatis inueniri poterit. Propoſito enim numero c quaeratur poteſtas ei proxima, ſiue maior ſiue minor: priori caſu u fiet numerus negatiuus, poſteriori affirmatiuus. Quod ſi vero ſeries reſultans non ſatis conuergere videatur, multiplicetur numerus c per quampiam poteſtatem puta per f^{ν}, ſi radix dignitatis ν extrahi debeat, & quaeratur numeri $f^{\nu} c$ radix, quae per f diuiſa dabit radicem numeri c quaeſitam.

tam. Quo maior autem accipitur numerus f, eo magis ſeries conuerget; idque imprimis, ſi quaepiam ſimilis poteſtas a^ν non multum ab $f^\nu c$ diſcrepet.

EXEMPLUM I.

Quaeratur radix quadrata ex numero 2.

Si ſine vlteriori praeparatione ponatur $a = 1$ & $u = 1$ fiet

$$\sqrt{2} = 1 + \frac{1}{2.2} + \frac{1.3}{2.4.2^2} + \frac{1.3.5}{2.4.6.2^3} + \&c.$$

quae etſi iam ſatis conuergit, tamen praeſtabit numerum 2 ante per quadratum quodpiam vti 25 multiplicare, vt productum 50 ab alio quadrato 49 minime diſcrepet. Hancobrem quaeratur radix quadrata ex 50, quae per 5 diviſa dabit $\sqrt{2}$. Erit autem tum $a = 7$ & $u = 1$, vnde fiet:

$$\sqrt{50} = 5\sqrt{2} = 7\left(1 + \frac{1}{2.50} + \frac{1.3}{2.4.50^2} + \frac{1.3.5}{2.4.6.50^3} + \&c.\right)$$

ſeu

$$\sqrt{2} = \tfrac{7}{5}\left(1 + \frac{1}{100} + \frac{1.3}{100.200} + \frac{1.3.5}{100.200.300} + \&c.\right)$$

quae ad computum in fractionibus decimalibus inſtituendum eſt aptiſſima.

Erit

Erit enim

$$\frac{7}{5} = 1,4000000000000$$

$$\frac{7}{5}\cdot\frac{1}{100} = 140000000000$$

$$\frac{7}{5}\cdot\frac{1}{100}\cdot\frac{3}{200} = 2100000000$$

$$\frac{7}{5}\cdot\frac{1}{100}\cdot\frac{3}{200}\cdot\frac{5}{300} = 35000000$$

$$\text{praec. in } \frac{7}{400} = 612500$$

$$\text{praec. in } \frac{9}{500} = 11025$$

$$\text{praac. in } \frac{11}{600} = 202$$

$$3$$

$$\text{Ergo } \sqrt{2} = 1,4142135623730$$

EXEMPLUM II.

Quaeratur radix cubica ex 3.

Multiplicetur 3 per cubum 8, & quaeratur radix cubica ex 24, erit enim $\sqrt[3]{24} = 2\sqrt[3]{3}$. Ponatur ergo

$a = 3$ & $u = -3$, eritque

$$\sqrt[3]{24} = 3\left(1 - \frac{1.3}{3.24} + \frac{1.4.3^2}{3.6.24^2} - \frac{1.4.7.3^3}{3.6.9.24^3} + \&c.\right)$$

&

$$\sqrt[3]{3} = \frac{3}{2}\left(1 - \frac{1}{3.8} + \frac{1.4}{3.6.8^2} - \frac{1.4.7}{3.6.9.8^3} + \&c.\right)$$

ſeu

$$\sqrt[3]{3} = \frac{3}{2}\left(1 - \frac{1}{24} + \frac{1}{24}\cdot\frac{4}{48} - \frac{1}{24}\cdot\frac{4}{48}\cdot\frac{7}{72} + \&c.\right)$$

 quae

quae ſeries iam vehementer conuergit, cum quilibet terminus plusquam octies minor ſit praecedente. Sin autem 3 multiplicetur per cubum 729 fiet 2187, &

$$\sqrt[3]{2187} = \sqrt[3]{(13^3 - 10)} = 9\sqrt[3]{3}.$$

Erit ergo ob $a = 13$ & $u = -10$.

$$\sqrt[3]{3} = \tfrac{13}{9}\left(1 - \frac{1.10}{3.2187} + \frac{1.4.10^2}{3.6.2187^2} - \frac{1.4.7.10^3}{3.6.9.2187^3} + \&c.\right)$$

cuius quiuis terminus plusquam ducenties minor eſt quam praecedens.

77. Euolutio binomii poteſtatis tum late patet, vt omnes functiones algebraicae in ea comprehendi queant. Si enim verbi gratia quaeratur valor huius functionis $V(a + 2bx + cxx)$ per ſeriem expreſſus, hoc per praecedentes formulas, duos terminos tanquam vnum conſiderando fieri poterit. Deinde vero haec explicatio fieri poterit ope expreſſionis primum traditae: nam ſi ponatur $V(a + 2bx + cxx) = y$, quia poſito $x = 0$ fit $y = Va$, erit $A = Va$, & cum differentialia ipſius y ita ſe habeant:

$$\frac{dy}{dx} = \frac{b + cx}{V(a + 2bx + cxx)}$$

$$\frac{ddy}{dx^2} = \frac{ac - bb}{(a + 2bx + cxx)^{\frac{3}{2}}}$$

$$\frac{d^3y}{dx^3} = + \frac{3(bb - ac)(b + cx)}{(a + 2bx + cxx)^{\frac{5}{2}}}$$

$$\frac{d^4y}{dx^4} = \frac{3(bb - ac)(ac - 5bb - 8bcx - 4ccxx)}{(a + 2bx + cxx)^{\frac{7}{2}}}$$

&c. Ex

Ex his ergo obtinebitur:

$$V(a+2bx+cxx)-\frac{(b+cx)x}{V(a+2bx+cxx)}-\frac{(bb-ac)xx}{2(a+2bx+cxx)^{\frac{3}{2}}}-\frac{(bb-ac)(b+cx)x^3}{2(a+2bx+cxx)^{\frac{5}{2}}}$$
$$-\frac{(bb-ac)(5bb-ac+8bcx+4ccxx)x^4}{8(a+2bx+cxx)^{\frac{7}{2}}}-\&c.=Va.$$

Quodſi ergo vbique per $V(a+2bx+cxx)$ multiplicetur ſeries fiet rationalis, eritque

$$Va(a+2bx+cxx)=a+2bx+cxx-(b+cx)x-\frac{(bb-ac)xx}{2(a+2bx+cxx)}-$$
$$\frac{(bb-ac)(b+cx)x^3}{2(a+2bx+cxx)^2}-\frac{(bb-ac)(5bb-ac+8bcx+4ccxx)x^4}{8(a+2bx+cxx)^3}-\&c.$$

ſiue

$$V(a+2bx+cxx)=Va+\frac{bx}{Va}-\frac{(bb-ac)xx}{2(a+2bx+cxx)Va}-\frac{(bb-ac)(b+cx)x^3}{2(a+2bx+cxx)^2Va}-\&c.$$

78. Tranſeamus ergo ad functiones tranſcendentes, quas loco y ſubſtituamus. Sit itaque primum $y=lx$, ac poſito $x+\omega$ loco x fiet $z=l(x+\omega)$. Sint autem hi logarithmi quicunque, qui ad hyperbolicos rationem teneant $n,:1$, eritque pro logarithmis hyperbolicis $n=1$, & pro tabularibus erit $n=0,434294481903$2. Hinc differentialia ipſius $y=lx$ erunt:

$$\frac{dy}{dx}=\frac{n}{x};\quad \frac{ddy}{dx^2}=-\frac{n}{x^2};\quad \frac{d^3y}{dx^3}=\frac{2n}{x^3};\quad \&c.$$

ex quibus conficitur:

$$l(x+\omega)=lx+\frac{n\omega}{x}-\frac{n\omega^2}{2x^2}+\frac{n\omega^3}{3x^3}-\frac{n\omega^4}{4x^4}+\&c.$$

Si-

Simili modo ſi ω ſtatuatur negatiuum, erit:

$$l(x-\omega) = lx - \frac{n\omega}{x} - \frac{n\omega^2}{2x^2} - \frac{n\omega^3}{3x^3} - \frac{n\omega^4}{4x^4} - \&c.$$

Quodſi ergo haec ſeries a priori ſubtrahatur, fiet

$$l\frac{x+\omega}{x-\omega} = 2n\left(\frac{\omega}{x} + \frac{\omega^3}{3x^3} + \frac{\omega^5}{5x^5} + \frac{\omega^7}{7x^7} + \&c.\right)$$

79. Si in ſerie primum inuenta:

$$l(x+\omega) = lx + \frac{n\omega}{x} - \frac{n\omega^2}{2x^2} + \frac{n\omega^3}{3x^3} - \frac{n\omega^4}{4x^4} + \&c.$$

ponatur $\omega = \frac{xx}{u-x}$; erit $x+\omega = \frac{ux}{u-x}$; &

$$l(x+\omega) = lu + lx - l(u-x) = lx + \frac{nx}{u-x} - \frac{nxx}{2(u-x)^2} + \&c.$$

atque

$$l(u-x) = lu - \frac{nx}{u-x} + \frac{nxx}{2(u-x)^2} - \frac{nx^3}{3(u-x)^3} + \&c.$$

ſumtoque x negatiuo habebitur:

$$l(u+x) = lu + \frac{nx}{u+x} + \frac{nxx}{2(u+x)^2} + \frac{nx^3}{3(u+x)^3} + \frac{nx^4}{4(u+x)^4}.$$

Harum ergo ſerierum ope logarithmi expedite inueniri poterunt, ſi quidem ſeries valde conuergant. Huiusmodi autem erunt ſequentes, quae ex inuentis facile deducuntur:

$$l(x+1) = lx + n\left(\frac{1}{x} - \frac{1}{2xx} + \frac{1}{3x^3} - \frac{1}{4x^4} + \&c.\right)$$

$$l(x-1 = lx - n\left(\frac{1}{x} + \frac{1}{2xx} + \frac{1}{3x^3} + \frac{1}{4x^4} + \&c.\right)$$

quae

quae duae ſeries, cum tantum ſignis a ſe inuicem diſcrepent, ſi ad calculum reuocentur, ex logarithmo numeri x cognito, eadem opera logarithmi amborum numerorum $x+1$ & $x-1$ reperientur. Deinde ex reliquis ſeriebus erit:

$$l(x+1) = l(x-1) + 2n\left(\frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \frac{1}{7x^7} + \&c.\right)$$

$$l(x-1) = lx - n\left(\frac{1}{x-1} - \frac{1}{2(x-1)^2} + \frac{1}{3(x-1)^3} - \frac{1}{4(x-1)^4} + \&c.\right)$$

$$l(x+1) = lx + n\left(\frac{1}{x+1} + \frac{1}{2(x+1)^2} + \frac{1}{3(x+1)^3} + \frac{1}{4(x+1)^4} + \&c.\right)$$

80. Ex dato ergo logarithmo numeri x, logarithmi numerorum contiguorum $x+1$ & $x-1$ facile inveniri poterunt; quin etiam ex logarithmo numeri $x-1$ logarithmus numeri binario maioris & viciſſim eruetur. Quod quamuis in Introductione vberius ſit oſtenſum, tamen hic quaedam exempla adiungemus.

EXEMPLUM I.

Ex dato numeri 10 *logarithmo hyperbolico*, qui eſt 2,302585092994o, *logarithmos hyperbolicos numerorum* 11 *&* 9 *inuenire.*

Quoniam haec quaeſtio logarithmos hyperbolicos ſpectat, erit $n = 1$; ideoque habebuntur hae ſeries:

$$l11 = l10 + \frac{1}{10} - \frac{1}{2.10^2} + \frac{1}{3.10^3} - \frac{1}{4.10^4} + \frac{1}{5.10^5} - \&c.$$

$$l9 = l10 - \frac{1}{10} - \frac{1}{2.10^2} - \frac{1}{3.10^3} - \frac{1}{4.10^4} - \frac{1}{5.10^5} - \&c.$$

Ad quarum ſerierum ſummas inueniendas, colligantur termini pares & impares ſeorſim, eritque

$\frac{1}{10} = 0{,}1000000000000$	$\frac{1}{2.10^2} = 0{,}0050000000000$
$\frac{1}{3.10^3} = 0{,}0003333333333$	$\frac{1}{4.10^4} = 0{,}0000250000000$
$\frac{1}{5.10^5} = 0{,}0000020000000$	$\frac{1}{6.10^6} = 0{,}0000001666666$
$\frac{1}{7.10^7} = 0{,}0000000142857$	$\frac{1}{8.10^8} = 0{,}0000000012500$
$\frac{1}{9.10^9} = 0{,}0000000001111$	$\frac{1}{10.10^{10}} = 0{,}0000000000100$
$\frac{1}{11.10^{11}} = 0{,}0000000000009$	$\frac{1}{12.10^{12}} = 0{,}0000000000001$
ſumma $= 0{,}1003353477310$	ſumma $= 0{,}0050251679267$

Summa vtriusque erit . . . 0,1053605156577

Differentia ambarum erit 0,0953101798043

Iam eſt $l10 = 2{,}3025850929940$

Ergo erit $l11 = 2{,}3978952727983$

& $l9 = 2{,}1972245773363$

Hinc porro erit $l3 = 1{,}0986122886681$

& $l99 = 4{,}5951198501346$

EX-

EXEMPLUM II.

Ex logarithmo hyperbolico numeri 99 *nunc inuento inuenire logarithmum numeri* 101.

Adhibeatur ad hoc ſeries ſupra inuenta:

$$l(x+1) = l(x-1) + \frac{2}{x} + \frac{2}{3x^3} + \frac{2}{5x^5} + \frac{2}{7x^7} + \&c.$$

in qua fiat $x = 100$; eritque:

$$l101 = l99 + \frac{2}{100} + \frac{2}{3.\,100^3} + \frac{2}{5.\,100^5} + \frac{2}{7.\,100^7} + \&c.$$

cuius ſeriei ſumma ex his quatuor terminis colligitur $= 0,0200006667066$, quae ad $l\,99$ addita dabit $l101 = 4,6151205168412$.

EXEMPLUM III.

Ex dato logarithmo tabulari numeri 10, qui eſt $=1$, *inuenire logarithmos numerorum* 11 & 9.

Quoniam hic logarithmos communes tabulares quaerimus, erit $n = 0,4342944819032$, poſito ergo $x = 10$ erit:

$$l11 = l10 + \frac{n}{10} + \frac{n}{2.\,10^2} - \frac{n}{3.\,10^3} + \frac{n}{4.\,10^4} + \&c.$$

$$l\,9 = l10 - \frac{n}{10} - \frac{n}{2.\,10^2} - \frac{n}{3.\,10^3} - \frac{n}{4.\,10^4} - \&c.$$

 Col-

Colligantur termini pares & impares ſeorſim:

$\frac{n}{10} = 0{,}0434294481903$	$\frac{n}{2.10^2} = 0{,}0021714724095$
$\frac{n}{3.10^3} = 0{,}0001447648273$	$\frac{n}{4.10^4} = 0{,}0000108573620$
$\frac{n}{5.10^5} = 0{,}0000008685889$	$\frac{n}{6.10^6} = 0{,}0000000723824$
$\frac{n}{7.10^7} = 0{,}0000000062042$	$\frac{n}{8.10^8} = 0{,}0000000005428$
$\frac{n}{9.10^9} = 0{,}0000000000482$	$\frac{n}{10.10^{10}} = 0{,}0000000000043$
$\frac{n}{11.10^{11}} = 0{,}0000000000005$	$\frac{n}{12.10^{12}} = 0{,}0000000000000$
ſumma $= 0{,}0435750878593$	ſumma $= 0{,}0021824027010$

Aggregatum ambarum eſt $= 0{,}0457574905603$

Differentia earum eſt $= 0{,}0413926851583$

Cum ergo ſit $l10 = 1{,}0000000000000$

Erit $l11 = 1{,}0413926851583$

& $l9 = 0{,}9542425094396$

Hinc $l3 = 0{,}4771212547198$

& $l99 = 1{,}9956351945979$

EX-

EXEMPLUM IV.

Ex logarithmo tabulari numeri 99 *hic inuentis inuenire logarithmum tabularem numeri* 101.

Adhibendo hic eandem ſeriem, qua in Exemplo ſecundo vſi ſumus, habebimus:

$$l101 = l99 + 2n\left(\frac{1}{100} + \frac{1}{3.\,100^3} + \frac{1}{5.\,100^5} + \&c.\right)$$

cuius ſeriei poſito pro n valore debito, ſumma mox reperietur $= 0{,}0086861791849$ quae addita, ad $l99 = 1{,}9956351945979$ oritur.

$$l101 = 2{,}0043213637829$$

81. Tribuamus nunc in expreſſione noſtra generali y valorem exponentialem, ſitque $y = a^x$, poſito $x + \omega$ loco x; erit $z = a^{x+\omega}$, cuius valor ob differentialia:

$$\frac{dy}{dx} = a^x\, la; \quad \frac{ddy}{dx^2} = a^x (la)^2; \quad \frac{d^3y}{dx^3} = a^x (la)^3; \quad \&c.$$

erit

$$a^{x+\omega} = a^x\left(1 + \frac{\omega\, la}{1} + \frac{\omega^2 (la)^2}{1.\,2} + \frac{\omega^3 (la)^3}{1.\,2.\,3} + \&c.\right)$$

quae ſi diuidatur per a^x prodibit ſeries valores quantitatis exponentialis exprimens, quam ſupra in Introductione iam elicuimus: nempe

$$a^\omega = 1 + \frac{\omega\, la}{1} + \frac{\omega^2 (la)^2}{1.\,2} + \frac{\omega^3 (la)^3}{1.\,2.\,3} + \frac{\omega^4 (la)^4}{1.2.3.4} + \&c.$$

Simili modo ſumto ω negatiuo erit:

$$a^{-\omega} = 1 - \frac{\omega la}{1} + \frac{\omega^2 (la)^2}{1.\,2} - \frac{\omega^3 (la)^3}{1.\,2.\,3} + \frac{\omega^4 (la)^4}{1.2.3.4} - \&c.$$

ex quarum combinatione oritur:

$$\frac{a^{\omega} + a^{-\omega}}{2} = 1 + \frac{\omega^2 (la)^2}{1.\,2} + \frac{\omega^4 (la)^4}{1.2.3.4} + \frac{\omega^6 (la)^6}{1.2\ldots 6} + \&c.$$

$$\frac{a^{\omega} - a^{-\omega}}{2} = \frac{\omega la}{1} + \frac{\omega^3 (la)^3}{1.\,2.\,3} + \frac{\omega^5 (la)^5}{1.2.3.4.5} + \&c.$$

vbi notandum eſt la denotare logarithmum hyperbolicum numeri a.

82. Huius formulae ope ex dato quouis logarithmo numerus ei conueniens reperiri poterit. Sit enim propoſitus logarithmus quicunque u ad canonem, in quo numeri a logarithmus $= 1$ ſtatuitur, pertinens. Quaeratur in eodem canone logarithmus x proxime ad u accedens, ſitque $u = x + \omega$; numerus autem logarithmo x conueniens ſit $= y = a^x$, erit numerus logarithmo $u = x + \omega$ reſpondens $= a^{x+\omega} = z$; fietque

$$z = y\left(1 + \frac{\omega la}{1} + \frac{\omega^2 (la)^2}{1.\,2} + \frac{\omega^3 (la)^3}{1.\,2.\,3} + \frac{\omega^4 (la)^4}{1.2.3.4} + \&c.\right)$$

quae ſeries ob ω numerum valde paruum, vehementer conuerget, cuius vſum ſequenti exemplo declaremus.

EXEMPLUM.

Quaeratur numerus iſti binarii poteſtati $2^{2^{24}}$ *aequalis.*

Cum ſit $2^{24} = 16777216$, erit $2^{2^{24}} = 2^{16777216}$, ſu-

ſumendisque logarithmis vulgaribus, erit huius numeri logarithmus $= 16777216\, l2$. Cum autem ſit:

$$l2 = 0,30102999566398119521373889$$

numeri quaeſiti logarithmus erit:

$$5050445,259733675932039063$$

cuius characteriſtica indicat numerum quaeſitum exprimi 5050446 figuris, quae cum omnes exhiberi nequeant, ſufficiet figuras initiales aſſignaſſe, quae ex mantiſſa

$$,259733675932039063 = u$$

inueſtigari debent. Ex tabulis autem colligitur, numerum cuius logarithmus proxime ad hunc accedat fore $18 . 101 = 1,818$; qui ponatur y; cuius logarithmus

$x = 0,2595938788859486 44$, vnde erit

$\omega = 0,000139797046090419$. Cum iam ſit

$a = 10$ erit

$la = 2,30258509299404568401799 14$ &

$\omega la = 0,000321894594372398$ Deinde erit

$$\begin{array}{rr}
y = & 1,818000000000000000 \\
\frac{\omega la}{1} y = & 585204372569020 \\
\frac{\omega^2 (la)^2}{1.2} y = & 94187062064 \\
\frac{\omega^3 (la)^3}{1.2.3} y = & 10106100 \\
\frac{\omega^4 (la)^4}{1.2.3.4} y = & 813 \\
\hline
 & 1818585298569737997
\end{array}$$

hae-

haeque ſunt figurae initiales numeri quaeſiti, cuius omnes figurae excepta forte vltima ſunt iuſtae.

83. Conſideremus quantitates tranſcendentes a circulo pendentes, ſitque vti perpetuo ponimus, radius circuli $= 1$, atque y denotet arcum circuli cuius ſinus $= x$ ſeu ſit $y = \text{A ſin}\, x$. Ponatur $x + \omega$ loco x, eritque $z = \text{A ſin}\,(x + \omega)$: ad quem valorem exprimendum quaerantur differentialia ipſius y:

$$\frac{dy}{dx} = \frac{1}{\sqrt{(1-xx)}}$$

$$\frac{ddy}{dx^2} = \frac{+x}{(1-xx)^{\frac{3}{2}}}$$

$$\frac{d^3y}{dx^3} = \frac{1+2xx}{(1-xx)^{\frac{5}{2}}}$$

$$\frac{d^4y}{dx^4} = \frac{9x+6x^3}{(1-xx)^{\frac{7}{2}}}$$

$$\frac{d^5y}{dx^5} = \frac{9+72x^2+24x^4}{(1-xx)^{\frac{9}{2}}}$$

$$\frac{d^6y}{dx^6} = \frac{225x+600x^3+120x^5}{(1-xx)^{\frac{11}{2}}} \qquad \&c.$$

Ex his ergo inuenitur:

$$\text{A ſin}(x+\omega) = \text{A ſin}\, x + \frac{\omega}{\sqrt{(1-xx)}} + \frac{\omega^2 x}{2(1-xx)^{\frac{3}{2}}} + \frac{\omega^3(1+2xx)}{6(1-xx)^{\frac{5}{2}}}$$

$$+ \frac{\omega^4(9x+6x^3)}{24(1-xx)^{\frac{7}{2}}} + \frac{\omega^5(9+72x^2+24x^4)}{120(1-xx)^{\frac{9}{2}}} + \&c.$$

84.

84. Si ergo cognitus fuerit arcus, cuius ſinus eſt $= x$, huius formulae beneficio inueniri poterit arcus, cuius ſinus eſt $x + \omega$, ſi fuerit ω quantitas valde parua. Series autem cuius ſumma addi debet, exprimetur in partibus radii, quae ad arcum facile reducentur: vti ex hoc exemplo intelligetur.

EXEMPLUM.

Quaeratur arcus circuli, cuius ſinus eſt

$$= \tfrac{1}{3} = 0,3333333333.$$

Quaeratur ex tabulis ſinuum arcus, cuius ſinus ſit proxime minor, quam $\frac{1}{3}$, qui erit $19^\circ, 28^I$, cuius ſinus eſt $= 0,3332584$. Statuatur ergo $19^\circ, 28^I, = A\,\text{ſin}\, x = y$. erit $x = 0,3332584$, & $\omega = 0,0000749$, atque ex tabulis $\sqrt{(1 - xx)} = \text{coſ}\, y = 0,9428356$. Erit ergo arcus quaeſitus z, cuius ſinus $= \frac{1}{3}$ proponitur

$$= 19^\circ, 28^I + \frac{\omega}{\text{coſ}\, y} + \frac{\omega\omega\, \text{ſin}\, y}{2\, \text{coſ}\, y^3},$$

quae expreſſio iam ſufficit; erit ergo per logarithmos calculum inſtituendo:

$$l\omega = 5,8744818$$
$$l\cos y = 9,9744359$$

$$l\frac{\omega}{\cos y} = 5,9000459 \;;\; \frac{\omega}{\cos y} = 0,0000794412$$
$$l\frac{\omega^2}{\cos y} = 1,8000918$$
$$l\frac{\sin y}{\cos y} = 9,5483452$$
$$1,3484370$$
$$l2 = 0,3010300$$

$$l\frac{\omega^2 \sin y}{2\cos y^3} = 1,0474070 \;;\; \frac{\omega^2 \sin y}{2\cos y^3} = 0,0000000011$$

Summa $= 0,0000794423$

qui eſt valor arcus ad $19°, 28^{I}$ addendi, ad quem in minutis ſecundis exprimendum, ſumamus eius logarithmum

qui eſt	5,9000518
a quo ſubtrahatur	4,6855749
	1,2144769
cui log. reſpondet num. $=$	16,38615

qui eſt numerus minutorum ſecundorum; fractionem vero in tertiis & quartis exprimendo fiet arcus quaeſitus

$$= 19°, 28^{I}, 16^{II}, 23^{III}, 10^{IV}, 8^{V}, 24^{VI}.$$

85. Simili modo expreſſio pro coſinibus eruetur; poſito enim $y = \text{A}\,\text{cof}\,x$; quia eſt $dy = \frac{-dx}{\sqrt{(1-xx)}}$, ſeries ante inuenta inuariata manebit, dummodo eius ſigna permutentur. Erit itaque

$$\text{A}\,\text{cof}(x+\omega) = \text{A}\,\text{cof}\,x - \frac{\omega}{\sqrt{(1-xx)}} - \frac{\omega^2 x}{2(1-xx)^{\frac{3}{2}}} - \frac{\omega^3(1+2xx)}{6(1-xx)^{\frac{5}{2}}}$$

$$- \frac{\omega^4(9x+6x^3)}{24(1-xx)^{\frac{7}{2}}} - \frac{\omega^5(9+72x^2+24x^4)}{120(1-xx)^{\frac{9}{2}}} + \&c.$$

quae ſeries pariter ac praecedens vehementer ſemper conuerget, ſi ex tabulis ſinuum proxime veri anguli excerpantur, ita vt plerumque vnicus terminus primus $\frac{\omega}{\sqrt{(1-xx)}}$ ſufficiat. Interim tamen ſi x fuerit ipſi 1 ſeu ſinui toti proxime aequalis, tum ob denominatores admodum paruos illa ſeries conuergentiam amittit. His igitur caſibus, quibus x non multum ab 1 deficit, quoniam tum differentiae fiunt minimae, commodius vtemur ſolita interpolatione.

86. Ponamus quoque pro y arcum cuius tangens datur, ſitque $y = \text{A}\,\text{tang}\,x$ & $z = \text{A}\,\text{tang}\,(x+\omega)$ ita vt ſit

$$z = y + \frac{\omega\,dy}{dx} + \frac{\omega^2\,ddy}{2\,dx^2} + \frac{\omega^3\,d^3y}{6\,dx^3} + \&c.$$

Ad quos terminos indagandos quaerantur ipſius y ſingula differentialia:

 dy

$$\frac{dy}{dx} = \frac{1}{1+xx}$$
$$\frac{ddy}{dx^2} = \frac{-2x}{(1+xx)^2}$$
$$\frac{d^3y}{dx^3} = \frac{-2+6xx}{(1+xx)^3}$$
$$\frac{d^4y}{dx^4} = \frac{24x-24x^3}{(1+xx)^4}$$
$$\frac{d^5y}{dx^5} = \frac{24-240x^2+120x^4}{(1+xx)^5}$$
$$\frac{d^6y}{dx^6} = \frac{-720x+2400x^3-720x^5}{(1+xx)^6}$$
&c.

vnde colligitur fore:

$$A\,\mathrm{tang}\,(x+\omega) = A\,\mathrm{tang}\,x +$$
$$\frac{\omega}{1(1+xx)} - \frac{\omega^2 x}{(1+xx)^2} + \frac{\omega^3}{(1+xx)^3}(xx-\tfrac{1}{3}) - \frac{\omega^4}{(1+xx)^4}(x^3-x) +$$
$$\frac{\omega^5}{(1+xx)^5}(x^4-2x^2+\tfrac{1}{5}) - \frac{\omega^6}{(1+xx)^6}(x^5-\tfrac{10}{3}x^3+x) + \&c.$$

87. Haec ſeries, cuius lex progreſſionis non adeo manifeſta eſt, transmutari poteſt in aliam formam, cuius progreſſio ſtatim in oculos incurrit. Ponatur in hunc finem $A\,\mathrm{tang}\,x = 90^\circ - u$, vt ſit $x = \cot u = \frac{\mathrm{coſ}\,u}{\mathrm{ſin}\,u}$; erit $1+xx = \frac{1}{\mathrm{ſin}\,u^2}$, vnde fit $\frac{dy}{dx} = \frac{1}{1+xx} = \mathrm{ſin}\,u$.

Cum

Cum deinde ſit $dx = \frac{-du}{\text{ſin}\, u^2}$, ſeu $du = -dx\, \text{ſin}\, u^2$, fiet vlteriora differentialia ſumendo:

$$\frac{ddy}{dx} = 2du\, \text{ſin}\, u\, \text{coſ}\, u = du\, \text{ſin}\, 2u = -dx\, \text{ſin}\, u\, \text{ſin}\, 2u$$

ideoque $\frac{ddy}{1\, dx^2} = +\, \text{ſin}\, u^2.\, \text{ſin}\, 2u.$

$$\frac{d^3y}{2\, dx^2} = -du\, \text{ſin}\, u.\, \text{coſ}\, u.\, \text{ſin}\, 2u - du\, \text{ſin}\, u^2\, \text{coſ}\, 2u = -du\, \text{ſin}\, u.\, \text{ſin}\, 3u$$
$$= dx\, \text{ſin}\, u^3\, \text{ſin}\, 3u$$

ideoque $\frac{d^3y}{1.2\, dx^3} = +\, \text{ſin}\, u^3.\, \text{ſin}\, 3u$

$$\frac{d^4y}{1.2.3\, dx^3} = du\, \text{ſin}\, u^2.(\text{coſ}\, u.\, \text{ſin}\, 3u + \text{ſin}\, u.\, \text{coſ}\, 3u) = du\, \text{ſin}\, u^2.\, \text{ſin}\, 4u$$
$$= -dx\, \text{ſin}\, u^4.\, \text{ſin}\, 4u$$

ideoque $\frac{d^4y}{1.2.3\, dx^4} = -\, \text{ſin}\, u^4.\, \text{ſin}\, 4u$

$$\frac{d^5y}{1.2.3.4\, dx^4} = -du\, \text{ſin}\, u^3(\text{coſ}\, u\, \text{ſin}\, 4u + \text{ſin}\, u.\, \text{coſ}\, 4u) = -du.\, \text{ſin}\, u.\, \text{ſin}\, 5u$$
$$= +\, dx\, \text{ſin}\, u^5.\, \text{ſin}\, 5u$$

ideoque $\frac{d^5y}{1.2.3.4\, dx^5} = +\, \text{ſin}\, u^5.\, \text{ſin}\, 5u$

&c.

Ex quibus colligitur fore:

$$A\, tg(x+\omega) = A\, tg\, x + \frac{\omega}{1}\, \text{ſn}\, u.\, \text{ſn}\, u - \frac{\omega^2}{2}\, \text{ſn}\, u^2.\, \text{ſn}\, u + \frac{\omega^3}{3}\, \text{ſn}\, u^3.\, \text{ſn}\, 3u$$
$$- \frac{\omega^4}{4}\, \text{ſn}\, u^4\, \text{ſn}\, 4u + \frac{\omega^5}{5}\, \text{ſn}\, u^5.\, \text{ſn}\, 5u - \frac{\omega^6}{6}\, \text{ſn}\, u^6\, \text{ſn}\, 6u + \&c.$$

vbi cum ſit $A\, tg\, x = y$ & $A\, tg\, x = 90° - u$, erit $y = 90° - u$.

88. Si ponatur $A\cot x = y$ & $A\cot(x+\omega) = z$; erit

$$z = y + \frac{\omega\,dy}{dx} + \frac{\omega^2\,ddy}{1.2\,dx^2} + \frac{\omega^3\,d^3y}{1.2.3\,dx^3} + \frac{\omega^4\,d^4y}{1.2.3.4\,dx^4} + \&c.$$

Cum autem ſit $dy = \frac{-\,dx}{1+xx}$, termini huius ſeriei congruent praeter primum cum ante inuentis, exceptis tantum ſignis. Quare ſi ponatur, vt ante $A\tang x = 90° - u$, ſeu $A\cot x = u$, vt ſit $u = y$; erit:

$$A\cot(x+\omega) = A\cot x - \frac{\omega}{1}\,\mathrm{ſin}\,u.\mathrm{ſin}\,u + \frac{\omega^2}{2}\,\mathrm{ſin}\,u^2.\mathrm{ſin}\,2u - \frac{\omega^3}{3}\,\mathrm{ſin}\,u^3.\mathrm{ſin}\,3u$$
$$+ \frac{\omega^4}{4}\,\mathrm{ſin}\,u^4.\mathrm{ſin}\,4u - \frac{\omega^5}{5}\,\mathrm{ſin}\,u^5.\mathrm{ſin}\,5u + \&c.$$

quae expreſſio immediate ex praecedente ſequitur: quia enim eſt $A\cot(x+\omega) = 90 - A\tang(x+\omega)$ & $A\cot x = 90 - A\tang x$; erit

$$A\cot(x+\omega) - A\cot x = -A\tang(x+\omega) + A\tang x.$$

89. Ex his expreſſionibus multa egregia corollaria conſequuntur, prout loco x & ω dati valores ſubſtituantur. Sit igitur primum $x = 0$; & cum ſit $u = 90° - A\tang x$ fiet $u = 90°$; atque $\mathrm{ſin}\,u = 1$; $\mathrm{ſin}\,2u = 0$; $\mathrm{ſin}\,3u = -1$; $\mathrm{ſin}\,4u = 0$; $\mathrm{ſin}\,5u = 1$; $\mathrm{ſin}\,6u = 0$; $\mathrm{ſin}\,7u = -1$; &c. vnde fiet

$$A\tang\omega = \frac{\omega}{1} - \frac{\omega^3}{3} + \frac{\omega^5}{5} - \frac{\omega^7}{7} + \frac{\omega^9}{9} - \frac{\omega^{11}}{11} + \&c.$$

quae eſt notiſſima ſeries exprimens arcum, cuius tangens eſt $= \omega$.

Sit

Sit $x = 1$, erit A tang. $x = 45°$, ideoque $u = 45°$, hinc $\text{ſin}\, u = \frac{1}{\sqrt{2}}$; $\text{ſin}\, 2u = 1$; $\text{ſin}\, 3u = \frac{1}{\sqrt{2}}$; $\text{ſin}\, 4u = 0$; $\text{ſin}\, 5u = -\frac{1}{\sqrt{2}}$; $\text{ſin}\, 6u = -1$; $\text{ſin}\, 7u = -\frac{1}{\sqrt{2}}$; $\text{ſin}\, 8u = 0$; $\text{ſin}\, 9u = \frac{1}{\sqrt{2}}$ &c. Ex quibus fit:

$$\mathrm{Atg}(1+\omega) = 45° + \frac{\omega}{2} - \frac{\omega^2}{2.2} + \frac{\omega^3}{3.4} - \frac{\omega^5}{5.8} + \frac{\omega^6}{6.8} - \frac{\omega^7}{7.16}$$
$$+ \frac{\omega^9}{9.32} - \frac{\omega^{10}}{10.32} + \frac{\omega^{11}}{11.64} - \frac{\omega^{13}}{13.128} + \frac{\omega^{14}}{14.128} - \&c.$$

Si igitur ſit $\omega = -1$; ob A tang $(1+\omega) = 0$, & $45° = \frac{\pi}{4}$ fiet:

$$\frac{\pi}{4} = \frac{1}{1.2} + \frac{1}{2.2} + \frac{1}{3.2^2} - \frac{1}{5.2^3} - \frac{1}{6.2^3} - \frac{1}{7.2^4}$$
$$+ \frac{1}{9.2^5} + \frac{1}{10.2^5} + \frac{1}{11.2^6} - \&c.$$

qui valor ſi loco arcus $45°$ ſubſtituatur in illa expreſſione erit: A tang $(1+\omega) =$

$$\frac{\omega+1}{1.2} - \frac{\omega^2+1}{2.2} + \frac{\omega^3+1}{3.2^2} - \frac{\omega^5-1}{5.2^3} + \frac{\omega^6-1}{6.2^3} - \frac{\omega^7-1}{7.2^4} + \&c.$$

Illa autem ſeries maxime eſt idonea ad valorem ipſius $\frac{\pi}{4}$ proxime inueniendum.

90. Cum ſit

$$\frac{\pi}{4} = \frac{1}{1.2} + \frac{1}{2.2} + \frac{1}{3.2^2} - \frac{1}{5.2^3} - \frac{1}{6.2^3} - \frac{1}{7.2^4} + \&c.$$

ter-

termini autem in denominatoribus habentes 2, 6, 10, &c.

$\frac{1}{2.2} - \frac{1}{6.2^3} + \frac{1}{10.2^5} - \frac{1}{14.2^7} + \&c.$ exprimunt $\frac{1}{2}\mathrm{Atg}\frac{1}{2}$;

erit:

$$\frac{\pi}{4} = \tfrac{1}{2}\mathrm{Atg}\tfrac{1}{2} + \frac{1}{1.2} + \frac{1}{3.2^2} - \frac{1}{5.2^3} - \frac{1}{7.2^4} + \frac{1}{9.2^5} + \frac{1}{11.2^6}\ \&c.$$

In altera autem formula poſito ω negatiuo, cum ſit

$$\mathrm{Atg}(1-\omega) = \begin{cases} +\frac{1}{1.2} + \frac{1}{2.2} + \frac{1}{3.2^2} - \frac{1}{5.2^3} - \frac{1}{6.2^3} - \frac{1}{7.2^4} + \&c. \\ -\frac{\omega}{1.2} - \frac{\omega^2}{2.2} - \frac{\omega^3}{3.2^2} + \frac{\omega^5}{5.2^3} + \frac{\omega^6}{6.2^3} + \frac{\omega^7}{7.2^4} - \&c. \end{cases}$$

ſi fiat $\omega = \frac{1}{2}$; erit:

$$\mathrm{Atg}\tfrac{1}{2} = \begin{cases} \frac{1}{1.2} + \frac{1}{2.2} + \frac{1}{3.2^2} + \frac{1}{5.2^3} - \frac{1}{6.2^3} - \frac{1}{7.2^4} + \&c. \\ -\frac{1}{1.2^2} - \frac{1}{2.2^3} - \frac{1}{3.2^5} + \frac{1}{5.2^8} + \frac{1}{6.2^9} + \frac{1}{7.2^{11}} - \&c. \end{cases}$$

& terminis per 2, 6, 10, &c. diuiſis ſeorſim ſumtis erit:

$$\mathrm{Atg}\tfrac{1}{2} = \tfrac{1}{2}\mathrm{Atg}\tfrac{1}{2} + \frac{1}{1.2} + \frac{1}{3.2^2} - \frac{1}{5.2^3} - \frac{1}{7.2^4} + \frac{1}{9.2^5} + \&c.$$

$$- \tfrac{1}{2}\mathrm{Atg}\tfrac{1}{8} - \frac{1}{1.2^2} - \frac{1}{3.2^5} + \frac{1}{5.2^8} + \frac{1}{7.2^{11}} - \frac{1}{9.2^{14}} - \&c.$$

ideoque

$$\tfrac{1}{2}\mathrm{Atang}\tfrac{1}{2} = \frac{1}{1.2} + \frac{1}{3.2^2} - \frac{1}{5.2^3} + \frac{1}{7.2^4} + \&c.$$

$$- \tfrac{1}{2}\mathrm{Atg}\tfrac{1}{8} - \frac{1}{1.2^2} - \frac{1}{3.2^5} + \frac{1}{5.2^8} + \frac{1}{7.2^{11}} - \&c.$$

qui

qui valor ſi in ſuperiore ſerie ſubſtituatur, atque A tang $\frac{1}{8}$ ipſe in ſeriem conuertatur, reperietur:

$$\frac{\pi}{4} = \begin{cases} 1 + \frac{1}{3.2^1} - \frac{1}{5.2^2} - \frac{1}{7.2^3} + \frac{1}{9.2^4} + \&c. \\ -\frac{1}{1.2^2} - \frac{1}{3.2^5} + \frac{1}{5.2^8} + \frac{1}{7.2^{11}} - \frac{1}{9.2^{14}} - \&c. \\ -\frac{1}{1.2^4} + \frac{1}{3.2^{10}} - \frac{1}{5.2^{16}} + \frac{1}{7.2^{22}} - \frac{1}{9.2^{28}} + \&c. \end{cases}$$

90. Sequuntur hae multaeque aliae ſeries ex poſitione $x = 1$: ſin autem ponamus $x = \sqrt{3}$, vt ſit $A\,tang\,x = 60°$, fiet $u = 30$, & $ſin\,u = \frac{1}{2}$; $ſin\,2u = \frac{\sqrt{3}}{2}$; $ſin\,3u = 1$; $ſin\,4u = \frac{\sqrt{3}}{2}$; $ſin\,5u = \frac{1}{2}$; $ſin\,6u = 0$; $ſin\,7u = -\frac{1}{2}$; &c. vnde erit:

$$A\,tang(\sqrt{3}+\omega) = 60° + \frac{\omega}{1.2^2} - \frac{\omega^2\sqrt{3}}{2.2^2} + \frac{\omega^3}{3.2^3} - \frac{\omega^4\sqrt{3}}{4.2^5} + \frac{\omega^5}{5.2^6} - \frac{\omega^7}{7.2^8} + \frac{\omega^8\sqrt{3}}{8.2^9} - \frac{\omega^9}{9.2^9} + \frac{\omega^{10}\sqrt{3}}{10.2^{11}} - \frac{\omega^{11}}{11.2^{12}} + \&c.$$

Sin autem ponatur $x = \frac{1}{\sqrt{3}}$, vt ſit $A\,tg\,x = 30°$; erit $u = 60°$; atque $ſin\,u = \frac{\sqrt{3}}{2}$; $ſin\,2u = \frac{\sqrt{3}}{2}$; $ſin\,3u = 0$; $ſin\,4u = -\frac{\sqrt{3}}{2}$; $ſin\,5u = -\frac{\sqrt{3}}{2}$; $ſin\,6u = 0$; $ſin\,7u = \frac{\sqrt{3}}{2}$; &c.

quibus valoribus ſubſtitutis erit:

$$\mathrm{Atg}\left(\frac{1}{\sqrt{3}}+\omega\right)=30^\circ+\frac{3\omega}{1.2^2}-\frac{3\omega^2\sqrt{3}}{2.2^3}+\frac{3^2\omega^4\sqrt{3}}{4.2^5}-\frac{3^3\omega^5}{5.2^6}+\&c.$$

ſi igitur ſit $\omega=-\frac{1}{\sqrt{3}}$, ob $30^\circ=\frac{\pi}{6}$; erit:

$$\frac{\pi}{6\sqrt{3}}=\frac{1}{1.2^2}+\frac{1}{2.2^3}-\frac{1}{4.2^5}-\frac{1}{5.2^6}+\frac{1}{7.2^8}+\frac{1}{8.2^9}-\&c.$$

91. Reſumamus expreſſionem generalem inuentam:

$$\mathrm{A\,tang}\,(x+\omega)=\mathrm{A\,tang}\,x$$

$$+\frac{\omega}{1}\,\mathrm{ſin}\,u.\,\mathrm{ſin}\,u-\frac{\omega^2}{2}\,\mathrm{ſin}\,u^2.\mathrm{ſin}\,2u+\frac{\omega^3}{3}\,\mathrm{ſin}\,u^3.\,\mathrm{ſin}\,3u-\&c.$$

ac ponamus $\omega=-x$, vt ſit $\mathrm{A\,tang}(x+\omega)=0$, eritque

$$\mathrm{A\,tang}\,x=$$

$$\frac{x}{1}\,\mathrm{ſin}\,u.\,\mathrm{ſin}\,u+\frac{x^2}{2}\,\mathrm{ſin}\,u^2.\,\mathrm{ſin}\,2u+\frac{x^3}{3}\,\mathrm{ſin}\,u^3.\,\mathrm{ſin}\,3u+\&c.$$

Cum autem ſit $\mathrm{A\,tang}\,x=90-u=\frac{\pi}{2}-u$;

erit: $x=\cot u=\frac{\mathrm{coſ}\,u}{\mathrm{ſin}\,u}$. Quamobrem erit:

$$\frac{\pi}{2}=u+\mathrm{coſ}\,u.\mathrm{ſin}\,u+\tfrac{1}{2}\,\mathrm{cſ}\,u^2.\mathrm{ſn}\,2u+\tfrac{1}{3}\,\mathrm{cſ}\,u^3.\mathrm{ſn}\,3u+\tfrac{1}{4}\,\mathrm{cſ}\,u^4.\mathrm{ſn}\,4u+\&c.$$

quae ſeries eo magis eſt notatu digna, quod quicunque arcus loco u accipiatur, valor ſeriei ſemper prodeat idem $=\frac{\pi}{2}$. Sin autem ſit $\omega=-2x$, ob $\mathrm{Atg}(-x)=-\mathrm{Atg}\,x$; fiet:

$$2\,\mathrm{A\,tang}\,x=$$

$$\frac{2x}{1}\,\mathrm{ſin}\,u.\,\mathrm{ſin}\,u+\frac{4x^2}{2}\,\mathrm{ſin}\,u^2.\,\mathrm{ſin}\,2u+\frac{8x^3}{3}\,\mathrm{ſin}\,u^3.\,\mathrm{ſin}\,3u+\&c.$$

Cum

Cum autem ſit $A\,\mathrm{tang}\,x = \frac{\pi}{2} - u$ & $x = \frac{\cos u}{\sin u}$, erit:

$$\pi = 2u + \frac{2}{1}\cos u.\sin u + \frac{2^2}{2}\cos u^2.\sin 2u + \frac{2^3}{3}\cos u^3.\sin 3u + \&c.$$

Sit $u = 45° = \frac{\pi}{4}$; erit $\cos u = \frac{1}{\sqrt{2}}$; $\sin u = \frac{1}{\sqrt{2}}$; $\sin 2u = 1$; $\sin 3u = \frac{1}{\sqrt{2}}$; $\sin 4u = 0$; $\sin 5u = \frac{-1}{\sqrt{2}}$; $\sin 6u = -1$; $\sin 7u = \frac{-1}{\sqrt{2}}$; $\sin 8u = 0$; $\sin 9u = \frac{1}{\sqrt{2}}$; eritque

$$\frac{\pi}{2} = \frac{1}{1} + \frac{2}{2} + \frac{2}{3} - \frac{2^2}{5} - \frac{2^3}{6} - \frac{2^3}{7} + \frac{2^4}{9} + \frac{2^5}{10} + \frac{2^5}{11} - \&c.$$

quae ſeries etſi diuergit, tamen ob ſimplicitatem eſt notatu digna.

92. Ponatur in expreſſione generali inuenta:

$$\omega = -x - \frac{1}{x} = \frac{-1}{\sin u.\cos u}, \text{ ob } x = \frac{\cos u}{\sin u}; \text{ erit:}$$

$$A\,\mathrm{tang}(x+\omega) = A\,\mathrm{tang} - \frac{1}{x} = -A\,\mathrm{tang}\frac{1}{x} = -\frac{\pi}{2} + A\,\mathrm{tang}\,x.$$

Hinc ergo obtinebitur ſequens expreſſio:

$$\frac{\pi}{2} = \frac{\sin u}{1\cos u} + \frac{\sin 2u}{2\cos u^2} + \frac{\sin 3u}{3\cos u^3} + \frac{\sin 4u}{4\cos u^3} + \frac{\sin 5u}{5\cos u^5} + \&c.$$

quae poſito $u = 45°$ dat eandem ſeriem, quam vltimo loco inuenimus. Sin autem ponamus $\omega = -\sqrt{(1+xx)}$ ob $x = \frac{\cos u}{\sin u}$, fiet $\omega = -\frac{1}{\sin u}$, &

 A tang

$$\text{A tang}(x - \sqrt{(1+xx)}) - \text{A tang}(\sqrt{(1+xx)} - x)$$
$$= -\tfrac{1}{2}\text{A tang}\frac{1}{x} = -\tfrac{1}{2}\left(\frac{\pi}{2} - \text{A tang}\,x\right) = -\tfrac{1}{2}u,$$

& $\text{A tang}\,x = \frac{\pi}{2} - u.$ Hancobrem erit:

$$\frac{\pi}{2} = \tfrac{1}{2}u + \tfrac{1}{1}\,\text{ſin}\,u + \tfrac{1}{2}\,\text{ſin}\,2u + \tfrac{1}{3}\,\text{ſin}\,3u + \tfrac{1}{4}\,\text{ſin}\,4u + \&c.$$

Quodſi haec aequatio differentietur erit:

$$0 = \tfrac{1}{2} + \text{coſ}\,u + \text{coſ}\,2u + \text{coſ}\,3u + \text{coſ}\,4u + \text{coſ}\,5u + \&c.$$

cuius ratio ex natura ſerierum recurrentium intelligitur.

93. Si ſimili modo ſeries ante inuentae differentientur, nouae ſeries ſummabiles reperientur. Ac primo quidem ex ſerie:

$$\text{A tang}(1+\omega) = \frac{\pi}{4} + \frac{\omega}{2} - \frac{\omega^2}{2.2} + \frac{\omega^3}{3.4} - \frac{\omega^5}{5.8} + \frac{\omega^6}{6.8} - \&c.$$

ſequitur

$$\frac{1}{2+2\omega+\omega^2} = \tfrac{1}{2} - \frac{\omega}{2} + \frac{\omega^2}{4} - \frac{\omega^4}{8} + \frac{\omega^5}{8} - \frac{\omega^6}{16} + \frac{\omega^8}{32} - \&c.$$

quae oritur ex euolutione fractionis $\frac{2-2\omega+\omega^2}{4+\omega^4} = \frac{1}{2+2\omega+\omega^2}$.

Deinde iſta ſeries:

$$\frac{\pi}{2} = u + \text{coſ}\,u\,\text{ſin}\,u + \tfrac{1}{2}\,\text{cſ}u^2.\text{ſn}\,2u + \tfrac{1}{3}\,\text{cſ}u^3\,\text{ſn}\,3u + \tfrac{1}{4}\,\text{cſ}u^4\,\text{ſn}\,4u + \&c.$$

per differentiationem dabit:

$$0 = 1 + \text{coſ}\,2u + \text{coſ}\,u.\text{coſ}\,3u + \text{coſ}\,u^2.\text{coſ}\,4u + \text{coſ}\,u^3\,\text{coſ}\,5u + \&c.$$

Denique ſeries $\frac{\pi}{2} = \frac{\text{ſin}\,u}{\text{coſ}\,u} + \frac{\text{ſin}\,2u}{2\,\text{cſ}u^2} + \frac{\text{ſin}\,3u}{3\,\text{cſ}u^3} + \frac{\text{ſin}\,4u}{4\,\text{cſ}u^4} + \&c.$

dat

dat $o = \frac{1}{\text{coſ}u^2} + \frac{\text{coſ}u}{\text{coſ}u^3} + \frac{\text{coſ}2u}{\text{coſ}u^4} + \frac{\text{coſ}3u}{\text{coſ}u^5} + \frac{\text{coſ}4u}{\text{coſ}u^6} + \&c.$

ſeu $o = 1 + \frac{\text{coſ}u}{\text{coſ}u} + \frac{\text{coſ}2u}{\text{coſ}u^2} + \frac{\text{coſ}3u}{\text{coſ}u^3} + \frac{\text{coſ}4u}{\text{coſ}u^4} + \frac{\text{coſ}5u}{\text{coſ}u^5} + \&c.$

94. Imprimis autem expreſſio inuenta:

$$\text{A tang}(x+\omega) =$$

$$\text{A tg}\,x + \frac{\omega}{1}\,\text{ſin}\,u.\text{ſin}\,u - \frac{\omega^3}{2}\,\text{ſin}\,u^2.\text{ſin}\,u + \frac{\omega^3}{3}\,\text{ſin}\,u^3.\text{ſin}\,3u - \&c.$$

exiſtente $x = \cot u$ ſeu $u = \text{A}\cot x = 90^\circ - \text{A tang}\,x$ inſeruiet ad angulum ſeu arcum datae cuique tangenti reſpondentem inueniendum. Sit enim propoſita tangens $= t$, quaeraturque in tabulis tangens ad hanc proxime accedens $= x$, cui reſpondeat arcus $= y$; eritque $u = 90^\circ - y$. Tum ponatur $x + \omega = t$, ſeu $\omega = t - x$; eritque arcus quaeſitus:

$$= y + \frac{\omega}{1}\,\text{ſin}\,u.\,\text{ſin}\,u - \frac{\omega^2}{2}\,\text{ſin}\,u^2.\,\text{ſin}\,2u + \&c.$$

quae regula tum praecipue eſt vtilis, cum tangens propoſita fuerit admodum magna, ac propterea arcus quaeſitus parum a 90° diſcrepet. His enim caſibus ob tangentes vehementer increſcentes, ſolita methodus interpolationum nimium a veritate abducit. Sit ergo propoſitum hoc exemplum.

EXEMPLUM.

Quaeratur arcus, cuius tangens ſit $= 100$, *poſito radio* $= 1$.

Arcus proxime quaeſito aequalis eſt $89°, 25'$, cuius tangens eſt $x = 98,217943$ ſecund. quae ſubtrahatur a $t = 100,00000$

remanebit $\omega = 1,782057$

Deinde cum ſit $y = 89°, 25'$, erit $u = 0°, 35'$, $2u = 1°, 10'$, $3u = 1°, 45'$, &c. Iam ſinguli termini per logarithmos inueſtigentur.

Ad	$l\omega =$	0, 2509215
add.	$l\,\text{ſin}\,u =$	8, 0077867
	$l\,\text{ſin}\,u =$	8, 0077867
	$l\,\omega\,\text{ſin}\,u.\,\text{ſin}\,u =$	6, 2664949
		4, 6855749
	ſubtr. $=$	1, 5809200
Ergo	$\omega\,\text{ſin}\,u.\,\text{ſin}\,u =$	38,09956 ſecund.
Ad	$l\,\omega\,\text{ſin}\,u^2 =$	6, 2664949
add.	$l\omega =$	0, 2509215
	$l\,\text{ſin}\,2u =$	8, 3087941
		4, 8262105
	ſubr. $l\,2 =$	0, 3010300
	$l\,\frac{1}{2}\,\omega^2\,\text{ſin}\,u^2.\,\text{ſin}\,2u =$	4, 5251805
	ſubtr.	4, 6855749
	Remanet	9, 8396056
Ergo	$\frac{1}{2}\,\omega^2\,\text{ſin}\,u^2.\,\text{ſin}\,2u =$	0, 69120 ſecund.

Porro

Porro ad	$l\omega^3 =$	0, 7527645
add.	$l \text{ ſin } u^3 =$	4, 0233601
	$l \text{ ſin } 3u =$	8, 4848479
		3, 2609725
ſubtr.	$l3 =$	0, 4771213
		2, 7838512
ſubtr.		4, 6855749
		8, 0982763

Ergo $\frac{1}{3}\omega^3 \text{ ſin } u^3 \text{ ſin } 3u =$ 0, 01254 ſecund.

Denique ad	$l\omega^4 =$	1, 0036860
add.	$l \text{ ſin } u^4 =$	2, 0311468
	$l \text{ ſin } 4u =$	8, 6097341
		1, 6445669
ſubtr.	$l4 =$	0, 6020600
		1, 0425069
ſubtr.		4, 6855749
		6, 3569320

Ergo $\frac{1}{4}\omega^4 \text{ ſin } u^4 \text{ ſin } 4u =$ 0, 00023 ſecund.

Hinc:

Hinc:

Termini addendi	Termini ſubtrahendi
38, 09956	0, 69120
0, 01254	0, 00023
38, 11210	0, 69143
ſubtr. 0, 69143	

$37,42067 = 37^{II}, 25^{III}, 14^{IV}, 24^{V}, 36^{VI}$.

Quocirca arcus, cuius tangens centies ſuperat radium erit: $89^{\circ}, 25^{I}, 37^{II}, 25^{III}, 14^{IV}, 24^{V}, 36^{VI}$, neque error ad minuta quarta aſcendit; ſed in minutis tantum quintis ineſſe poteſt, ex quo vere hunc angulum pronunciare poterimus $= 89^{\circ}, 25^{I}, 37^{II}, 25^{III}, 14^{IV}$. Si tangens adhuc maior proponatur, etiamſi fortaſſe ω maius prodeat, tamen ob u angulum adhuc minorem, aeque expedite arcus definiri poterit.

95. Cum hic pro y arcum circuli ſubſtituerimus, nunc functiones reciprocas in locum y ponamus, cuiusmodi ſunt $\operatorname{ſin} x$, $\operatorname{coſ} x$, $\operatorname{tang} x$, $\cot x$, &c. Sit igitur $y = \operatorname{ſin} x$, poſitoque $x + \omega$ loco x, fiet: $z = \operatorname{ſin}(x+\omega)$, atque aequatio

$$z = y + \frac{\omega\, dy}{dx} + \frac{\omega^2\, ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^2} + \frac{\omega^4 d^4y}{24dx^4} + \&c.$$

ob $\frac{dy}{dx} = \operatorname{coſ} x$; $\frac{ddy}{dx^2} = -\operatorname{ſin} x$; $\frac{d^3y}{dx^3} = -\operatorname{coſ} x$; &c. dabit

$$\operatorname{ſin}(x+\omega) = \operatorname{ſin} x + \omega \operatorname{coſ} x - \tfrac{1}{2}\omega^2 \operatorname{ſin} x - \tfrac{1}{6}\omega^3 \operatorname{coſ} x + \tfrac{1}{24}\omega^4 \operatorname{ſin} x + \&c.$$

&

& ſumto ω negatiuo erit:

$$\text{ſin}(x-\omega) = \text{ſn}\,x - \omega\,\text{cſ}\,x - \tfrac{1}{2}\omega^2\,\text{ſn}\,x + \tfrac{1}{6}\omega^3\,\text{cſ}\,x + \tfrac{1}{24}\omega^4\,\text{ſn}\,x - \&c.$$

Quod ſi vero ſtatuatur $y = \text{coſ}\,x$,

ob $\frac{dy}{dx} = -\,\text{ſn}\,x$; $\frac{ddy}{dx^2} = -\,\text{cſ}\,x$; $\frac{d^3y}{dx^3} = \text{ſn}\,x$; $\frac{d^4y}{dx^4} = \text{cſ}\,x$; &c.

erit:

$$\text{coſ}(x+\omega) = \text{cſ}\,x - \omega\,\text{ſn}\,x - \tfrac{1}{2}\omega^2\,\text{cſ}\,x + \tfrac{1}{6}\omega^3\,\text{ſn}\,x + \tfrac{1}{24}\omega^4\,\text{cſ}\,x - \&c.$$

& facto ω negatiuo erit:

$$\text{coſ}(x-\omega) = \text{cſ}\,x + \omega\,\text{ſn}\,x - \tfrac{1}{2}\omega^2\,\text{cſ}\,x - \tfrac{1}{6}\omega^3\,\text{ſn}\,x + \tfrac{1}{24}\omega^4\,\text{cſ}\,x + \&c.$$

96. Vſus harum formularum eximius eſt cum in condendis, tum interpolandis tabulis ſinuum & coſinuum. Si enim cogniti fuerint ſinus & coſinus cuiuspiam arcus x, ex iis facili negotio ſinus & coſinus angulorum $x+\omega$ & $x-\omega$ inueniri poſſunt, ſiquidem differentia ω fuerit ſatis exigua: hoc enim caſu ſeries inuentae vehementer couergunt. Ad hoc vero neceſſe eſt, vt arcus ω in partibus radii exprimatur; quod cum arcus 180° ſit:

3, 14159265358979323846

facile fiet: erit enim diuiſione per 180 inſtituta

arcus $1^\circ = 0,017453292519943295769$
arcus $1' = 0,000290888208665721596$
arcus $10'' = 0,000048481368110953599$

EXEMPLUM I.

Inuenire ſinus & coſinus angulorum 45°, 1′, *&* 44°, 59′, *ex datis ſinu & coſinu anguli* 45°, *quorum vterque eſt*

$$= \frac{1}{\sqrt{2}} = 0,7071067811865.$$

 Cum

Cum igitur ſit:

$\sin x = \cos x = 0,7071067811865$

atque $\omega = 0,0002908882086$

erit ad multiplicationes facilius inſtituendas:

$2\omega = 0,0005817764173$
$3\omega = 0,0008726646259$
$4\omega = 0,0011635528346$
$5\omega = 0,0014544410432$
$6\omega = 0,0017453292519$
$7\omega = 0,0020362174605$
$8\omega = 0,0023271056692$
$9\omega = 0,0026179938779$

Ergo $\omega \sin x$ & $\omega \cos x$ hoc modo inuenietur:

7 .	0,00020362174605
0 .	. . . .
7 .	0,0000020362174
1 .	. 2908882
0 .	. . .
6 .	. 174532
7 .	. 20362
8 .	. 2372
1 .	. . 29
1 .	. . 2
8 .	. . 2
6 .	. . 0

$\omega \sin x = \omega \cos x = 0,00020568902490$

Ergo $\frac{1}{2}\omega \cos x = 0,00010284451245$

per

per ω . 1 . 0, 00000002908882
0 . . . .
2 . . 58177
8 . . 23271
4 . . 1163
4 . . 116
5 . . . 14

$\frac{1}{2}\omega^2$coſx = 0, 00000002991625

$\frac{1}{6}\omega^2$coſx = 0, 00000000997208

per ω . 9 . 0, 00000000000261
9 . . . 26
7 . . . 2

$\frac{1}{6}\omega^3$coſx = 0, 00000000000290

Ergo ad ſin 45°, 1′, inueniendum:

Ad ſin$x$ = 0, 7071067811865
add. ωcoſx = 2056890249

0, 7073124702114
ſubtr. $\frac{1}{2}\omega^2$ſin$x$ = 299162

0, 7073124402952
ſubtr. $\frac{1}{6}\omega^3$coſx = 29

ſin 45°, 1′, = 0, 7073124402923 = coſ 44°, 59′.

At ad coſ 45°, 1', inueniendum:

$$
\begin{array}{rrl}
 & A\cos x = & 0,7071067811865 \\
\text{ſubr.} & \omega \sin x = & 2056890249 \\
\hline
 & & 0,7069010921616 \\
\text{ſubtr.} & \tfrac{1}{2}\omega^2 \cos x = & 299162 \\
\hline
 & & 0,7069010622454 \\
\text{add.} & \tfrac{1}{6}\omega^3 \sin x = & 29 \\
\hline
\end{array}
$$

$$\cos 45°, 1', = 0,7069010622483 = \sin 44°, 59'.$$

EXEMPLUM II.

Ex datis ſinu & coſinu arcus 67°, 30', *inuenire ſinus & coſinus arcuum* 67°, 31', & 67°, 29'.

Abſoluamus hunc calculum in fractionibus decimalibus, tantum ad 7 notas, vti tabulae vulgares conſtrui ſolent, ſicque negotium facile per logarithmos conficietur. Cum ſit $x = 67°, 30'$, & $\omega = 0,000290888$; erit: $l\omega = 6,4637259$ &

$$
\begin{array}{rl@{\quad;\quad}rl}
l\sin x = & 9,9656153 & l\cos x = & 9,5828397 \\
l\omega = & 6,4637259 & l\omega = & 6,4637259 \\
\hline
l\omega \sin x = & 6,4293412 & l\omega \cos x = & 6,0465656 \\
l\tfrac{1}{2}\omega = & 6,1626959 & l\tfrac{1}{2}\omega = & 6,1626959 \\
\hline
l\tfrac{1}{2}\omega^2 \sin x = & 2,5920371 & l\tfrac{1}{2}\omega^2 \cos x = & 2,2092615
\end{array}
$$

ergo:

ergo:

$\omega\,\text{ſin}\,x = 0{,}00026874$; $\omega\,\text{coſ}\,x = 0{,}00011232$

$\frac{1}{2}\omega^2\,\text{ſin}\,x = 0{,}00000004$; $\frac{1}{2}\omega^2\,\text{coſ}\,x = 0{,}00000001$

vnde fit:

$\text{ſin}\,67^\circ, 31^{\text{I}} = 0{,}9239908$; $\text{coſ}\,67^\circ, 31^{\text{I}} = 0{,}3824147$

$\text{ſin}\,67^\circ, 29^{\text{I}} = 0{,}9237681$; $\text{coſ}\,67^\circ, 29^{\text{I}} = 0{,}3829522$

vbi nequidem terminis $\frac{1}{2}\omega^2\,\text{ſin}\,x$ & $\frac{1}{2}\omega^2\,\text{coſ}\,x$ erat opus.

97. Ex ſeriebus quas ſupra inuenimus:

$\text{ſin}(x+\omega) = \text{ſn}\,x + \omega\,\text{cſ}\,x - \frac{1}{2}\omega^2\,\text{ſn}\,x - \frac{1}{6}\omega^3\,\text{cſ}\,x + \frac{1}{24}\omega^4\,\text{ſn}\,x + \&c.$

$\text{coſ}(x+\omega) = \text{cſ}\,x - \omega\,\text{ſn}\,x - \frac{1}{2}\omega^2\,\text{cſ}\,x + \frac{1}{6}\omega^3\,\text{ſn}\,x + \frac{1}{24}\omega^4\,\text{cſ}\,x - \&c.$

$\text{ſin}(x-\omega) = \text{ſn}\,x - \omega\,\text{cſ}\,x - \frac{1}{2}\omega^2\,\text{ſn}\,x + \frac{1}{6}\omega^3\,\text{cſ}\,x + \frac{1}{24}\omega^4\,\text{ſn}\,x - \&c.$

$\text{coſ}(x-\omega) = \text{cſ}\,x + \omega\,\text{ſn}\,x - \frac{1}{2}\omega^2\,\text{cſ}\,x - \frac{1}{6}\omega^3\,\text{ſn}\,x + \frac{1}{24}\omega^4\,\text{cſ}\,x + \&c.$

ſequitur per combinationem fore:

$$\frac{\text{ſin}(x+\omega) + \text{ſin}(x-\omega)}{2} =$$

$$\text{ſin}\,x - \tfrac{1}{2}\omega^2\,\text{ſin}\,x + \tfrac{1}{24}\omega^4\,\text{ſin}\,x - \tfrac{1}{720}\omega^6\,\text{ſn}\,x + \&c. = \text{ſn}\,x\,\text{cſ}\,\omega$$

Et $$\frac{\text{ſin}(x+\omega) - \text{ſin}(x-\omega)}{2} =$$

$$\omega\,\text{coſ}\,x - \tfrac{1}{6}\omega^3\,\text{coſ}\,x + \tfrac{1}{120}\omega^5\,\text{coſ}\,x - \&c. = \text{coſ}\,x\,\text{ſin}\,\omega$$

vnde prodeunt ſeries pro ſinibus & coſinibus iam ſupra inuentae:

$$\text{coſ}\,\omega = 1 - \tfrac{1}{2}\omega^2 + \tfrac{1}{24}\omega^4 - \tfrac{1}{720}\omega^6 + \&c.$$

$$\text{ſin}\,\omega = \omega - \tfrac{1}{6}\omega^3 + \tfrac{1}{120}\omega^5 - \tfrac{1}{5040}\omega^7 + \&c.$$

quae eaedem ſeries ex primis ponendo $x = 0$ conſequuntur; cum enim ſit $\text{coſ}\,x = 1$ & $\text{ſin}\,x = 0$ prima ſeries *ſin* ω, ſecunda vero *coſ* ω exhibebit.

98. Ponamus nunc quoque $y = \tang x$, vt ſit $z = \tang(x + \omega)$, erit ob

$$y = \frac{\sin x}{\cos x}; \quad \frac{dy}{dx} = \frac{1}{\cos x^2}; \quad \frac{ddy}{2dx^2} = \frac{\sin x}{\cos x^3};$$

$$\frac{d^3y}{2dx^3} = \frac{1}{\cos x^2} + \frac{3\sin x^2}{\cos x^4} = \frac{3}{\cos x^4} - \frac{2}{\cos x^2};$$

$$\frac{d^4y}{2.4dx^4} = \frac{3\sin x}{\cos x^5} - \frac{\sin x}{\cos x^3};$$

$$\frac{d^5y}{2.4dx^5} = \frac{15}{\cos x^6} - \frac{15}{\cos x^4} + \frac{2}{\cos x^2};$$

vnde ſequitur fore:

$$\tang(x+\omega) = \mathrm{tg}\, x + \frac{\omega}{\cos x^2} + \frac{\omega^2 \sin x}{\cos x^3} + \frac{\omega^3}{\cos x^4} + \frac{\omega^4 \sin x}{\cos x^5} - \frac{2\omega^3}{3\cos x^2} - \frac{\omega^4 \sin x}{3\cos x^3};$$

cuius formulae ope ex data cuiusuis anguli tangente inveniri poſſunt tangentes angulorum proximorum. Quia vero ſuperior ſeries eſt geometrica, ea in vnam ſummam collecta erit:

$$\tang(x+\omega) = \mathrm{tg}\, x + \frac{\omega + \omega^2 \mathrm{tg}\, x}{\cos x^2 - \omega^2} - \frac{2\omega^3}{3\cos x^2} - \frac{\omega^4 \sin x}{3\cos x^3} \;\&c.$$

ſeu $$\tang(x+\omega) = \frac{\sin x \cos x + \omega}{\cos x^2 - \omega^2} - \frac{2\omega^3}{3\cos x^2} - \frac{\omega^4 \sin x}{3\cos x^3} \;\&c.$$

quae formula in hunc finem commodius adhibetur.

99. Similes expreſſiones quoque pro logarithmis ſinuum, coſinuum & tangentium inueniri poſſunt. Sit enim $y =$ logarithmo ſinus anguli x, quod ita exprimamus

mus $y = l\,\text{ſin}\,x$, & $z = l\,\text{ſin}(x + \omega)$, ob $\frac{py}{dx} = \frac{n\,\text{coſ}\,x}{\text{ſin}\,x}$; erit: $\frac{ddy}{dx^2} = \frac{-n}{\text{ſin}\,x^2}$; $\frac{d^3y}{dx^3} = \frac{+n\,\text{coſ}\,x}{\text{ſin}\,x^3}$ &c. vnde fiet:

$$z = l\,\text{ſin}\,(x + \omega) = l\,\text{ſin}\,x$$

$$+ \frac{n\omega\,\text{coſ}\,x}{\text{ſin}\,x} - \frac{n\omega^2}{2\,\text{ſin}\,x^2} + \frac{n\omega^3\,\text{coſ}\,x}{3\,\text{ſin}\,x^3}\ \&c.$$

vbi n denotat numerum, per quem logarithmi hyperbolici multiplicari debent, vt prodeant logarithmi propoſiti. Sin autem ſit

$$y = l\,\text{tang}\,x \quad \& \quad z = l\,\text{tang}\,(x + \omega)$$

fiet:

$$\frac{dy}{dx} = \frac{n}{\text{ſin}\,x\,\text{coſ}\,x} = \frac{2n}{\text{ſin}\,2x};\quad \frac{ddy}{2\,dx^2} = \frac{-2n\,\text{coſ}\,2x}{(\text{ſin}\,2x)^2};$$

ideoque

$$l\,\text{tang}\,(x + \omega) = l\,\text{tang}\,x + \frac{2n\omega}{\text{ſin}\,2x} - \frac{2n\omega^2\,\text{coſ}\,2x}{(\text{ſin}\,2x)^2}\ \&c.$$

quarum formularum ope logarithmi ſinuum & tangentium interpolari poſſunt.

100. Ponamus denotare y arcum cuius ſinus logarithmus ſit $= x$, ſeu vt ſit $y = A.\,l\,\text{ſin}\,x$, & z eſſe arcum, cuius ſinus logarithmus ſit $= x + \omega$, ſeu $z = A.\,l\,\text{ſin}\,(x + \omega)$; erit $x = l\,\text{ſin}\,y$, & $\frac{dx}{dy} = \frac{n\,\text{coſ}\,y}{\text{ſin}\,y}$, vnde $\frac{dy}{dx} = \frac{\text{ſin}\,y}{n\,\text{coſ}\,y}$; erit:

ddy

$$\frac{ddy}{dx} = \frac{dy}{n\,\mathrm{coſ}\,y^2} = \frac{dx\,\mathrm{ſin}\,y}{n^2\,\mathrm{coſ}\,y^3}\,; \quad \text{ergo} \quad \frac{ddy}{dx^2} = \frac{\mathrm{ſin}\,y}{n^2\,\mathrm{coſ}\,y^3}:$$

Conſequenter

$$z = y + \frac{\omega\,\mathrm{ſin}\,y}{n\,\mathrm{coſ}\,y} + \frac{\omega^2\,\mathrm{ſin}\,y}{2n^2\,\mathrm{coſ}\,y^3} + \&c.$$

Simili modo ſi logarithmus coſinus detur, expreſſio reperietur. Sin autem ſit

$$y = A.\,l\,\mathrm{tang}\,x \quad \& \quad z = A.\,l\,\mathrm{tang}\,(x+\omega).$$

Eum ſit $x = l\,\mathrm{tang}\,y$; fiet:

$$\frac{dx}{dy} = \frac{n}{\mathrm{ſin}\,y\,\mathrm{coſ}\,y}, \quad \& \quad \frac{dy}{dx} = \frac{\mathrm{ſin}\,y\,\mathrm{coſ}\,y}{n} = \frac{\mathrm{ſin}\,2y}{2n}\,;$$

quare

$$\frac{ddy}{dx} = \frac{2\,dy\,\mathrm{coſ}\,2y}{2n} = \frac{dx\,\mathrm{ſin}\,2y\,\mathrm{coſ}\,2y}{2nn}$$

&

$$\frac{ddy}{dx^2} = \frac{\mathrm{ſin}\,2y\,\mathrm{coſ}\,2y}{2nn} = \frac{\mathrm{ſin}\,4y}{4nn}\,; \quad \frac{d^3y}{dx^3} = \frac{\mathrm{ſin}\,2y.\,\mathrm{coſ}\,4y}{2n^3}\ \&c.$$

hinc

$$z = y + \frac{\omega\,\mathrm{ſin}\,2y}{2n} + \frac{\omega^2\,\mathrm{ſin}\,2y\,\mathrm{coſ}\,2y}{4nn} + \frac{\omega^3\,\mathrm{ſin}\,2y.\,\mathrm{coſ}\,4y}{12n^3} + \&c.$$

101. Quoniam vſus harum expreſſionum in condendis tabulis logarithmorum ſinuum & tangentium ex antecedentibus facile perſpici poteſt, his diutius non immorabimur. Conſideremus ergo adhuc huiusmodi valorem:

$y =$

$y = e^x \text{ſin}\, nx$; ſitque $z = e^{x+\omega} \text{ſin}\, n(x+\omega)$

quia eſt

$$\frac{dy}{dx} = e^x (\text{ſin}\, nx + n \text{coſ}\, nx)$$

$$\frac{ddy}{dx^2} = e^x ((1 - nn) \text{ſin}\, nx + 2n \text{coſ}\, nx)$$

$$\frac{d^3y}{dx^3} = e^x ((1 - 3nn) \text{ſin}\, nx + n(3 - nn) \text{coſ}\, nx)$$

$$\frac{d^4y}{dx^4} = e^x ((1 - 6nn + n^4) \text{ſin}\, nx + n(4 - 4nn) \text{coſ}\, nx)$$

$$\frac{d^5y}{dx^5} = e^x ((1 - 10nn + 5n^4) \text{ſin}\, nx + n(5 - 10nn + n^4) \text{coſ}\, nx)$$

His ſubſtitutis & diuiſione per e^x inſtituta erit:

$$e^{\omega} \text{ſin}\, n(x+\omega) = \text{ſin}\, nx + \omega \text{ſin}\, nx + \frac{(1-nn)}{2} \omega^2 \text{ſin}\, nx$$

$$+ n\omega \text{coſ}\, nx + \frac{2n\omega^2}{2} \text{coſ}\, nx$$

$$+ \frac{(1-3nn)}{6} \omega^3 \text{ſin}\, nx + \frac{(1-6nn+n^4)}{24} \omega^4 \text{ſin}\, nx + \&c.$$

$$+ \frac{n(3-nn)}{6} \omega^3 \text{coſ}\, nx + \frac{n(4-4nn)}{24} \omega^4 \text{coſ}\, nx + \&c.$$

102. Hinc plurima egregia corollaria deduci poſſunt; ſufficiat autem nobis haec annotaſſe.

Si fuerit $x = 0$ erit:

$$e^{\omega} \text{ſin}\, n\omega = n\omega$$

$$+ \frac{2n\omega^2}{2} + \frac{n(3-nn)}{6} \omega^3 + \frac{n(4-4nn)}{24} \omega^4 + \frac{n(5-10n^2+n^4)}{120} \omega^5 + \&c.$$

 Si

Si ſit $\omega = -x$, ob $\text{ſin}\, n(x+\omega) = 0$; erit:

$$\text{tang}\; nx = \frac{nx - \frac{2n}{2}x^2 + \frac{n(3-nn)}{6}x^3 - \frac{n(4-4nn)}{24}x^4 + \frac{n(5-10n^2+n^4)}{120}x^5}{1 - x + \frac{(1-nn)}{2}x^2 - \frac{(1-3nn)}{6}x^3 + \frac{(1-6nn+n^4)}{24}x^4 - \&c.}$$

Generaliter vero ſi ſit $n = 1$ habebitur:

$$e^{\omega}\,\text{ſin}(x+\omega) = \text{ſin}\,x\,(1 + \omega - \tfrac{1}{3}\omega^3 - \tfrac{1}{6}\omega^4 - \tfrac{1}{30}\omega^5 + \tfrac{1}{630}\omega^7 + \&c.)$$
$$+\, \omega\,\text{coſ}\,x\,(1 + \omega + \tfrac{1}{3}\omega^2 - \tfrac{1}{30}\omega^4 - \tfrac{1}{90}\omega^5 - \tfrac{1}{630}\omega^6 + \&c.)$$

Sin autem ſit $n = 0$, ob $\text{ſin}\, n(x+\omega) = n(x+\omega)$, & $\text{ſin}\, nx = nx$, atque $\text{coſ}\, nx = 1$, ſi vbique per n diuidatur, prodibit:

$$e^{\omega}(x+\omega) = x + \omega x + \tfrac{1}{2}\omega^2 x + \tfrac{1}{6}\omega^3 x + \tfrac{1}{24}\omega^4 x + \&c.$$
$$+\, \omega + \omega^2 + \tfrac{1}{2}\omega^3 + \tfrac{1}{6}\omega^4 + \tfrac{1}{24}\omega^5 + \&c.$$

cuius ſeriei ratio eſt manifeſta.

CAPUT

CAPUT V.

INUESTIGATIO SUMMAE SERIERUM EX TERMINO GENERALI.

103.

Sit Seriei cuiusque terminus generalis $= y$, reſpondens indici x, ita vt y ſit functio quaecunque ipſius x. Sit porro Sy ſumma ſeu terminus ſummatorius ſeriei, exprimens aggregatum omnium terminorum a primo ſeu alio termino fixo vsque ad y incluſiue. Computabimus autem ſummas ſerierum a termino primo, vnde ſi ſit $x = 1$, dabit y terminum primum, atque Sy hunc y terminum primum exhibebit: ſin autem ponatur $x = 0$, terminus ſummatorius Sy in nihilum abire debet, propterea quod nulli termini ſummandi adſunt. Quocirca terminus ſummatorius Sy eiusmodi erit functio ipſius x, quae euaneſcat poſito $x = 0$.

104. Si terminus generalis y ex pluribus partibus conſtet, vt ſit $y = p + q + r +$ &c. tum ipſa ſeries conſiderari poterit tanquam conflata ex pluribus aliis ſeriebus, quarum termini generales ſint p, q, r, &c. Hinc ſi ſingularum iſtarum ſerierum ſummae fuerint cognitae, ſimul ſeriei propoſitae ſumma poterit aſſignari; erit enim aggregatum ex ſummis ſingularum ſerierum. Hancobrem ſi ſit $y = p + q + r +$ &c. erit $Sy = Sp + Sq + Sr +$ &c. Cum igitur ſupra exhi-

buerimus ſummas ſerierum, quarum termini generales ſint quaecunque poteſtates ipſius x, habentes exponentes integros affirmatiuos; hinc cuiusque ſeriei, cuius terminus generalis eſt $ax^\alpha + bx^\beta + cx^\gamma +$ &c. denotantibus α, β, γ, &c. numeros integros affirmatiuos, ſeu cuius terminus generalis eſt functio rationalis integra ipſius x, terminus ſummatorius inueniri poterit.

105. Sit in ſerie, cuius terminus generalis ſeu exponenti x reſpondens eſt $= y$, terminus hunc praecedens ſeu exponenti $x-1$ reſpondens $= v$, quoniam v oritur ex y, ſi loco x ſcribatur $x-1$; erit:

$$v = y - \frac{dy}{dx} + \frac{ddy}{2dx^2} - \frac{d^3y}{6dx^3} + \frac{d^4y}{24dx^4} - \frac{d^5y}{120dx^5} + \&c.$$

Si igitur y fuerit terminus generalis huius ſeriei

$$\begin{matrix} 1 & 2 & 3 & 4 & \ldots & x-1 & x \\ a+ & b+ & c+ & d+ & \ldots & +v & +y \end{matrix}$$

huiusque ſeriei terminus indici o reſpondens fuerit $= A$, erit v, quatenus eſt functio ipſius x, terminus generalis huius ſeriei:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & \ldots & x \\ A+ & a+ & b+ & c+ & d+ & \ldots & +v \end{matrix}$$

vnde ſi Sv denotet ſummam huius ſeriei, erit $Sv = Sy - y + A$. Sicque poſito $x = 0$, quia fit $Sy = 0$ & $y = A$, quoque Sv euaneſcet.

106. Cum igitur ſit $v = y - \frac{dy}{dx} + \frac{ddy}{2dx^2} - \frac{d^3y}{6dx^3} + \&c.$ erit per ante oſtenſa:

$Sv =$

$$Sv = Sy - S\frac{dy}{dx} + S\frac{ddy}{2dx^2} - S\frac{d^3y}{6dx^3} + S\frac{d^4y}{24dx^4} - \&c.$$

atque ob $Sv = Sy - y + A$, erit:

$$y - A = S\frac{dy}{dx} - S\frac{ddy}{2dx^2} + S\frac{d^3y}{6dx^3} - S\frac{d^3y}{24dx^4} + \&c.$$

ideoque habebitur:

$$S\frac{dy}{dx} = y - A + S\frac{ddy}{2dx^2} - S\frac{d^3y}{6dx^3} + S\frac{d^4y}{24dx^4} - \&c.$$

Si ergo habeantur termini ſummatorii ſerierum, quarum termini generales ſunt $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$, &c. ex iis obtinebitur terminus ſummatorius ſeriei, cuius terminus generalis eſt $\frac{dy}{dx}$. Quantitas vero conſtans A ita debet eſſe comparata, vt facto $x = 0$ terminus ſummatorius $S\frac{dy}{dx}$ euaneſcat; hacque conditione facilius determinatur, quam ſi diceremus, eam eſſe terminum indici o reſpondentem in ſerie, cuius terminus generalis ſit $= y$.

107. Ex hoc fonte ſummae poteſtatum numerorum naturalium inueſtigari ſolent. Sit enim $y = x^{n+1}$; quoniam fit $\frac{dy}{dx} = (n+1)x^n$; $\frac{ddy}{2dx^2} = \frac{(n+1)n}{1.\ 2}x^{n-1}$;

$$\frac{d^3y}{6dx^3} = \frac{(n+1)n(n-1)}{1.\ 2.\ 3}x^{n-2}; \quad \frac{d^4y}{24dx^4} = \frac{(n+1)n(n-1)(n-2)}{1.\ 2.\ 3.\ 4}x^{n-3}$$

&c.

 erit

erit his valoribus ſubſtitutis:

$$(n+1)Sx^n = x^{n+1} - A + \frac{(n+1)n}{1.\ 2} Sx^{n-1} - \frac{(n+1)n(n-1)}{1.\ 2.\ 3} Sx^{n-2} + \&c.$$

atque ſi vtrinque per $n+1$ diuidatur; erit:

$$Sx^n = \frac{1}{n+1} x^{n+1} + \frac{n}{2} Sx^{n-1} - \frac{n(n-1)}{2.\ 3} Sx^{n-2} + \frac{n(n-1)(n-2)}{2.\ 3.\ 4} Sx^{n-3} + \&c. - \text{Conſt.}$$

quae conſtans ita accipi debet, vt poſito $x = 0$, totus terminus ſummatorius euaneſcat. Ope huius ergo formulae ex iam cognitis ſummis poteſtatum inferiorum, quarum termini generales ſunt x^{n-1}, x^{n-2}, &c. inueniri poterit ſumma poteſtatum ſuperiorum termino generali x^n expreſſarum.

108. Si in hac expreſſione n denotet numerum integrum affirmatiuum, numerus terminorum erit finitus. Atque adeo hinc ſumma infinitarum poteſtatum ſi $n = 0$, abſolute cognoſcetur; erit enim: $S.x^0 = x$. Hacque cognita ad ſuperiores progredi licebit, poſito enim $n = 1$; fiet:

$S.x^1 = \frac{1}{2} x^2 + \frac{1}{2} Sx^0 = \frac{1}{2} x^2 + \frac{1}{2} x$

ſi porro ponatur $n = 2$ prodibit:

$S.x^2 = \frac{1}{3} x + Sx - \frac{1}{3} Sx^0 = \frac{1}{3} x^3 + \frac{1}{2} x^2 + \frac{1}{6} x$; deinde

$S.x^3 = \frac{1}{4} x^4 + \frac{3}{2} Sx^2 - Sx + \frac{1}{4} Sx^0 = \frac{1}{4} x^4 + \frac{1}{2} x^3 + \frac{1}{4} x^2$

$S.x^4 = \frac{1}{5} x^5 + \frac{4}{2} Sx^3 - \frac{4}{2} Sx^2 + Sx - \frac{1}{5} Sx^0$ ſiue

$S.x^4 = \frac{1}{5} x^5 + \frac{1}{2} x^4 + \frac{1}{3} x^3 - \frac{1}{30} x.$

Sic-

Sicque porro quarumvis poteſtatum ſuperiorum ſummae ſucceſſiuae ex inferioribus colligentur; hoc autem facilius per ſequentes modos praeſtabitur.

109. Quoniam ſupra inuenimus eſſe:

$$S\frac{dy}{dx} = y + \tfrac{1}{2}S\frac{ddy}{dx^2} - \tfrac{1}{6}S\frac{d^3y}{dx^3} + \tfrac{1}{24}S\frac{d^4y}{dx^4} - \tfrac{1}{120}S\frac{d^5y}{dx^5} + \&c.$$

Si ponamus $\frac{dy}{dx} = z$; fiet $\frac{ddy}{dx^2} = \frac{dz}{dx}$; $\frac{d^3y}{dx^3} = \frac{ddz}{dx^2}$, &c. tum vero ob $dy = zdx$, erit y quantitas, cuius differentiale eſt $= zdx$, quod hoc modo indicamus, vt ſit $y = \int zdx$. Quanquam autem haec inuentio ipſius y ex dato z a calculo integrali pendet, tamen hic iam iſta forma $\int zdx$ vti poterimus, ſi quidem pro z alias ipſius x functiones non ſubſtituamus, niſi eiusmodi, vt functio illa, cuius differentiale eſt $= zdx$, ex praecedentibus exhiberi queat. His igitur valoribus ſubſtitutis erit:

$$Sz = \int zdx + \tfrac{1}{2}S\frac{dz}{dx} - \tfrac{1}{6}S\frac{ddz}{dx^2} + \tfrac{1}{24}S\frac{d^3z}{dx^3} - \&c.$$

adiiciendo eiusmodi conſtantem, vt poſito $x = 0$ ipſa ſumma Sz euaneſcat.

110. Subſtituendo autem loco y in ſuperiori expreſſione litteram z, vel quod eodem redit differentiando iſtam aequationem erit:

$$S\frac{dz}{dx} = z + \tfrac{1}{2}S\frac{ddz}{dx^3} - \tfrac{1}{6}S\frac{d^3z}{dx^3} + \tfrac{1}{24}S\frac{d^4z}{dx^4} - \&c.$$

ſin

ſin autem loco y ponatur $\frac{dz}{dx}$; erit:

$$S\frac{ddz}{dx^2} = \frac{dz}{dx} + \tfrac{1}{2}S\frac{d^3z}{dx^3} - \tfrac{1}{6}S\frac{d^4z}{dx^4} + \tfrac{1}{24}S\frac{d^5z}{dx^5} - \&c.$$

Similique modo ſi pro y ſucceſſiue ponantur valores $\frac{ddz}{dx^2}$; $\frac{d^3z}{dx^3}$; &c. reperietur:

$$S\frac{d^3z}{dx^3} = \frac{ddz}{dx^2} + \tfrac{1}{2}S\frac{d^4z}{dx^4} - \tfrac{1}{6}S\frac{d^5z}{dx^5} + \tfrac{1}{24}S\frac{d^6z}{dx^6} - \&c.$$

$$S\frac{d^4z}{dx^4} = \frac{d^3z}{dx^4} + \tfrac{1}{2}S\frac{d^5z}{dx^5} - \tfrac{1}{6}S\frac{d^6z}{dx^6} + \tfrac{1}{24}S\frac{d^7z}{dx^7} - \&c.$$

ſicque porro in infinitum.

111. Si nunc iſti valores pro $S\frac{dz}{dx}$; $S\frac{ddz}{dx}$; $S\frac{d^3z}{dx^3}$; &c. ſucceſſiue ſubſtituantur in expreſſione:

$$Sz = \int z\,dx + \tfrac{1}{2}S\frac{dz}{dx} - \tfrac{1}{6}S\frac{ddz}{dx^2} + \tfrac{1}{24}S\frac{d^3z}{dx^3} - \&c.$$

inuenietur expreſſio pro Sz, quae conſtabit ex his terminis $\int z\,dx$; z; $\frac{dz}{dx}$; $\frac{ddz}{dx^2}$; $\frac{d^3z}{dx^3}$; &c. quorum coefficientes facilius ſequenti modo inueſtigabuntur. Ponatur

$$Sz = \int z\,dz + \alpha z + \frac{\beta dz}{dx} + \frac{\gamma ddz}{dx^2} + \frac{\delta d^3z}{dx^3} + \frac{\varepsilon d^4z}{dx^4} + \&c.$$

atque pro his terminis ſui valores ſubſtituantur, quos obtinent ex praecedentibus ſeriebus, ex quibus eſt:

$\int z dx$

$$\int z\,dx = Sz - \frac{1}{2}S\frac{dz}{dx} + \frac{1}{6}S\frac{ddz}{dx^2} - \frac{1}{24}S\frac{d^3z}{dx^3} + \frac{1}{120}S\frac{d^4z}{dx^4} - \&c.$$

$$\alpha z = +\alpha S\frac{dz}{dx} - \frac{\alpha}{2}S\frac{ddz}{dx^2} + \frac{\alpha}{6}S\frac{d^3z}{dx^3} - \frac{\alpha}{24}S\frac{d^4z}{dx^4} + \&c.$$

$$\frac{\beta dz}{dx} = \beta S\frac{ddz}{dx^2} - \frac{\beta}{2}S\frac{d^3z}{dx^3} + \frac{\beta}{6}S\frac{d^4z}{dx^4} - \&c.$$

$$\frac{\gamma ddz}{dx^2} = \gamma S\frac{d^3z}{dx^3} - \frac{\gamma}{2}S\frac{d^4z}{dx^4} + \&c.$$

$$\frac{\delta d^3z}{dx^3} = \delta S\frac{d^4z}{dx^4} - \&c.$$

qui valores additi, cum producere debeant Sz, coefficientes α, β, γ, δ, &c. ex ſequentibus aequationibus definientur:

$$\alpha - \frac{1}{2} = 0$$

$$\beta - \frac{\alpha}{2} + \frac{1}{6} = 0$$

$$\gamma - \frac{\beta}{2} + \frac{\alpha}{6} - \frac{1}{24} = 0$$

$$\delta - \frac{\gamma}{2} + \frac{\beta}{6} - \frac{\alpha}{24} + \frac{1}{120} = 0$$

$$\varepsilon - \frac{\delta}{2} + \frac{\gamma}{6} - \frac{\beta}{24} + \frac{\alpha}{120} - \frac{1}{720} = 0$$

$$\zeta - \frac{\varepsilon}{2} + \frac{\delta}{6} - \frac{\gamma}{24} + \frac{\beta}{120} - \frac{\alpha}{720} + \frac{1}{5040} = 0$$

&c.

112. Ex his ergo aequationibus ſucceſſiue valores omnium litterarum α, β, γ, δ, &c. definiri poterunt, reperietur autem:

$$\alpha = \frac{1}{2}$$

$$\beta = \frac{\alpha}{2} - \frac{1}{6} = \frac{1}{12}$$

$$\gamma = \frac{\beta}{2} - \frac{\alpha}{6} + \frac{1}{24} = 0$$

$$\delta = \frac{\gamma}{2} - \frac{\beta}{6} + \frac{\alpha}{24} - \frac{1}{120} = -\frac{1}{720}$$

$$\varepsilon = \frac{\delta}{2} - \frac{\gamma}{6} + \frac{\beta}{24} - \frac{\alpha}{120} + \frac{1}{720} = 0$$

&c.

ſicque vlterius progrediendo reperientur continuo termini alterni euaneſcentes. Litterae ergo ordine tertia, quinta, ſeptima, &c. omnesque impares erunt $= 0$, excepta prima, quo ipſo haec valorum ſeries contra legem continuitatis impingere videtur. Quamobrem eo magis neceſſe eſt, vt rigide demonſtretur, omnes terminos impares praeter primum neceſſario euaneſcere.

113. Quoniam ſingulae litterae ſecundum legem conſtantem ex praecedentibus determinantur, eae ſeriem recurrentem inter ſe conſtituent. Ad quam explicandam concipiatur iſta ſeries:

$$1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \zeta u^6 + \&c.$$

cu-

cuius valor ſit $=V$; atque manifeſtum eſt hanc ſeriem recurrentem oriri ex euolutione huius fractionis:

$$V = \frac{1}{1 - \frac{1}{2}u + \frac{1}{6}u^2 - \frac{1}{24}u^3 + \frac{1}{120}u^4 - \&c.}$$

atque ſi iſta fractio alio modo in ſeriem infinitam ſecundum poteſtates ipſius u progredientem reſolui queat, neceſſe eſt, vt ſemper eadem ſeries:

$$V = 1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \&c.$$

reſultet: hocque modo alia lex, qua iſti iidem valores α, β, γ, δ, &c. determinantur, eruetur.

114. Quoniam, ſi e denotet numerum, cuius logarithmus hyperbolicus vnitati aequatur, erit:

$$e^{-u} = 1 - u + \frac{1}{2}u^2 - \frac{1}{6}u^3 + \frac{1}{24}u^4 - \frac{1}{120}u^5 + \&c.$$

erit: $\frac{1-e^{-u}}{u} = 1 - \frac{1}{2}u + \frac{1}{6}u^2 - \frac{1}{24}u^3 + \frac{1}{120}u^4 - \&c.$

ideoque $V = \frac{u}{1-e^{-u}}$. Nunc extinguatur ex ſerie ſecundus terminus $\alpha u = \frac{1}{2}$, vt ſit:

$$V - \frac{1}{2}u = 1 + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \zeta u^6 + \&c.$$

erit: $V - \frac{1}{2}u = \frac{\frac{1}{2}u(1+e^{-u})}{1-e^{-u}}$. Multiplicentur numerator ac denominator per $e^{\frac{1}{2}u}$, eritque

$$V - \frac{1}{2}u = \frac{u(e^{\frac{1}{2}u} + e^{-\frac{1}{2}u})}{2(e^{\frac{1}{2}u} - e^{-\frac{1}{2}u})},$$

& quantitatibus $e^{\frac{1}{2}u}$ & $e^{-\frac{1}{2}u}$ in ſeries conuerſis fiet:

$$V - \tfrac{1}{2}u = \frac{1 + \frac{u^2}{2.4} + \frac{u^4}{2.4.6.8} + \frac{u^6}{2.4.6.8.10.12} + \&c.}{2(\frac{1}{2} + \frac{u^2}{2.4.6} + \frac{u^4}{2.4.6.8.10} + \&c.)}$$

ſiue

$$V - \tfrac{1}{2}u = \frac{1 + \frac{u^2}{2.4} + \frac{u^4}{2.4.6.8} + \frac{u^6}{2.4...12} + \frac{u^8}{2.4...16} + \&c.}{1 + \frac{u^2}{4.6} + \frac{u^4}{4.6.8.10} + \frac{u^6}{4.6...14} + \frac{u^8}{4.6...18} + \&c.}$$

115. Cum igitur in hac fractione poteſtates impares prorſus deſint, in eius quoque euolutione poteſtates impares omnino nullae ingredientur; quare cum $V - \frac{1}{2}u$ aequetur iſti ſeriei:

$$1 + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \zeta u^6 + \&c.$$

coefficientes imparium poteſtatum γ, ε, η, ι, &c. omnes euaneſcent. Sicque ratio manifeſta eſt, cur in ſerie:

$$1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \&c.$$

termini ordine pares omnes praeter ſecundum ſint $= 0$, neque tamen lex continuitatis vim patiatur. Erit ergo $V = 1 + \frac{1}{2}u + \beta u^2 + \delta u^4 + \zeta u^6 + \theta u^8 + \varkappa u^{10} + \&c.$ litteriſque β, δ, ζ, θ, $\varkappa$, &c. per euolutionem ſuperioris fractionis determinatis, obtinebimus terminum ſummatorium Sz ſeriei, cuius terminus generalis eſt $= z$, indici x reſpondens, hoc modo expreſſum:

$$Sz = \int z\,dx + \tfrac{1}{2}z + \frac{\beta\,dz}{dx} + \frac{\delta\,d^3z}{dx^3} + \frac{\zeta\,d^5z}{dx^5} + \frac{\theta\,d^7z}{dx^7} + \&c.$$

116.

116. Quia ſeries $1 + \beta u^2 + \delta u^4 + \zeta u^6 + \theta u^8 + \&c.$ oritur ex euolutione huius fractionis:

$$\frac{1 + \frac{u^2}{2.4} + \frac{u^4}{2.4.6.8} + \frac{u^6}{2.4.6.8.10.12} + \&c.}{1 + \frac{u^2}{4.6} + \frac{u^4}{4.6.8.10} + \frac{u^6}{4.6.8.10.12.14} + \&c.}$$

litterae β, δ, ζ, θ, &c. hanc legem tenebunt, vt ſit:

$$\beta = \frac{1}{2.4} - \frac{1}{4.6}$$

$$\delta = \frac{1}{2.4.6.8} - \frac{\beta}{4.6} - \frac{1}{4.6.8.10}$$

$$\zeta = \frac{1}{2.4.6.12} - \frac{\delta}{4.6} - \frac{\beta}{4.6.8.10} - \frac{1}{4.6\ldots14}$$

$$\theta = \frac{1}{2.4\ldots16} - \frac{\zeta}{4.6} - \frac{\delta}{4.6.8.10} - \frac{\beta}{4.6\ldots14} - \frac{1}{4.6\ldots18}.$$

Hi autem valores alternatiue fiunt affirmatiui & negatiui.

117. Si igitur harum litterarum alternae capiantur negatiue, ita vt ſit:

$$Sz = \int z\,dx + \tfrac{1}{2}z - \frac{\beta dz}{dx} + \frac{\delta d^3z}{dx^3} - \frac{\zeta d^5z}{dx^5} + \frac{\theta d^7z}{dx^7} - \&c.$$

litterae β, δ, ζ, θ, &c. definientur ex hac fractione:

$$\frac{1 - \frac{u^2}{2.4} + \frac{u^4}{2.4.6.8} - \frac{u^6}{2.4\ldots12} + \frac{u^8}{2.4\ldots16} - \&c.}{1 - \frac{u^2}{4.6} + \frac{u^4}{4.6.8.10} - \frac{u^6}{4.6\ldots14} + \frac{u^8}{4.6\ldots18} - \&c.}$$

 eam

eam euoluendo in ſeriem:

$$1 + \beta u^2 + \delta u^4 + \zeta u^6 + \theta u^8 + \&c.$$

quocirca erit:

$$\beta = \frac{1}{4.6} - \frac{1}{2.4}$$

$$\delta = \frac{\beta}{4.6} - \frac{1}{4.6.8.10} + \frac{1}{2.4.6.8}$$

$$\zeta = \frac{\delta}{4.6} - \frac{\beta}{4.6.8.10} + \frac{1}{4.6\ldots14} - \frac{1}{2.4\ldots12}$$

&c.

nunc autem omnes termini fient negatiui.

118. Ponamus ergo $\beta = -A$; $\delta = -B$; $\zeta = -C$; &c. vt ſit:

$$Sz = \int z\,dx + \tfrac{1}{2}z + \frac{A\,dz}{dx} - \frac{B\,d^3z}{dx^3} + \frac{C\,d^5z}{dx^5} - \frac{D\,d^7z}{dx^7} + \&c.$$

atque ad litteras A, B, C, D, &c. definiendas conſideretur haec ſeries:

$$1 - Au^2 - Bu^4 - Cu^6 - Du^8 - Eu^{10} - \&c.$$

quae oritur ex euolutione huius fractionis:

$$\frac{1 - \frac{u^2}{2.4} + \frac{u^4}{2.4.6.8} - \frac{u^6}{2.4\ldots12} + \frac{u^8}{2.4\ldots16} - \&c.}{1 - \frac{u^2}{4.6} + \frac{u^4}{4.6.8.10} - \frac{u^6}{4.6\ldots18} + \frac{u^8}{4.6\ldots18} - \&c.}$$

vel conſideretur iſta ſeries:

$$\frac{1}{u} - Au - Bu^3 - Cu^5 - Du^7 - Eu^9 - \&c. = s$$

quae oritur ex euolutione huius fractionis:

$s =$

$$s = \frac{1 - \frac{u^2}{2.4} + \frac{u^4}{2.4.6.8} - \frac{u^6}{2.4\ldots12} + \&c.}{u - \frac{u^3}{4.6} + \frac{u^5}{4.6.8.10} - \frac{u^7}{4.6\ldots14} + \&c.}$$

Cum autem ſit:

$$\operatorname{cof} \tfrac{1}{2} u = 1 - \frac{u^2}{2.4} + \frac{u^4}{2.4.6.8} - \frac{u^6}{2.4\ldots12} + \&c.$$

$$\operatorname{fin} \tfrac{1}{2} u = \frac{u}{2} - \frac{u^3}{2.4.6} + \frac{u^5}{2.4.6.8.10} - \frac{u^7}{2.4\ldots14} + \&c.$$

ſequitur fore: $s = \frac{\operatorname{cof} \frac{1}{2} u}{2 \operatorname{fin} \frac{1}{2} u} = \frac{1}{2} \cot. \frac{1}{2} u.$

Quare ſi cotangens arcus $\frac{1}{2} u$ in ſeriem conuertatur, cuius termini ſecundum poteſtates ipſius u procedant, ex ea cognoſcentur valores litterarum A, B, C, D, E, &c.

119. Cum igitur ſit $s = \frac{1}{2} \cot \frac{1}{2} u$; erit $\frac{1}{2} u =$ A cot $2s$, & differentiando erit $\frac{1}{2} du = \frac{-2 ds}{1 + 4ss}$ ſeu $4ds + du + 4ss du = 0$, ſiue $\frac{4ds}{du} + 1 + 4ss = 0.$

Quia autem eſt: $s = \frac{1}{u} - Au - Bu^3 - Cu^5 - \&c.$

erit:

$$\frac{4ds}{du} = -\frac{4}{uu} - 4A - 3.4Bu^2 - 5.4Cu^4 - 7.4Du^6 - \&c.$$
$$1 = \quad\quad 1$$
$$4ss = \frac{4}{uu} - 8A - 8Bu^2 - 8Cu^4 - 8Du^6 - \&c.$$
$$+ 4A^2u^2 + 8ABu^4 + 8ACu^6 + \&c.$$
$$+ 4BBu^6 + \&c.$$

per-

perductis his terminis homogeneis ad cyphram fiet:

$$A = \frac{1}{12}$$

$$B = \frac{A^2}{5}$$

$$C = \frac{2AB}{7}$$

$$D = \frac{2AC + BB}{9}$$

$$E = \frac{2AD + 2BC}{11}$$

$$F = \frac{2AE + 2BD + CC}{13}$$

$$G = \frac{2AF + 2BE + 2CD}{15}$$

$$H = \frac{2AG + 2BF + 2CE + DD}{17}$$

&c.

Ex quibus formulis iam manifeſto liquet, ſingulos hos valores eſſe affirmatiuos.

120. Quoniam vero denominatores horum valorum fiunt vehementer magni, calculumque non mediocriter impediunt; loco litterarum A, B, C, D, &c.

has

has nouas introducamus:

$$A = \frac{\alpha}{1.2.3}$$

$$B = \frac{\beta}{1.2.3.4.5}$$

$$C = \frac{\gamma}{1.2.3\ldots.7}$$

$$D = \frac{\delta}{1.2.3\ldots..9}$$

$$E = \frac{\varepsilon}{1.2.3\ldots\ldots.11} \quad \&c.$$

Atque reperietur fore:

$$\alpha = \frac{1}{2}$$

$$\beta = \frac{2}{3}\alpha^2$$

$$\gamma = 2.\frac{3}{3}\alpha\beta$$

$$\delta = 2.\frac{4}{3}\alpha\gamma + \frac{8.7}{4.5}\beta^2$$

$$\varepsilon = 2.\frac{5}{3}\alpha\delta + 2.\frac{10.9.8}{1\ldots.5}\beta\gamma$$

$$\zeta = 2.\frac{12}{1.2.3}\alpha\varepsilon + 2.\frac{12.11.10}{1\ldots..5}\beta\delta + \frac{12.11.10.9.8}{1\ldots\ldots.7}\gamma\gamma.$$

$$\eta = 2.\frac{14}{1.2.3}\alpha\zeta + 2.\frac{14.13.12}{1\ldots..5}\beta\varepsilon + 2.\frac{14.13.12.11.10}{1\ldots\ldots..7}\gamma\delta$$

&c.

121. Commodius autem vtemur his formulis:

$$\alpha = \frac{1}{2}$$

$$\beta = \frac{4}{3} \cdot \frac{\alpha\alpha}{2}$$

$$\gamma = \frac{6}{3} \cdot \alpha\beta$$

$$\delta = \frac{8}{3} . \alpha\gamma + \frac{8.\ 7.\ 6}{3.\ 4.\ 5} . \frac{\beta\beta}{2}$$

$$\varepsilon = \frac{10}{3} . \alpha\delta + \frac{10.\ 9.\ 8}{3.\ 4.\ 5} . \beta\gamma$$

$$\zeta = \frac{12}{3} . \alpha\varepsilon + \frac{12.11.10}{3.\ 4.\ 5} . \beta\delta + \frac{12.11.10.9.8}{3.\ 4.\ 5.\ 6.\ 7} . \frac{\gamma\gamma}{2}$$

$$\eta = \frac{14}{3} . \alpha\zeta + \frac{14.13.12}{3.\ 4.\ 5} . \beta\varepsilon + \frac{14.13.12.11.10}{3.\ 4.\ 5.\ 6.\ 7} . \gamma\delta$$

$$\theta = \frac{16}{3}\ \alpha\eta + \frac{16.15.14}{3.\ 4.\ 5} . \beta\zeta + \frac{16.15\ .\ .\ 12}{3.4.\ .\ .\ .\ 7} . \gamma\varepsilon + \frac{16.15\ .\ .\ 10}{3.4.\ .\ .\ .\ .\ 9} . \frac{\delta\delta}{2}$$

&c.

Ex hac igitur lege, ſecundum quam calculus non difficulter inſtituitur, ſi inuenti fuerint valores litterarum α, β, γ, δ, &c. tum ſeriei cuiuscunque, cuius terminus generalis ſeu indici x conueniens fuerit $= z$, terminus ſummatorius ita exprimetur, vt ſit:

$$Sz = \int z dx + \tfrac{1}{2} z + \frac{\alpha dz}{1.2.3.dx} - \frac{\beta d^3 z}{1.2.3.4.5\, dx^3} + \frac{\gamma d^5 z}{1.2\ldots 7 dx^5}$$

$$- \frac{\delta d^7 z}{1.2\ldots\ldots 9 dx^7} + \frac{\varepsilon d^9 z}{1.2\ldots 11 dx^9} - \frac{\zeta d^{11} z}{1.2\ldots 13 dx^{11}} + \&c.$$

iſtae autem litterae α, β, γ, δ, &c. ſequentes valores habere inuentae ſunt:

ſiue

$\alpha = \frac{1}{2}$	$1.2\,\alpha = 1$
$\beta = \frac{1}{6}$	$1.2.3\,\beta = 1$
$\gamma = \frac{1}{6}$	$1.2.3.4\,\gamma = 4$
$\delta = \frac{3}{10}$	$1.2.3.4.5\,\delta = 36$
$\varepsilon = \frac{5}{6}$	$1.2.3\,.\,.\,6\,\varepsilon = 600$
$\zeta = \frac{691}{210}$	$1.2.3\,.\,.\,7\,\zeta = 24.691$
$\eta = \frac{35}{2}$	$1.2.3\,.\,.\,8\,\eta = 20160.35$
$\theta = \frac{3617}{30}$	$1.2.3\,.\,.\,9\,\theta = 12096.3617$
$\iota = \frac{43867}{42}$	$1.2.3\,.\,.\,10\,\iota = 86400.43867$
$\varkappa = \frac{1222277}{110}$	$1.2.3\,.\,.\,11\,\varkappa = 362880.1222277$
$\lambda = \frac{854513}{6}$	$1.2.3\,.\,.\,12\,\lambda = 79833600.854513$
$\mu = \frac{1181820455}{546}$	$1.2.3\,.\,.\,13\,\mu = 11404800.1181820455$
$\nu = \frac{76977927}{2}$	$1.2.3\,.\,.\,14\,\nu = 109109145600.76977927$
$\xi = \frac{23749461029}{30}$	$1.2.3\,.\,.\,15\,\xi = 43589145600.23749461029$
$\pi = \frac{3615841276005}{462}$	$1.2.3\,.\,.\,16\,\pi = 45287424000.8615841276005$
&c.	122.

122. Numeri iſti per vniuerſam ſerierum doctrinam ampliſſimum habent vſum. Primum enim ex his numeris formari poſſunt vltimi termini in ſummis poteſtatum parium, quos non aeque ac reliquos terminos ex ſummis praecedentium reperiri poſſe ſupra annotauimus. In poteſtatibus enim paribus poſtremi ſummarum termini ſunt x per certos numeros multiplicati; qui numeri pro poteſtatibus II; IV; VI; VIII; &c. ſunt $\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}$, &c. ſignis alternantibus affecti. Oriuntur autem hi numeri ſi valores litterarum α, β, γ, δ, &c. ſupra inuenti reſpectiue diuidantur per numeros impares 3, 5, 7, &c. vnde iſti numeri, qui ab Inuentore *Iacobo Bernoullio* vocari ſolent Bernoulliani erunt:

$$\frac{\alpha}{3} = \frac{1}{6} = \mathfrak{A}$$

$$\frac{\beta}{5} = \frac{1}{30} = \mathfrak{B}$$

$$\frac{\gamma}{7} = \frac{1}{42} = \mathfrak{C}$$

$$\frac{\delta}{9} = \frac{1}{30} = \mathfrak{D}$$

$$\frac{\varepsilon}{11} = \frac{5}{66} = \mathfrak{E}$$

$$\frac{\zeta}{13} = \frac{691}{2730} = \mathfrak{F}$$

$$\frac{\eta}{15} = \frac{7}{6} = \mathfrak{G}$$

$\theta =$

$$\frac{\theta}{17} = \frac{3617}{510} = \mathfrak{H}$$

$$\frac{\iota}{19} = \frac{43867}{798} = \mathfrak{I}$$

$$\frac{\kappa}{21} = \frac{174611}{330} = \mathfrak{K} = \frac{283.617}{330}$$

$$\frac{\lambda}{23} = \frac{854513}{138} = \mathfrak{L} = \frac{11.131.593}{2.3.23}$$

$$\frac{\mu}{25} = \frac{236364091}{2730} = \mathfrak{M} =$$

$$\frac{\nu}{27} = \frac{8553103}{2} = \mathfrak{N} = \frac{13.657931}{6}$$

$$\frac{\xi}{29} = \frac{23749461029}{870} = \mathfrak{O}$$

$$\frac{\pi}{31} = \frac{8615841276005}{14322} = \mathfrak{P}$$

&c.

123. Iſti igitur numeri Bernoulliani $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, &c. immediate ex ſequentibus aequationibus inueniri poterunt:

$$\mathfrak{A} = \frac{1}{6}$$

$$\mathfrak{B} = \frac{4.3}{1.2} \cdot \frac{1}{5} \mathfrak{A}^2$$

$$\mathfrak{C} = \frac{6.5}{1.2} \cdot \frac{2}{7} \mathfrak{A}\mathfrak{B}$$

$$\mathfrak{D} = \frac{8.7}{1.2} \cdot \frac{2}{9} \mathfrak{A}\mathfrak{C} + \frac{8.7.6.5}{1.2.3.4} \cdot \frac{1}{9} \mathfrak{B}^2$$

$$\mathfrak{E} = \frac{10.9}{1.2} \cdot \frac{2}{11} \mathfrak{A}\mathfrak{D} + \frac{10.9.8.7}{1.2.3.4} \cdot \frac{2}{11} \mathfrak{B}\mathfrak{C}$$

 $\mathfrak{F} =$

$$\mathfrak{F} = \frac{12.11}{1.2}\cdot\frac{2}{13}\mathfrak{A}\mathfrak{E} + \frac{12.11.10.9}{1.2.3.4}\cdot\frac{2}{13}\mathfrak{B}\mathfrak{D} + \frac{12.11.10.9.8.7}{1.2.3.4.5.6}\cdot\frac{1}{13}\mathfrak{C}^2$$

$$\mathfrak{G} = \frac{14.13}{1.2}\cdot\frac{2}{15}\mathfrak{A}\mathfrak{F} + \frac{14.13.12.11}{1.2.3.4}\cdot\frac{2}{15}\mathfrak{B}\mathfrak{E} + \frac{14.13.12.11.10.9}{1.2.3.4.5.6}\cdot\frac{2}{15}\mathfrak{C}\mathfrak{D}$$

&c.

quarum aequationum lex per ſe eſt manifeſta, ſi tantum notetur, vbi quadratum cuiuſpiam litterae occurrit, eius coefficientem duplo eſſe minorem, quam ſecundum regulam eſſe debere videatur. Reuera autem termini, qui continent producta ex diſparibus litteris, bis occurrere cenſendi ſunt, erit enim verbi gratia:

$$13\mathfrak{F} = \frac{12.11}{1.2}\mathfrak{A}\mathfrak{E} + \frac{12.11.10.9}{1.2.3.4}\mathfrak{B}\mathfrak{D} + \frac{12.11.10.9.8.7}{1.2.3.4.5.6}\mathfrak{C}\mathfrak{C} +$$
$$\frac{12.11.10 \ldots 5}{1.2.3 \ldots 8}\mathfrak{D}\mathfrak{B} + \frac{12.11.10 \ldots 11}{1.2.3 \ldots 3}\mathfrak{E}\mathfrak{A}$$

124. Deinde vero etiam iidem numeri α, β, γ, δ, &c. ingrediuntur in expreſſiones ſummarum ſerierum fractionum in hac forma generali:

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \&c.$$

quoties n eſt numerus par affirmatiuus, contentarum. Has enim ſummas in Introductione per poteſtates ſemiperipheriae circuli π radio exiſtente $= 1$ expreſſas dedimus, atque in harum poteſtatum coefficientibus iſti ipſi numeri α, β, γ, δ, &c. ingredi deprehenduntur. Quo autem haec conuenientia non caſu euenire, ſed neceſſario locum habere intelligatur, has easdem ſummas ſingu-

gulari modo inueſtigemus, quo lex ſummarum illarum facilius patebit. Quoniam ſupra inuenimus eſſe:

$$\frac{\pi}{n}\cot.\frac{m}{n}\pi = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \&c.$$

binis terminis coniungendis habebimus:

$$\frac{\pi}{n}\cot.\frac{m}{n}\pi = \frac{1}{m} - \frac{2m}{nn-m^2} - \frac{2m}{4n^2-m^2} - \frac{2m}{9n^2-m^2} - \frac{2m}{16n^2-m^2} - \&c.$$

vnde colligimus fore:

$$\frac{1}{n^2-m^2} + \frac{1}{4n^2-m^2} + \frac{1}{9n^2-m^2} + \frac{1}{16n^2-m^2} + \&c. = \frac{1}{2mm} - \frac{\pi}{2mn}\cot.\frac{m}{n}\pi$$

Statuamus nunc $n=1$, & pro m ponamus u; vt ſit:

$$\frac{1}{1-u^2} + \frac{1}{4-u^2} + \frac{1}{9-u^2} + \frac{1}{16-u^2} + \&c. = \frac{1}{2uu} - \frac{\pi}{2u}\cot.\pi u.$$

Reſoluantur ſingulae iſtae fractiones in ſeries:

$$\frac{1}{1-u^2} = 1 + u^2 + u^4 + u^6 + u^8 + \&c.$$

$$\frac{1}{4-u^2} = \frac{1}{2^2} + \frac{u^2}{2^4} + \frac{u^4}{2^6} + \frac{u^6}{2^8} + \frac{u^8}{2^{10}} + \&c.$$

$$\frac{1}{9-u^2} = \frac{1}{3^2} + \frac{u^2}{3^4} + \frac{u^4}{3^6} + \frac{u^6}{3^8} + \frac{u^8}{3^{10}} + \&c.$$

$$\frac{1}{16-u^2} = \frac{1}{4^2} + \frac{u^2}{4^4} + \frac{u^4}{4^6} + \frac{u^6}{4^8} + \frac{u^8}{4^{10}} + \&c.$$

&c.

125. Quod ſi ergo ponatur:

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c. = \mathfrak{a}$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \&c. = \mathfrak{b}$$

1 +

$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \&c. = \mathfrak{c}$$

$$1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \&c. = \mathfrak{d}$$

$$1 + \frac{1}{2^{10}} + \frac{1}{3^{10}} + \frac{1}{4^{10}} + \&c. = \mathfrak{e}$$

&c.

ſuperior ſeries transmutabitur in hanc:

$$\mathfrak{a} + \mathfrak{b}u^2 + \mathfrak{c}u^4 + \mathfrak{d}u^6 + \mathfrak{e}u^8 + \mathfrak{f}u^{10} + \&c. = \frac{1}{2uu} - \frac{\pi}{2u}\cot.\pi u.$$

Cum igitur in §. 118. litterae A, B, C, D, &c. ita comparatae ſint inuentae, vt poſito:

$$s = \frac{1}{u} - Au - Bu^3 - Cu^5 - Du^7 - Eu^9 - \&c.$$

ſit $s = \frac{1}{2}\cot.\frac{1}{2}u$, erit poſito πu loco $\frac{1}{2}u$ ſeu $2\pi u$ loco u

$$\tfrac{1}{2}\cot\pi u = \frac{1}{2\pi u} - A\pi u - 2^3 B\pi^3 u^3 - 2^5 C\pi^5 u^5 - 2^2 D\pi^7 u^7 - \&c.$$

vnde per $\frac{\pi}{u}$ multiplicando erit:

$$\frac{\pi}{2u}\cot.\pi u = \frac{1}{2uu} - 2A\pi^2 - 2^3 B\pi^4 u^2 - 2^5 C\pi^6 u^4 - \&c.$$

hincque ſequitur fore:

$$\frac{1}{2uu} - \frac{\pi}{2u}\cot.\pi u = 2A\pi^2 + 2^3 B\pi^4 u^2 + 2^5 C\pi^6 u^4 + 2^7 D\pi^8 u^6 + \&c.$$

Quia igitur modo inuenimus eſſe:

$$\frac{1}{2uu} - \frac{\pi}{2u}\cot.\pi u = \mathfrak{a} + \mathfrak{b}u^2 + \mathfrak{c}u^4 + \mathfrak{d}u^6 + \&c.$$

neceſſe eſt vt ſit:

$\mathfrak{a} =$

$$\mathfrak{a} = 2A\pi^2 = \frac{2\alpha}{1.2.3}.\pi^2 = \frac{2\mathfrak{A}}{1.2}.\pi^2$$

$$\mathfrak{b} = 2^3 B\pi^4 = \frac{2^3 \beta}{1.2.3.4.5}.\pi^4 = \frac{2^3 \mathfrak{B}}{1.2.3.4}.\pi^4$$

$$\mathfrak{c} = 2^5 C\pi^6 = \frac{2^5 \gamma}{1.2.3....7}\pi^6 = \frac{2^5 \mathfrak{C}}{1.2.....6}\pi^6$$

$$\mathfrak{d} = 2^7 D\pi^8 = \frac{2^7 \delta}{1.2.3....9}\pi^8 = \frac{2^7 \mathfrak{D}}{1.2......8}\pi^8$$

$$\mathfrak{e} = 2^9 E\pi^{10} = \frac{2^9 \varepsilon}{1.2.3...11}\pi^{10} = \frac{2^9 \mathfrak{E}}{1.2...,10}\pi^{10}$$

$$\mathfrak{f} = 2^{11} F\pi^{12} = \frac{2^{11} \zeta}{1.2.3...13}\pi^{12} = \frac{2^{11} \mathfrak{F}}{1.2....12}\pi^{12}$$

&c.

126. Ex hoc ergo tam facili ratiocinio non ſolum omnes ſeries poteſtatum reciprocarum, quas §. praeced. exhibuimus, expedite ſummantur; ſed ſimul quoque perſpicitur, quemadmodum iſtae ſummae ex cognitis valoribus litterarum α, β, γ, δ, ε, &c. vel etiam ex numeris Bernoullianis $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. formentur. Quare cum iſtorum numerorum quindecim §. 122. definiuerimus, ex iis ſummae omnium poteſtatum parium vsque ad ſummam huius ſeriei incluſiue aſſignari poterunt:

$1 + \frac{1}{2^{30}} + \frac{1}{3^{30}} + \frac{1}{4^{30}} + \frac{1}{5^{30}} +$ &c. erit enim huius ſeriei ſumma $= \frac{2^{29}\pi}{1.2.3...31}\pi^{30} = \frac{2^{29}\mathfrak{P}}{1.2.....30}\pi^{30}$.

Atque ſi quis voluerit has ſummas vlterius determinare, id continuandis numeris α, β, γ, &c. vel his $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, &c. facillime praeſtabitur.

127. Origo ergo horum numerorum α, β, γ, δ, &c. vel inde formatorum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. potiſſimum debetur euolutioni cotangentis cuiusuis anguli in ſeriem infinitam. Cum enim ſit

$$\tfrac{1}{2}\cot.\tfrac{1}{2}u = \frac{1}{u} - Au - Bu^3 - Cu^5 - Du^7 - Eu^9 - \&c.$$

erit:

$$Au^2 + Bu^4 + Cu^6 + Du^8 + \&c. = 1 - \frac{u}{2}\cot.\tfrac{1}{2}u,$$

ſi igitur loco coefficientium A, B, C, D, &c. valores ipſorum ſubſtituantur, reperietur:

$$\frac{\alpha u^2}{1.2.3} + \frac{\beta u^4}{1.2\ldots5} + \frac{\gamma u^6}{1.2\ldots7} + \frac{\delta u^8}{1.2\ldots9} + \&c. = 1 - \frac{u}{2}\cot\tfrac{1}{2}u$$

atque numeros Bernoullianos adhibendo erit:

$$\frac{\mathfrak{A}u^2}{1.2} + \frac{\mathfrak{B}u^4}{1.2.3.4} + \frac{\mathfrak{C}u^6}{1.2\ldots6} + \frac{\mathfrak{D}u^8}{1.2\ldots8} + \&c. = 1 - \frac{u}{2}\cot\tfrac{1}{2}u$$

ex quibus ſeriebus per differentiationem innumerabiles aliae deduci poſſunt, ſicque infinitae ſeries ſummari, in quas iſti numeri notatu tantopere digni ingrediuntur.

128. Sumamus aequationem priorem, quam per u multiplicemus, vt ſit:

$$\frac{\alpha u^3}{1.2.3} + \frac{\beta u^5}{1.2\ldots5} + \frac{\gamma u^7}{1.2\ldots7} + \frac{\delta u^9}{1.2\ldots9} + \&c. = u - \frac{uu}{2}\cot\tfrac{1}{2}u$$

quae

quae differentiata ac per du diuiſa dat:

$$\frac{\alpha u^2}{1.2}+\frac{\beta u^4}{1.2.3.4}+\frac{\gamma u^6}{1.2\ldots6}+\frac{\delta u^8}{1.2\ldots8}+\&c.=1-u\cot\tfrac{1}{2}u+\frac{uu}{4(\sin\frac{1}{2}u)^2}.$$

&, ſi denuo differentietur erit:

$$\frac{\alpha u}{1}+\frac{\beta u^3}{1.2.3}+\frac{\gamma u^5}{1.2.3.4.5}+\&c.=-\cot\tfrac{1}{2}u+\frac{u}{(\sin\frac{1}{2}u)^2}-\frac{uu\cos\frac{1}{2}u}{4(\sin\frac{1}{2}u)^3}$$

Sin autem altera aequatio differentietur erit:

$$\frac{\mathfrak{A}u}{1}+\frac{\mathfrak{B}u^3}{1.2.3}+\frac{\mathfrak{C}u^5}{1.2\ldots5}+\frac{\mathfrak{D}u^7}{1.2\ldots7}=-\tfrac{1}{2}\cot\tfrac{1}{2}u+\frac{u}{4(\sin\frac{1}{2}u)^2}$$

Ex his ergo ſi ponatur $u=\pi$, ob $\cot\frac{1}{2}\pi=0$, & $\sin\frac{1}{2}u=1$, ſequuntur iſtae ſummationes:

$$1=\frac{\alpha\pi^2}{1.2.3}+\frac{\beta\pi^4}{1.2.3.4.5}+\frac{\gamma\pi^6}{1.2.3\ldots7}+\frac{\delta\pi^8}{1.2.3\ldots9}+\&c.$$

$$1+\frac{\pi^2}{4}=\frac{\alpha\pi^2}{1.2}+\frac{\beta\pi^4}{1.2.3.4}+\frac{\gamma\pi^6}{1.2.3\ldots6}+\frac{\delta\pi^8}{1.2.3\ldots8}+\&c.$$

$$\pi=\frac{\alpha\pi}{1}+\frac{\beta\pi^3}{1.2.3}+\frac{\gamma\pi^5}{1.2.3.4.5}+\frac{\delta\pi^7}{1.2.3\ldots7}\ \&c.$$

ſeu $$1=\alpha+\frac{\beta\pi^2}{1.2.3}+\frac{\gamma\pi^4}{1.2.3.4.5}+\frac{\delta\pi^6}{1.2.3\ldots7}+\&c.$$

a qua ſi prima ſubtrahatur remanebit:

$$\alpha=\frac{(\alpha-\beta)\pi^2}{1.2.3}+\frac{(\beta-\gamma)\pi^4}{1.2.3.4.5}+\frac{(\gamma-\delta)\pi^6}{1.2.3\ldots7}+\&c.$$

 Tum

Tum vero erit:

$$1 = \frac{\mathfrak{A}\pi^2}{1.2} + \frac{\mathfrak{B}\pi^4}{1.2.3.4} + \frac{\mathfrak{C}\pi^6}{1.2.3\ldots6} + \frac{\mathfrak{D}\pi^8}{1.2.3\ldots.8} + \&c.$$

$$\frac{\pi}{4} = \frac{\mathfrak{A}\pi}{1} + \frac{\mathfrak{B}\pi^3}{1.2.3} + \frac{\mathfrak{C}\pi^5}{1.2.3.4.5} + \frac{\mathfrak{D}\pi^7}{1.2.3\ldots.7} + \&c.$$

$$\text{ſeu } \frac{1}{4} = \frac{\mathfrak{A}}{1} + \frac{\mathfrak{B}\pi^2}{1.2.3} + \frac{\mathfrak{C}\pi^4}{1.2.3.4.5} + \frac{\mathfrak{D}\pi^6}{1.2.3\ldots.7} + \&c.$$

129. Ex tabula valorum numerorum $\alpha, \beta, \gamma, \delta$, &c. quam ſupra §. 121. exhibuimus, patet eos primum decreſcere tum vero iterum creſcere, & quidem in infinitum. Operae igitur pretium erit inueſtigare, in quanam ratione hi numeri, poſtquam iam vehementer longe fuerint continuati, vlterius progredi pergant. Sit igitur ϕ numerus quicunque huius ſeriei numerorum $\alpha, \beta, \gamma, \delta$, &c. longiſſime ab initio remotus, & ſit ψ numerorum ſequens. Quoniam per hos numeros ſummae poteſtatum reciprocarum definiuntur, ſit $2n$ exponens poteſtatis, in cuius ſumma numerus ϕ ingreditur, erit $2n+2$ exponens poteſtatis numero ψ reſpondens, atque numerus n iam erit vehementer magnus. Hinc ex §. 125. habebitur:

$$1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{n2}} + \&c. = \frac{2^{2n-1}\phi}{1.2.3\ldots(2n+1)}\pi^{2n}$$

$$1 + \frac{1}{2^{2n+2}} + \frac{1}{3^{2n+2}} + \frac{1}{4^{2n+2}} + \&c. = \frac{2^{2n+1}\psi}{1.2.3\ldots(2n+3)}\pi^{2n+2}$$

Quod

Quod ſi ergo haec per iſtam diuidatur, erit:

$$\frac{1 + \frac{1}{2^{2n+2}} + \frac{1}{3^{2n+2}} + \&c.}{1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \&c.} = \frac{4\psi}{(2n+2)(2n+3)} \frac{\pi^2}{\phi}$$

Quia vero n eſt numerus vehementer magnus, ob ſeriem vtramque proxime $=1$, erit:

$$\frac{\psi}{\phi} = \frac{(2n+2)(2n+3)}{4\pi^2} = \frac{nn}{\pi\pi}.$$

Cum igitur n deſignet, quotus ſit numerus ϕ a primo α computatus, ſe habebit hic numerus ϕ ad ſuum ſequentem ψ vt π^2 ad n^2, quae ratio, ſi n fuerit numerus infinitus, veritati penitus ſit conſentanea. Quoniam eſt fere $\pi\pi = 10$, ſi ponatur $n = 100$; erit terminus centeſimus circiter millies minor ſuo ſequente. Conſtituunt ergo numeri α, β, γ, δ, &c. pariter ac Bernoulliani 𝔄, 𝔅, ℭ, 𝔇, &c. ſeriem maxime diuergentem, quae etiam magis increſcat, quam vlla ſeries geometrica terminis creſcentibus procedens.

130. Inuentis ergo his valoribus numerorum α, β, γ, δ, &c. ſeu 𝔄, 𝔅, ℭ, 𝔇, &c. ſi proponatur ſeries, cuius terminus generalis z fuerit functio quaecunque ipſius indicis x, terminus ſummatorius Sz huius ſeriei ſequenti modo exprimetur, vt ſit:

 $Sz =$

$$
\begin{aligned}
Sz = & \int z\,dx + \tfrac{1}{2}z + \frac{1}{6}\cdot\frac{dz}{1.2\,dx} - \frac{1}{30}\cdot\frac{d^3z}{1.2.3.4\,dx^3} \\
& + \frac{1}{42}\cdot\frac{d^5z}{1.2.3\ldots5\,dx^5} - \frac{1}{30}\cdot\frac{d^7z}{1.2.3\ldots8\,dx^7} \\
& + \frac{5}{66}\cdot\frac{d^9z}{1.2.3\ldots10\,dx^9} - \frac{691}{2730}\cdot\frac{d^{11}z}{1.2.3\ldots12\,dx^{11}} \\
& + \frac{7}{6}\cdot\frac{d^{13}z}{1.2.3\ldots14\,dx^{13}} - \frac{3617}{510}\cdot\frac{d^{15}z}{1.2.3\ldots16\,dx^{15}} \\
& + \frac{43867}{798}\cdot\frac{d^{17}z}{1.2.3\ldots18\,dx^{17}} - \frac{174611}{330}\cdot\frac{d^{19}z}{1.2.3\ldots20\,dx^{19}} \\
& + \frac{854513}{138}\cdot\frac{d^{21}z}{1.2.3\ldots22\,dx^{21}} - \frac{236364091}{2730}\cdot\frac{d^{23}z}{1.2.3\ldots24\,dx^{23}} \\
& + \frac{8553103}{6}\cdot\frac{d^{25}z}{1.2.3\ldots26\,dx^{25}} - \frac{23749461029}{870}\cdot\frac{d^{27}z}{1.2.3\ldots28\,dx^{27}} \\
& + \frac{8615841276005}{14322}\cdot\frac{d^{29}z}{1.2.3\ldots30\,dx^{29}} - \&c.
\end{aligned}
$$

Si igitur innotuerit integrale $\int z\,dx$, ſeu quantitas illa cuius differentiale ſit $= z\,dx$, terminus ſummatorius ope continuae differentiationis inuenietur. Perpetuo autem notandum eſt ad hanc expreſſionem ſemper eiusmodi conſtantem addi oportere, vt ſumma fiat $= 0$, ſi index x ponatur in nihilum abire.

131. Si igitur z fuerit functio rationalis integra ipſius x, quia eius differentialia tandem euaneſcunt, terminus ſummatorius per expreſſionem finitam exprimetur; id quod ſequentibus exemplis illuſtrabimus.

EX-

EXEMPLUM I.

Quaeratur terminus ſummatorius huius ſeriei:

$$\overset{1}{1} + \overset{2}{9} + \overset{3}{25} + \overset{4}{49} + \overset{5}{81} + \ldots\ldots + \overset{x}{(2x-1)^2}$$

Quia hic eſt $z = (2x-1)^2 = 4xx - 4x + 1$;

erit $\int z\,dx = \frac{4}{3}x^3 - 2x^2 + x$,

ex huius enim differentiatione oritur:

$$4xx\,dx - 4x\,dx + dx = z\,dx.$$

Deinde vero per differentiationem erit:

$$\frac{dz}{dx} = 8x - 4$$

$$\frac{ddz}{dx^2} = 8$$

$$\frac{d^3z}{dx^3} = 0 \quad \&c.$$

Hinc erit terminus ſummatorius quaeſitus: $=$ $\frac{4}{3}x^3 - 2x^2 + x + 2xx - 2x + \frac{1}{2} + \frac{2}{3}x - \frac{1}{3} \pm$ Conſt. qua conſtante tolli debent termini $\frac{1}{2} - \frac{1}{3}$; vnde erit:

$$S(2x-1)^2 = \frac{4}{3}x^3 - \frac{1}{3}x = \frac{x}{3}(2x-1)(2x+1).$$

Sic erit poſito $x = 4$ ſumma 4 primorum terminorum

$$1 + 9 + 25 + 49 = \frac{4}{3}\cdot 7\cdot 9 = 84.$$

EXEMPLUM II.

Quaeratur terminus ſummatorius huius ſeriei:

$$\overset{1}{1} + \overset{2}{27} + \overset{3}{125} + \overset{4}{343} + \ldots\ldots \overset{x}{(2x-1)^3}$$

Quia

Quia eſt $z = (2x-1)^3 = 8x^3 - 12x^2 + 6x - 1$; erit:
$\int z\,dx = 2x^4 - 4x^3 + 3x^2 - x$; $\frac{dz}{dx} = 24x^2 - 24x + 6$;
$\frac{ddz}{dx^2} = 48x - 24$; $\frac{d^3z}{dx^3} = 48$; ſequentia euaneſcunt.
Quare erit

$$S(2x-1)^3 = 2x^4 - 4x^3 + 3x^2 - x \\ + 4x^3 - 6x^2 + 3x - \tfrac{1}{2} \\ + 2x^2 - 2x + \tfrac{1}{2} \\ - \tfrac{1}{15}$$

hoc eſt $S(2x-1)^3 = 2x^4 - x^2 = x^2(2xx-1)$. Sic erit poſito $x = 4$ $1 + 27 + 125 + 343 = 16.31 = 496$.

132. Ex hac inuenta generali expreſſione pro termino ſummatorio ſponte ſequitur ille terminus ſummatorius, quem ſuperiori parte pro poteſtatibus numerorum naturalium dedimus, cuiusque demonſtrationem ibi tradere non licuerat. Quod ſi enim ponamus $z = x^n$, erit vtique $\int z\,dx = \frac{1}{n+1} x^{n+1}$; differentialia vero ita ſe habebunt:

$$\frac{dz}{dx} = nx^{n-1}$$

$$\frac{ddz}{dx^2} = n(n-1)x^{n-2}$$

$$\frac{d^3z}{dx^3} = n(n-1)(n-2)x^{n-3}$$

$$\frac{d^5z}{dx^5} = n(n-1)(n-2)(n-3)(n-4)x^{n-5}$$

$$\frac{d^7z}{dx^7} = n(n-1)\ \ldots\ldots\ (n-6)x^{n-7} \quad \&c.$$

Ex

Ex his ergo deducetur ſequens terminus ſummatorius reſpondens termino generali x^n; ſcilicet

$$Sx^n = \frac{1}{n+1}x^{n+1} + \frac{1}{2}x^n + \frac{1}{6}.\frac{n}{2}x^{n-1}$$

$$- \frac{1}{30}.\frac{n(n-1)(n-2)}{2.\ 3.\ 4}x^{n-3}$$

$$+ \frac{1}{42}.\frac{n(n-1)(n-2)(n-3)(n-4)}{2.\ 3.\ 4.\ 5.\ 6}x^{n-5}$$

$$- \frac{1}{30}.\frac{n(n-1)\ .\ .\ .\ .\ (n-6)}{2.\ 3.\ .\ .\ .\ .\ .\ .\ .\ 8}x^{n-7}$$

$$+ \frac{5}{66}.\frac{n(n-1)\ .\ .\ .\ .\ (n-8)}{2.\ 3.\ .\ .\ .\ .\ .\ .\ .\ 10}x^{n-9}$$

$$- \frac{691}{2730}.\frac{n(n-1)\ .\ .\ .\ .\ .\ (n-10)}{2.\ 3\ .\ .\ .\ .\ .\ .\ .\ 12}x^{n-11}$$

$$+ \frac{7}{6}.\frac{n(n-1)\ .\ .\ .\ .\ .\ (n-12)}{2.\ 3\ .\ .\ .\ .\ .\ .\ .\ 14}x^{n-13}$$

$$- \frac{3617}{510}.\frac{n(n-1)\ .\ .\ .\ .\ .\ (n-14)}{2.\ 3\ .\ .\ .\ .\ .\ .\ .\ 16}x^{n-15}$$

$$+ \frac{43867}{798}.\frac{n(n-1)\ .\ .\ .\ .\ .\ (n-16)}{2.\ 3\ .\ .\ .\ .\ .\ .\ .\ 18}x^{n-17}$$

$$- \frac{174611}{330}.\frac{n(n-1)\ .\ .\ .\ .\ .\ (n-18)}{2.\ 3\ .\ .\ .\ .\ .\ .\ .\ 20}x^{n-19}$$

$$+ \frac{854513}{138}.\frac{n(n-1)\ .\ .\ .\ .\ .\ (n-20)}{2.\ 3\ .\ .\ .\ .\ .\ .\ .\ 22}x^{n-21}$$

$$- \frac{236364091}{2730}.\frac{n(n-1)\ .\ .\ .\ .\ .\ (n-22)}{2.\ 3\ .\ .\ .\ .\ .\ .\ .\ 24}x^{n-23}$$

$$+ \frac{8553103}{6}.\frac{n(n-1)\ .\ .\ .\ .\ .\ (n-24)}{2.\ 3\ .\ .\ .\ .\ .\ .\ .\ 26}x^{n-25}$$

$$-\frac{23749461029}{870}\cdot\frac{n(n-1)\;.\;.\;.\;.\;.\;(n-26)}{2.\;3\;.\;.\;.\;.\;.\;.\;.\;28}x^{n-27}$$

$$+\frac{8615841276005}{14322}\cdot\frac{n(n-1)\;.\;.\;.\;.\;.\;(n-28)}{2.\;3\;.\;.\;.\;.\;.\;.\;.\;30}x^{n-29}\ \&c.$$

quae expreſſio non differt ab ea, quam ſupra dedimus, niſi quod hic numeros Bernoullianos 𝔄, 𝔅, ℭ, &c. introduximus, cum ſupra vſi eſſemus numeris α, β, γ, δ, &c. interim tamen conſenſus ſponte elucet. Hinc ergo terminos ſummatorios omnium poteſtatum vsque ad poteſtatem trigeſimam incluſiue exhibere licuit; quae inveſtigatio, ſi alia via fuiſſet ſuſcepta, ſine longiſſimis & taedioſiſſimis calculis abſolui non potuiſſet.

133. Iam ſupra §. 59. ſimilem fere expreſſionem pro termino ſummatorio dedimus ex termino generali definiendo. Ea enim pariter ſecundum differentialia termini generalis procedebat; ab iſta autem in hoc potiſſimum erat diuerſa, quod illa non integrale $\int z\,dx$ requirebat, ſingula vero termini generalis differentialia per certas ipſius x functiones habebat multiplicata. Eandem igitur expreſſionem ſequenti modo ad naturam ſerierum magis accommodato denuo eliciamus, ex quo ſimul lex clarius patebit, ſecundum quam coefficientes illi differentialium progrediuntur. Sit igitur ſeriei terminus generalis z, functio quaecunque ipſius indicis x, terminus vero ſummatorius quaeſitus ſit s: qui quoniam vti vidimus eiusmodi erit functio ipſius x, vt euaneſcat poſito $x = 0$, erit per ea, quae ſupra de natura huiusmodi functionum demonſtrauimus:

s —

$$s - \frac{x\,ds}{1\,dx} + \frac{x^2\,dds}{1.2\,dx^2} - \frac{x^3\,d^3s}{1.2.3\,dx^3} + \frac{x^4\,d^4s}{1.2.3.4\,dx^4} - \&c. = 0.$$

134. Quia s denotat ſummam omnium terminorum ſeriei a primo vsque ad vltimum z, perſpicuum eſt ſi in s loco x ponatur $x-1$, tum priorem ſummam vltimo termino z mulctari: erit ſcilicet

$$s - z = s - \frac{ds}{dx} + \frac{dds}{2\,dx^2} - \frac{d^3s}{6\,dx^3} + \frac{d^4s}{24\,dx^4} - \&c.$$

ideoque $z = \frac{ds}{dx} - \frac{dds}{2\,dx^2} + \frac{d^3s}{6\,dx^3} - \frac{d^4s}{24\,dx^4} + \&c.$

quae aequatio modum ſuppeditat ex dato termino ſummatorio s definiendi terminum generalem, quod quidem per ſe eſt facillimum. Ex idonea autem combinatione huius aequationis cum ea, quam §. praeced. inuenimus, valor ipſius s per x & z determinari poterit. Ponamus in hunc finem eſſe:

$$s - Az + \frac{B\,dz}{dx} - \frac{C\,ddz}{dx^2} + \frac{D\,d^3z}{dx^3} - \frac{E\,d^4z}{dx^4} + \&c. = 0$$

vbi A, B, C, D, &c. denotent coefficientes neceſſarios ſiue conſtantes ſiue variabiles: nam cum ſit

$$z = \frac{ds}{dx} - \frac{dds}{2\,dx^2} + \frac{d^3s}{6\,dx^3} - \frac{d^4s}{24\,dx^4} + \frac{d^5s}{120\,dx^5} - \&c.$$

ſi hinc valores pro z, $\frac{dz}{dx}$, $\frac{ddz}{dx^2}$, $\frac{d^3z}{dx^3}$, &c. in ſuperiori aequatione ſubſtituantur, prodibit:

 $s = s$

$$s = s$$
$$-Az = -\frac{Ads}{dx} + \frac{Adds}{2dx^2} - \frac{Ad^3s}{6dx^3} + \frac{Ad^4s}{24dx^4} - \frac{Ad^5s}{120dx^5} + \&c.$$
$$+\frac{Bdz}{dx} = +\frac{Bdds}{dx^2} - \frac{Bd^3s}{2dx^3} + \frac{Bd^4s}{6dx^4} - \frac{Bd^5s}{24dx^5} + \&c.$$
$$-\frac{Cddz}{dx^2} = -\frac{Cd^3s}{dx^3} + \frac{Cd^4s}{4dx^4} - \frac{Cd^5s}{6dx^5} + \&c.$$
$$+\frac{Dd^3z}{dx^3} = +\frac{Dd^4s}{dx^4} - \frac{Dd^5s}{2dx^5} + \&c.$$
$$-\frac{Ed^4z}{dx^4} = -\frac{Ed^5s}{dx^5} + \&c.$$
&c.

quae igitur ſeries iunctim ſumtae aequales erunt nihilo.

135. Cum ergo ante inuenimus eſſe:

$$0 = s - \frac{xds}{dx} + \frac{x^2dds}{2dx^2} - \frac{x^3d^3s}{6dx^3} + \frac{x^4d^4s}{24dx^4} - \frac{x^5d^5s}{120dx^5} + \&c.$$

ſi ſuperior aequatio huic aequalis ſtatuatur, prodibunt ſequentes litterarum A, B, C, D, &c. denominationes:

$$A = x$$
$$B = \frac{x^2}{2} - \frac{A}{2}$$
$$C = \frac{x^3}{6} - \frac{B}{2} - \frac{A}{6}$$
$$D = \frac{x^4}{24} - \frac{C}{2} - \frac{B}{6} - \frac{A}{24}$$
$$E = \frac{x^5}{120} - \frac{D}{2} - \frac{C}{6} - \frac{B}{24} - \frac{A}{120} \quad \&c.$$

His

His igitur litterarum A, B, C, D, &c. valoribus inuentis, ex termino generali z terminus ſummatorius $s = Sz$ ita determinabitur, vt ſit:

$$Sz = Az - \frac{Bdz}{dx} + \frac{Cddz}{dx^2} - \frac{Dd^3z}{dx^3} + \frac{Ed^4z}{dx^4} - \frac{Fd^5z}{dx^5} + \&c.$$

136. Cum autem fiat:

$$A = x$$
$$B = \tfrac{1}{2}x^2 - \tfrac{1}{2}x$$
$$C = \tfrac{1}{6}x^3 - \tfrac{1}{4}x^2 + \tfrac{1}{12}x$$
$$D = \tfrac{1}{24}x^4 - \tfrac{1}{12}x^3 + \tfrac{1}{24}xx \qquad \&c.$$

patet hos coefficientes eſſe eosdem, quos ſupra §. 59. habuimus, vnde iſta termini ſummatorii expreſſio eadem eſt, quam ibi inuenimus; eritque propterea:

$$A = Sx^0 = S1$$
$$B = \tfrac{1}{1}Sx^1 - \tfrac{1}{1}x$$
$$C = \tfrac{1}{2}Sx^2 - \tfrac{1}{2}x^2$$
$$D = \tfrac{1}{6}Sx^3 - \tfrac{1}{6}x^3$$
$$E = \tfrac{1}{24}Sx^4 - \tfrac{1}{24}x^4 \qquad \&c.$$

Hinc ergo erit:

$$Sz = xz - \frac{dz}{dx}Sx + \frac{ddz}{2dx^2}Sx^2 - \frac{d^3z}{6dx^3}Sx^3 + \frac{d^4z}{24dx^4}Sx^4 - \&c.$$
$$+ \frac{xdz}{dx} - \frac{x^2ddz}{2dx^2} + \frac{x^3d^3z}{6dx^3} - \frac{x^4d^4z}{24dx^4} + \&c.$$

Quodſi autem in termino generali z ponatur $x = 0$, prodibit terminus indici $= 0$ reſpondens; qui ſi ponatur $= a$,

erit: $a = z - \frac{xdz}{dx} + \frac{x^2 ddz}{2dx^2} - \frac{x^3 d^3 z}{6dx^3} + \&c.$ ideoque

$$\frac{xdz}{dx} - \frac{x^2 ddz}{2dx^3} + \frac{x^3 d^3 z}{6dx^3} - \frac{x^4 d^4 z}{24dx^4} + \&c. = z - a,$$

quo valore ſubſtituto habebitur:

$$Sz = (x+1)z - a - \frac{dz}{dx}Sx + \frac{ddz}{2dx^2}Sx^2 - \frac{d^3 z}{6dx^3}Sx^3 + \frac{d^4 z}{24dx^4}Sx^4 - \&c.$$

Cognitis ergo ſummis poteſtatum, hinc pro quouis termino generali ei conueniens terminus ſummatorius exhiberi poteſt.

137. Quoniam ergo geminam inuenimus expreſſionem termini ſummatorii Sz pro termino generali z, earumque altera formulam integralem $\int z dx$ continet, ſi iſtae duae expreſſiones ſibi aequales ponantur, obtinebitur valor ipſius $\int z dx$ per ſeriem expreſſus. Cum enim ſit:

$$\int z dx + \tfrac{1}{2} z + \frac{\mathfrak{A} dz}{1.2\,dx} - \frac{\mathfrak{B} d^3 z}{1.2.3.4\,dx^3} + \frac{\mathfrak{C} d^5 z}{1.2\ldots 6dx^5} - \&c.$$

$$= (x+1)z - a - \frac{dz}{1\,dx}Sx + \frac{ddz}{1.2dx^2}Sx^2 - \frac{d^3 z}{1.2.3\,dx^3}Sx^3 + \&c.$$

erit:

$$\int z dx = (x+\tfrac{1}{2})z - a - \frac{dz}{dx}(Sx + \tfrac{1}{2}\mathfrak{A}) + \frac{ddz}{2dx^2}Sx^2 - \frac{d^3 z}{6dx^3}(Sx^3 - \tfrac{1}{4}\mathfrak{B})$$

$$+ \frac{d^4 z}{24dx^4}Sx^4 - \frac{d^5 z}{120dx^5}(Sx^5 + \tfrac{1}{6}\mathfrak{C}) + \frac{d^6 z}{720dx^6}Sx^6$$

$$- \frac{d^7 z}{5040dx^7}(Sx^7 - \tfrac{1}{8}\mathfrak{D}) + \&c.$$

vbi $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. denotant numeros Bernoullianos ſupra §. 122. exhibitos.

Sit

Sit v.gr. $z = xx$, fiet $a = 0$; $\frac{dz}{dx} = 2x$; & $\frac{ddz}{2dx^2} = 1$, hinc erit:

$$\int xx\,dx = (x+\tfrac{1}{2})\,xx - 2x(\tfrac{1}{2}xx+\tfrac{1}{2}x+\tfrac{1}{12}) + 1(\tfrac{1}{3}x^3+\tfrac{1}{2}x^2+\tfrac{1}{6}x)$$

ſeu $\int xx\,dx = \frac{1}{3}x^3$; dat autem $\frac{1}{3}x^3$ differentiatum vtique $xx\,dx$.

138. Noua ergo hinc patet via ad terminos ſummatorios ſerierum poteſtatum inueniendos; quoniam enim ex coefficientibus ante aſſumtis A, B, C, D, &c. hi termini ſummatorii facillime formantur, horum autem coefficientium quilibet ex praecedentibus conflatur; ſi in formulis §. 135. datis loco iſtarum litterarum valores in §. 136. traditi ſubſtituantur, erit:

$$Sx^1 - x = \tfrac{1}{2}xx - \tfrac{1}{2}x$$

$$Sx^2 - x^2 = \tfrac{1}{3}x^3 - \tfrac{1}{3}x - \tfrac{2}{2}(Sx - x)$$

$$Sx^3 - x^3 = \tfrac{1}{4}x^4 - \tfrac{1}{4}x - \tfrac{3}{2}(Sx^2 - x^2) - \frac{3.2}{2.3}(Sx^1 - x^1)$$

$$Sx^4 - x^4 = \tfrac{1}{5}x^5 - \tfrac{1}{5}x - \tfrac{4}{2}(Sx^3 - x^3) - \frac{4.3}{2.3}(Sx^2 - x^2) - \frac{4.3.2}{2.3.4}(Sx - x)$$

&c.

Hinc ergo ſummae poteſtatum ſuperiorum ex ſummis inferiorum formari poterunt.

139. Quod ſi vero legem, qua coefficientes A, B, C, D, &c. ſupra §. 135. progredi inuenti ſunt, attentius intueamur, eos ſeriem recurrentem conſtituere deprehendemus. Si enim euoluamus hanc fractionem:

$$\frac{x + \frac{1}{2}xxu + \frac{1}{6}x^3u^2 + \frac{1}{24}x^4u^3 + \frac{1}{120}x^5u^4 + \&c.}{1 + \frac{1}{2}u + \frac{1}{6}u^2 + \frac{1}{24}u^3 + \frac{1}{120}u^4 + \&c.}$$

ſecundum poteſtates ipſius u, hancque ſeriem reſultare ſumamus: A +

$$A + Bu + Cu^2 + Du^3 + Eu^4 + \&c.$$

erit vti ante inuenimus $A = x$; $B = \frac{1}{2}xx - \frac{1}{2}A$; &c. ſicque inuenta hac ſerie, obtinebuntur termini ſummatorii ſerierum poteſtatum. Illa autem fractio, ex cuius euolutione iſta ſeries naſcitur, tranſit in hanc formam: $\frac{e^{xu}-1}{e^u-1}$, quae ſi x fuerit numerus integer affirmatiuus, abit in $1 + e^u + e^{2u} + e^{3u} + \ldots + e^{(x-1)u}$; cum ergo ſit:

$$1 = 1$$

$$e^u = 1 + \frac{u}{1} + \frac{u^2}{1.2} + \frac{u^3}{1.2.3} + \frac{u^4}{1.2.3.4} + \&c.$$

$$e^{2u} = 1 + \frac{2u}{1} + \frac{4u^2}{1.2} + \frac{8u^3}{1.1.3} + \frac{16u^4}{1.2.3.4} + \&c.$$

$$e^{3u} = 1 + \frac{3u}{1} + \frac{9u^2}{1.2} + \frac{27u^3}{1.2.3} + \frac{81u^4}{1.2.3.4} + \&c.$$

$$e^{(x-1)u} = 1 + \frac{(x-1)u}{1} + \frac{(x-1)^2u^2}{1.2} + \frac{(x-1)^3u^3}{1.2.3} + \frac{(x-1)^4u^4}{1.2.3.4} + \&c.$$

ideoque erit:

$$A = x$$

$$B = S(x-1) = Sx - x$$

$$C = \tfrac{1}{2}S(x-1)^2 = \tfrac{1}{2}Sx^2 - \tfrac{1}{2}x^2$$

$$D = \tfrac{1}{6}S(x-1)^3 = \tfrac{1}{6}Sx^3 - \tfrac{1}{6}x^3 \quad \&c.$$

Vnde nexus horum coefficientium cum ſummis poteſtatum, ante iam obſeruatus, penitus confirmatur ac demonſtratur.

CAPUT VI.

DE SUMMATIONE PROGRESSIONUM PER SERIES INFINITAS.

140.

Expreſſio generalis, quam in Capite praecedente pro termino ſummatorio cuiusque ſeriei, cuius termiterminus generalis ſeu indici x reſpondens eſt $= z$, invenimus:

$$Sz = \int z\,dx + \tfrac{1}{2}z + \frac{\mathfrak{A}\,dz}{1.2\,dx} - \frac{\mathfrak{B}\,d^3z}{1.2.3.4\,dx^3} + \frac{\mathfrak{C}\,d^5z}{1.2\ldots6\,dx^5} - \&c.$$

proprie inſeruit ſeriebus ſummandis, quarum termini generales ſunt functiones quaecunque rationales integrae indicis x, quoniam his caſibus ad differentialia tandem euaneſcentia peruenitur. Sin autem z non fuerit eiusmodi functio ipſius x, tum eius differentialia in infinitum progrediuntur, ſicque reſultat ſeries infinita ſummam ſeriei propoſitae exprimens, & quidem ad datum vsque terminum, cuius index eſt $= x$. Quocirca progreſſionis propoſitae in infinitum continuatae ſumma prodibit, ſi ponatur $x = \infty$; hocque pacto alia inuenitur ſeries infinita priori aequalis.

141. Sin autem ponatur $x = 0$, tum expreſſio ſummam exhibens debet euaneſcere, vti iam annotauimus; quod niſi fiat, eiusmodi quantitas conſtans ad ſummam addi vel inde auferri debet, vt huic conditioni ſatisfiat.

Quo facto ſi ponatur $x=1$, ſumma inuenta praebebit terminum primum ſeriei: ſin $x=2$, aggregatum primi & ſecundi; ſin $x=3$, orietur aggregatum trium terminorum initialium ſeriei, & ita porro. His igitur caſibus, quia ſumma vnius, vel duorum, vel trium, &c. terminorum eſt cognita, ſeriei infinitae, qua iſta ſumma exprimitur, valor innoteſcet; ex hocque fonte innumerabiles ſeries ſummari poterunt.

142. Quoniam, ſi eiusmodi conſtans ſummae fuerit adiecta, vt ea euaneſcat poſito $x=0$, tum ſumma omnibus reliquis caſibus, quicunque numeri pro x ſubſtituantur, ſatisfacit; manifeſtum eſt, dummodo ſummae inuentae eiusmodi quantitas conſtans adiiciatur, vt vno quodam caſu vera ſumma indicetur, tum omnibus reliquis caſibus veram ſummam prodire debere. Quare ſi ponendo $x=0$, non pateat, cuiusmodi valorem expreſſio ſummae recipiat, neque igitur conſtans adiicienda hinc inueniri queat; tum alius quicunque numerus pro x ſtatui poterit, adiiciendaque conſtante effici vt debita ſumma indicetur: quod quomodo fieri debeat, ex ſequentibus magis fiet perſpicuum.

142. Conſideremus primum hanc progreſſionem harmonicam:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ \ldots\ +\frac{1}{x}=s,$$

cuius terminus generalis cum ſit $=\frac{1}{x}$, fiet $z=\frac{1}{x}$, &

ter-

terminus ſummatorius s ita inuenietur. Primo erit $\int z\,dx = \int \frac{dx}{x} = lx$; deinde differentialia ita ſe habebunt:

$$\frac{dz}{dx} = -\frac{1}{x^2}; \quad \frac{ddz}{2\,dx^2} = \frac{1}{x^3}; \quad \frac{d^3z}{6\,dx^3} = -\frac{1}{x^4};$$

$$\frac{d^4z}{24\,dx^4} = \frac{1}{x^5}; \quad \frac{d^5z}{120\,dx^6} = -\frac{1}{x^6}$$ &c. Hinc itaque erit:

$$s = lx + \frac{1}{2x} - \frac{\mathfrak{A}}{2x^2} + \frac{\mathfrak{B}}{4x^4} - \frac{\mathfrak{C}}{6x^6} + \frac{\mathfrak{D}}{8x^8} - \&c.$$
$$+ \text{Conſtante.}$$

Conſtans igitur hic addenda ex caſu $x = 0$ non poteſt definiri. Ponatur ergo $x = 1$, quia tum fit $s = 1$, erit $1 = \frac{1}{2} - \frac{\mathfrak{A}}{2} + \frac{\mathfrak{B}}{4} - \frac{\mathfrak{C}}{6} + \frac{\mathfrak{D}}{8} + \text{Conſt.}$ vnde fit iſta conſtans $= \frac{1}{2} + \frac{\mathfrak{A}}{2} - \frac{\mathfrak{B}}{4} + \frac{\mathfrak{C}}{6} - \frac{\mathfrak{D}}{8} + \&c.$ eritque ideo terminus ſummatorius quaeſitus:

$$s = lx + \frac{1}{2x} - \frac{\mathfrak{A}}{2x^2} + \frac{\mathfrak{B}}{4x^4} - \frac{\mathfrak{C}}{6x^6} + \frac{\mathfrak{D}}{8x^8} - \&c.$$
$$+ \frac{1}{2} + \frac{\mathfrak{A}}{2} - \frac{\mathfrak{B}}{4} + \frac{\mathfrak{C}}{6} - \frac{\mathfrak{D}}{8} + \&c.$$

143. Quoniam numeri Bernoulliani $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. conſtituunt ſeriem diuergentem, hic valor conſtantis cognoſci nequit. Sin autem loco x ſubſtituatur numerus maior, atque ſumma totidem terminorum actu quaeratur, valor conſtantis commode inueſtigabitur. Ponatur

 in

in hunc finem $x = 10$, decemque primis terminis colligendis reperietur eorum ſumma $=$

$$2,9289682539682539 68$$

cui aequalis eſſe debet expreſſio ſummae, ſi in ea ponatur $x = 10$, quae fit:

$$l10 + \frac{1}{20} - \frac{\mathfrak{A}}{200} + \frac{\mathfrak{B}}{40000} - \frac{\mathfrak{C}}{6000000} + \frac{\mathfrak{D}}{800000000} - \&c. + C.$$

ſumto ergo pro $l10$ logarithmo hyperbolico denarii & loco $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, &c. ſubſtitutis valoribus ſupra inuentis, reperietur conſtans illa:

$$C = 0,5772156649015325$$

qui numerus ergo exprimit ſummam ſeriei:

$$\frac{1}{2} + \frac{\mathfrak{A}}{2} - \frac{\mathfrak{B}}{4} + \frac{\mathfrak{C}}{6} - \frac{\mathfrak{D}}{8} + \frac{\mathfrak{E}}{10} - \&c.$$

144. Si pro x numeri non nimis magni ſubſtituantur, quia ſumma ſeriei facile actu inuenitur, obtinebitur ſumma ſeriei huius:

$$\frac{1}{2x} - \frac{\mathfrak{A}}{2x^2} + \frac{\mathfrak{B}}{4x^4} - \frac{\mathfrak{C}}{6x^6} + \frac{\mathfrak{D}}{8x^8} - \&c. = s - lx - C.$$

Sin autem x ſignificet numerum valde magnum, quia tum valor huius expreſſionis in infinitum excurrentis facile in fractionibus decimalibus aſſignatur, viciſſim ſumma ſeriei definietur. Ac primo quidem conſtat, ſi ſeries in infinitum continuetur, eius ſummam futuram eſſe infinite magnam; facto enim $x = \infty$ fit lx quoque infini-

nitus; etſi $\backsim$ ad x rationem infinite paruam teneat. Quo autem commodius ſumma quotcunque terminorum ſeriei aſſignari queat, valores litterarum 𝔄, 𝔅, ℭ, &c. in fractionibus decimalibus exprimamus:

$$
\begin{aligned}
\mathfrak{A} &= 0,166666666666 \\
\mathfrak{B} &= 0,033333333333 \\
\mathfrak{C} &= 0,023809523809\,5 \\
\mathfrak{D} &= 0,033333333333 \\
\mathfrak{E} &= 0,075757575757\,5 \\
\mathfrak{F} &= 0,253113553113\,5 \\
\mathfrak{G} &= 1,166666666666 \\
\mathfrak{H} &= 7,092156862745\,1 \quad \&c.
\end{aligned}
$$

vnde ergo erit:

$$
\begin{aligned}
\frac{\mathfrak{A}}{2} &= 0,083333333333\,3 \\
\frac{\mathfrak{B}}{4} &= 0,008333333333\,3 \\
\frac{\mathfrak{C}}{6} &= 0,003968253968\,2 \\
\frac{\mathfrak{D}}{8} &= 0,004166666666\,6 \\
\frac{\mathfrak{E}}{10} &= 0,007575757575\,7 \\
\frac{\mathfrak{F}}{12} &= 0,021092796092\,8 \\
\frac{\mathfrak{G}}{14} &= 0,083333333333\,3 \\
\frac{\mathfrak{H}}{16} &= 0,443259803921\,6 \quad \&c.
\end{aligned}
$$

EXEMPLUM I.

Inuenire ſummam mille terminorum ſeriei

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \&c.$$

Ponatur ergo $x = 1000$, & cum ſit

$$l10 = 2,3025850929940456840 \quad \text{erit}$$

$$\begin{array}{rrl}
 & lx = & 6,9077552789821 \\
 & \text{Conſt.} = & 0,5772156649015 \\
 & \frac{1}{2x} = & 0,0005000000000 \\
 & & 7,4849709438836 \\
\text{ſubt.} & \frac{\mathfrak{A}}{2xx} = & 0,0000000833333 \\
 & & 7,4849708605503 \\
\text{add.} & \frac{\mathfrak{B}}{4x^4} = & 0,0000000000000
\end{array}$$

Ergo $7,4849708605503$ eſt ſumma quaeſita mille terminorum, qui nequidem ſeptem vnitates cum ſemiſſe conficiunt.

EXEMPLUM II.

Inuenire ſummam millies mille terminorum ſeriei

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c.$$

Quia eſt $x = 1000000$, erit $lx = 6.l10$, ergo

$$\begin{array}{rl}
lx = & 13,8155105579642 \\
\text{Conſt.} = & 0,5772156649015 \\
\frac{1}{2x} = & 0,0000005000000 \\
 & 14,3927262228657 = \text{ſummae quaeſitae.}
\end{array}$$

145. Si ergo pro x ſtatuatur numerus vehementer magnus, ſumma ſatis exacte inuenitur ex ſolo primo termino lx conſtante C aucto: vnde egregia corollaria deduci poſſunt. Sic ſi x fuerit numerus vehementer magnus, ponaturque:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots \frac{1}{x} = s$$

$$\& \; 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \ldots + \frac{1}{x} + \ldots \frac{1}{x+y} = t$$

quia eſt proxime $s = lx + C$, & $t = l(x+y) + C$; erit $t - s = l(x+y) - lx = l\frac{x+y}{x}$, ideoque hic logarithmus proxime per ſeriem harmonicam finito terminorum numero conſtantem exprimetur hoc modo:

$$l\frac{x+y}{x} = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \ldots + \frac{1}{x+y}.$$

Accuratius autem hic logarithmus exhibebitur, ſi ſuperiores ſummae s & t exactius capiantur. Sic cum ſit

$$s = lx + C + \frac{1}{2x} - \frac{1}{12xx}, \quad \&$$

$$t = l(x+y) + C + \frac{1}{2(x+y)} - \frac{1}{12(x+y)^2}; \text{ erit}$$

$$t - s = l\frac{x+y}{x} - \frac{1}{2x} + \frac{1}{2(x+y)} + \frac{1}{12xx} - \frac{1}{12(x+y)^2},$$

ideoque

$$l\frac{x+y}{x} = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \ldots + \frac{1}{x+y} + \frac{1}{2x} - \frac{1}{2(x+y)} - \frac{1}{12xx} + \frac{1}{12(x+y)^2}$$

Sin

Sin autem ſit numerus tam magnus, vt bini termini vltimi reiici queant, erit proxime:

$$l\frac{x+y}{x}=\frac{1}{x+1}+\frac{1}{x+2}+\frac{1}{x+3}+\ldots+\frac{1}{x+y}+\frac{1}{2}\left(\frac{1}{x}-\frac{1}{x+y}\right).$$

145. Ex hac quoque ſerie harmonica deriuare poterimus ſummam huius ſeriei, in qua tantum numeri impares occurrunt:

$$\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}\ldots+\frac{1}{2x+1}.$$

Cum enim omnibus terminis capiendis ſit:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\ldots\ldots+\frac{1}{2x}+\frac{1}{2x+1}=$$

$$l(2x+1)+C+\frac{1}{2(2x+1)}-\frac{\mathfrak{A}}{2(2x+1)^2}+\frac{\mathfrak{B}}{4(2x+1)^4}-\frac{\mathfrak{C}}{6(2x+1)^6}+\&c.$$

terminorum vero ordine parium:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots\ldots\ldots+\frac{1}{2x}$$

ſumma ſit ſemiſſis ſuperioris nempe:

$$\tfrac{1}{2}C+\tfrac{1}{2}lx+\frac{1}{4x}-\frac{\mathfrak{A}}{4x^2}+\frac{\mathfrak{B}}{8x^4}-\frac{\mathfrak{C}}{12x^6}+\frac{\mathfrak{D}}{16x^8}-\&c.$$

erit hac ſerie ab illa ablata:

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots\ldots+\frac{1}{2x+1}=$$

$$\tfrac{1}{2}C+l\frac{2x+1}{\sqrt{x}}+\frac{1}{2(2x+1)}-\frac{\mathfrak{A}}{2(2x+1)^2}+\frac{\mathfrak{B}}{4(2x+1)^4}-\&c.$$

$$-\frac{1}{4x}+\frac{\mathfrak{A}}{4x^2}-\frac{\mathfrak{B}}{8x^4}+\&c.$$

146.

146. Poteſt vero etiam per eandem expreſſionem generalem ſumma cuiusque ſeriei harmonicae inueniri; ſit enim:

$$\frac{1}{m+n}+\frac{1}{2m+n}+\frac{1}{3m+n}+\frac{1}{4m+n}+\ \ldots\ +\frac{1}{mx+n}=s,$$

quia eſt terminus generalis $z=\frac{1}{mx+n}$, erit:

$$\int z\,dx=\frac{1}{m}l(mx+n)\ ;\quad \frac{dz}{dx}=-\frac{m}{(mx+n)^2}$$

$$\frac{ddz}{2dx^2}=\frac{mm}{(mx+n)^3}\ ;\quad \frac{d^3z}{6dx^3}=-\frac{m^3}{(mx+n)^4}$$

$$\frac{d^4z}{24dx^4}=\frac{m^4}{(mx+n)^5}\ ;\quad \frac{d^5z}{120dx^5}=-\frac{m^5}{(mx+n)^6}\ \&c.$$

Ex his ergo reperitur:

$$s=D+\frac{1}{m}l(mx+n)+\frac{1}{2(mx+n)}-\frac{\mathfrak{A}m}{2(mx+n)^2}+\frac{\mathfrak{B}m^3}{4(mx+n)^4}$$
$$-\frac{\mathfrak{C}m^5}{6(mx+n)^6}+\frac{\mathfrak{D}m^7}{8(mx+n)^8}-\&c.$$

Poſito ergo $x=0$, fiet conſtans illa addenda:

$$D=-\frac{1}{m}ln-\frac{1}{2n}+\frac{\mathfrak{A}m}{2n^2}-\frac{\mathfrak{B}m^3}{4n^4}+\frac{\mathfrak{C}m^5}{6n^6}-\&c.$$

147. Si vero ſit $n=0$, quoniam ſeriei:

$$\frac{1}{m}+\frac{1}{2m}+\frac{1}{3m}+\frac{1}{4m}+\ \ldots\ldots\ldots\ +\frac{1}{mx}$$

ſumma eſt $=\frac{1}{m}C+\frac{1}{m}lx+\frac{1}{2mx}-\frac{\mathfrak{A}}{2mx^2}+\frac{\mathfrak{B}}{4mx^4}-\&c.$

 at

at vero huius ſeriei:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots\ldots + \frac{1}{mx}$$

ſumma eſt $= C + lmx + \frac{1}{2mx} - \frac{\mathfrak{A}}{2m^2x^2} + \frac{\mathfrak{B}}{4m^4x^4} - \&c.$ ſi ab hac ſerie illa m vicibus ſumta ſubtrahatur vt prodeat haec ſeries:

$$1 + \tfrac{1}{2} + \ldots + \frac{1}{m} + \ldots + \frac{1}{2m} + \ldots + \frac{1}{3m} + \ldots + \frac{1}{mx}$$
$$\quad - \frac{m}{m} \qquad - \frac{m}{2m} \qquad - \frac{m}{3m} \qquad - \frac{m}{mx}$$

eius ſumma erit $= lm + \frac{1}{2mx} - \frac{\mathfrak{A}}{2m^2x^2} + \frac{\mathfrak{B}}{4m^4x^4} - \&c.$

$$- \frac{1}{2x} + \frac{\mathfrak{A}}{2xx} - \frac{\mathfrak{B}}{4x^4} + \&c.$$

atque ſi ſtatuatur $x = \infty$ ſumma erit $= lm$. Hinc pro m ponendo numeros 2, 3, 4, &c. erit:

$$l2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \tfrac{1}{7} - \tfrac{1}{8} + \&c.$$
$$l3 = 1 + \tfrac{1}{2} - \tfrac{2}{3} + \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{2}{6} + \tfrac{1}{7} + \tfrac{1}{8} - \tfrac{2}{9}\ \&c.$$
$$l4 = 1 + \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{3}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} - \tfrac{3}{8} + \&c.$$
$$l5 = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} - \tfrac{4}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} + \tfrac{1}{9} - \tfrac{4}{10}\ \&c.$$

&c.

148. Relicta autem ſerie harmonica progrediamur ad ſeriem quadratorum reciprocam, ſitque:

$$s = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots\ldots + \frac{1}{xx},$$

in

in qua cum ſit terminus generalis $z = \frac{1}{xx}$, erit $\int z\,dx = -\frac{1}{x}$, & differentialia ipſius z ita ſe habebunt

$$\frac{dz}{2\,dx} = -\frac{1}{x^3};\ \frac{ddz}{2.3dx^2} = \frac{1}{x^4};\ \frac{d^3z}{2.3.4dx^3} = -\frac{1}{x^5};\ \&c.$$

vnde erit ſumma

$$s = C - \frac{1}{x} + \frac{1}{2xx} - \frac{\mathfrak{A}}{x^3} + \frac{\mathfrak{B}}{x^5} - \frac{\mathfrak{C}}{x^7} + \frac{\mathfrak{D}}{x^9} - \frac{\mathfrak{E}}{x^{11}} + \&c.$$

in qua conſtans addenda C ex vno caſu, quo ſumma conſtat, eſt definienda. Ponamus ergo $x = 1$, quia fit $s = 1$ debet eſſe:

$$C = 1 + 1 - \tfrac{1}{2} + \mathfrak{A} - \mathfrak{B} + \mathfrak{C} - \mathfrak{D} + \mathfrak{E} - \&c.$$

quae ſeries autem cum ſit maxime diuergens, valorem conſtantis C non oſtendit. Quia autem ſupra demonſtrauimus ſummam huius ſeriei in infinitum continuatae eſſe $= \frac{\pi\pi}{6}$; facto $x = \infty$, ſi ponatur $s = \frac{\pi\pi}{6}$, fiet $C = \frac{\pi\pi}{6}$, ob reliquos terminos omnes euaneſcentes. Erit ergo

$$1 + 1 - \tfrac{1}{2} + \mathfrak{A} - \mathfrak{B} + \mathfrak{C} - \mathfrak{D} + \mathfrak{E} - \&c. = \frac{\pi\pi}{6}.$$

149. Sin autem ſumma huius ſeriei cognita non fuiſſet, valor conſtantis illius C ex alio quopiam caſu, quo ſumma actu eſt inuenta, determinari deberet. Hunc in finem ponamus $x = 10$, atque decem terminis actu addendis reperietur:

 $s = 1,$

$$s = 1,549767731166540690$$ tum eſt

add. $\frac{1}{x} = 0,1$

ſubtr. $\frac{1}{2xx} = 0,005$

$$1,644767731166540690$$

add. $\frac{\mathfrak{A}}{x^3} = 0,000166666666666666$

$$1,644934397833207356$$

ſubtr. $\frac{\mathfrak{B}}{x^5} = 0,000000333333333333$

$$1,644934064499874023$$

add. $\frac{\mathfrak{C}}{x^7} = 0,000000002380952381$

$$1,644934066880826404$$

ſubtr. $\frac{\mathfrak{D}}{x^9} = 0,000000000033333333$

$$1,644934066847493071$$

add. $\frac{\mathfrak{E}}{x^{11}} = 0,000000000000757575$

$$1,644934066848250646$$

ſubtr. $\frac{\mathfrak{F}}{x^{13}} = 0,000000000000025311$

$$1,644934066848225335$$

add. $\frac{\mathfrak{G}}{x^{15}} = 0,000000000000001166$

ſubtr. $\frac{\mathfrak{H}}{x^{17}} = 71$

$$1,644934066848226430 = C.$$

Hicque numerus ſimul eſt valor expreſſionis $\frac{\pi\pi}{6}$, quemadmodum ex valore ipſius π cognito calculum inſtituenti patebit. Vnde ſimul intelligitur, etiamſi ſeries $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, &c. diuergat, tamen hoc modo veram prodire ſummam.

150.

150. Sit nunc $z = \frac{1}{x^3}$; atque

$$s = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \dots\dots\dots + \frac{1}{x^3},$$

quia eſt

$$\int z\,dx = -\frac{1}{2xx}; \quad \frac{dz}{1.2.3\,dx} = -\frac{1}{2x^4}; \quad \frac{ddz}{1.2.3.4\,dx^2} = \frac{1}{2x^5}$$

$$\frac{d^3z}{1.2\ldots5\,dx^3} = -\frac{1}{2x^6}; \quad \frac{d^5z}{1.2\ldots7\,dx^5} = -\frac{1}{2x^8}; \quad \&c.$$

erit

$$s = C - \frac{1}{2xx} + \frac{1}{2x^3} - \frac{3\mathfrak{A}}{2x^4} + \frac{5\mathfrak{B}}{2x^6} - \frac{7\mathfrak{C}}{2x^8} + \&c.$$

hincque poſito $x = 1$, ob $s = 1$, fiet:

$$C = 1 + \tfrac{1}{2} - \tfrac{1}{2} + \tfrac{3}{2}\mathfrak{A} - \tfrac{5}{2}\mathfrak{B} + \tfrac{7}{2}\mathfrak{C} - \tfrac{9}{2}\mathfrak{D} + \&c.$$

atque iſte valor ipſius C ſimul oſtendet ſummam ſeriei propoſitae in infinitum continuatae. Quoniam vero ſummae poteſtatum imparium non aeque ac parium conſtant, iſte ipſius C valor ex cognita ſumma aliquot terminorum definiri debet. Sit ergo $x = 10$, erit:

$$C = s + \frac{1}{2xx} - \frac{1}{2x^3} + \frac{3\mathfrak{A}}{2x^4} - \frac{5\mathfrak{B}}{2x^4} + \frac{7\mathfrak{C}}{2x^8} - \&c.$$

Eſt vero ad computum facilius inſtituendum:

$$\frac{3\mathfrak{A}}{2} = 0,250000000000$$

$$\frac{5\mathfrak{B}}{2} = 0,083333333333$$

$$\frac{7\mathfrak{C}}{2} = 0,083333333333$$

$$\frac{9\mathfrak{D}}{2} = 0,150000000000$$

$$\frac{11\mathfrak{E}}{2} = 0,416666666666$$

$$\frac{13\mathfrak{F}}{2} = 1,645238095238$$

$$\frac{15\mathfrak{G}}{2} = 8,750000000000$$

$$\frac{17\mathfrak{H}}{2} = 60,283333333333 \quad \&c.$$

Hinc ergo fient termini ad s addendi:

$$\frac{1}{2xx} = 0,005000000000000000$$

$$\frac{3\mathfrak{A}}{2x^4} = 0,000025000000000000$$

$$\frac{7\mathfrak{C}}{2x^8} = 0,000000000833333333$$

$$\frac{11\mathfrak{E}}{2x^{12}} = 0,000000000000416666$$

$$\frac{15\mathfrak{G}}{2x^{16}} = 0,000000000000000875$$

$$0,005025000833750875$$

ter-

termini autem ſubtrahendi ſunt :

$$\frac{1}{2x^3} = 0,000500000000000000$$

$$\frac{5\mathfrak{B}}{2x^6} = 0,000000083333333333$$

$$\frac{9\mathfrak{D}}{2x^{10}} = 0,000000000015000000$$

$$\frac{13\mathfrak{F}}{2x^{14}} = 0,000000000000016452$$

$$\frac{17\mathfrak{H}}{2x^{18}} = 0,000000000000000060$$

$$0,000500083348349845$$

$$\text{ab}: \; 0,005025000833750875$$

$$0,004524917485401030$$

$$s = 1,197531985674193251$$

$$C = 1,202056903159594281.$$

151. Si hoc modo vlterius progrediamur, inueniemus ſummas omnium ſerierum poteſtatum reciprocarum in fractionibus decimalibus expreſſas :

1+

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\&c.=1,6449340668482264=\frac{2\mathfrak{A}}{1.\,2}\pi^2$$

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\&c.=1,2020569031595942$$

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\&c.=1,0823232337111381=\frac{2^3\mathfrak{B}}{1.2.3.4}\pi^4$$

$$1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\&c.=1,0369277551068632$$

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\&c.=1,0173430619844491=\frac{2^5\mathfrak{C}}{1.2\ldots6}\pi^6$$

$$1+\frac{1}{2^7}+\frac{1}{3^7}+\frac{1}{4^7}+\&c.=1,0083492773866018$$

$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\&c.=1,0040773561979443=\frac{2^7\mathfrak{D}}{1.2\ldots8}\pi^8$$

$$1+\frac{1}{2^9}+\frac{1}{3^9}+\frac{1}{4^9}+\&c.=1,0020083928260822$$

$$1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\&c.=1,0009945751278180=\frac{2^9\mathfrak{E}}{1.2\ldots10}\pi^{10}$$

$$1+\frac{1}{2^{11}}+\frac{1}{3^{11}}+\frac{1}{4^{11}}+\&c.=1,0004941886041094$$

$$1+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{4^{12}}+\&c.=1,0002460865533080=\frac{2^{11}\mathfrak{F}}{1.2\ldots12}\pi^{12}$$

$$1+\frac{1}{2^{13}}+\frac{1}{3^{13}}+\frac{1}{4^{13}}+\&c.=1,0001227233475857$$

$$1+\frac{1}{2^{14}}+\frac{1}{3^{14}}+\frac{1}{4^{14}}+\&c.=1,0000612481350587=\frac{2^{13}\mathfrak{G}}{1.2\ldots14}\pi^{14}$$

$$1+\frac{1}{2^{15}}+\frac{1}{3^{15}}+\frac{1}{4^{15}}+\&c.=1,0000305882363070$$

$$1+\frac{1}{2^{16}}+\frac{1}{3^{16}}+\frac{1}{4^{16}}+\&c.=1,0000152822594086=\frac{2^{15}\mathfrak{H}}{1.2\ldots16}\pi^{16}$$

&c. 152.

152. Ex his ergo viciſſim ſummae illarum ſerierum infinitarum numeris Bernoullianis conſtantium exhiberi poterunt. Erit enim:

$$1+0-\tfrac{1}{2}+\frac{\mathfrak{A}}{2}-\frac{\mathfrak{B}}{4}+\frac{\mathfrak{C}}{6}-\frac{\mathfrak{D}}{8}+\&c.=0,57721\&c.$$

$$1+1-\tfrac{1}{2}+\mathfrak{A}-\mathfrak{B}+\mathfrak{C}-\mathfrak{D}+\&c.=\frac{2\mathfrak{A}}{1.2}\pi^2$$

$$1+\tfrac{1}{2}-\tfrac{1}{2}+\frac{3\mathfrak{A}}{2}-\frac{5\mathfrak{B}}{2}+\frac{7\mathfrak{C}}{2}-\frac{9\mathfrak{D}}{2}+\&c.=1,2020\ \&c.$$

$$1+\tfrac{1}{3}-\tfrac{1}{2}+\frac{3.4\mathfrak{A}}{2.3}-\frac{5.6\mathfrak{B}}{2.3}+\frac{7.8\mathfrak{C}}{2.3}-\frac{9.10\mathfrak{D}}{2.3}+\&c.=\frac{2^3\mathfrak{B}}{1.2.3.4}\pi^4$$

$$1+\tfrac{1}{4}-\tfrac{1}{2}+\frac{3.4.5\mathfrak{A}}{2.3.4}-\frac{5.6.7\mathfrak{B}}{2.3.4}+\frac{7.8.9\mathfrak{C}}{2.3.4}-\frac{9.10.11\mathfrak{D}}{2.3.4}+\&c.=1,0369\ \&c.$$

$$1+\tfrac{1}{5}-\tfrac{1}{2}+\frac{3.4.5.6\mathfrak{A}}{2.3.4.5}-\frac{5.6.7.8\mathfrak{B}}{2.3.4.5}+\frac{7.8.9.10\mathfrak{C}}{2.3.4.5}-\&c.=\frac{2^5\mathfrak{C}}{1.2\ldots6}\pi^6$$

&c.

Harum ergo ſerierum alternae ope quadraturae circuli ſummari poſſunt; a quanam vero quantitate transcendente reliquae pendeant, adhuc non conſtat: neque enim ad poteſtates ipſius π exponentes impares habentes revocari poſſunt, ita vt coefficientes eſſent numeri rationales. Quo autem ſaltem proxime appareat, quales futuri ſint coefficientes poteſtatum ipſius π pro exponentibus imparibus, tabellam ſequentem adiunximus:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\&c. \text{ in infin.} = \frac{\pi}{0,0000} = \infty$$

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\&c. \;.\;.\; = \frac{\pi^2}{6,0000} \quad \text{vere}$$

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\&c. \;.\;.\; = \frac{\pi^3}{25,79436} \quad \text{prox.}$$

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\&c. \;.\;.\; = \frac{\pi^4}{90,00000} \quad \text{vere}$$

$$1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\&c. \;.\;.\; = \frac{\pi^5}{295,1215} \quad \text{prox.}$$

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\&c. \;.\;.\; = \frac{\pi^6}{945,000} \quad \text{vere}$$

$$1+\frac{1}{2^7}+\frac{1}{3^7}+\frac{1}{4^7}+\&c. \;.\;.\; = \frac{\pi^7}{2995,286} \quad \text{prox.}$$

$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\&c. \;.\;.\; = \frac{\pi^8}{9450,000} \quad \text{vere}$$

$$1+\frac{1}{2^9}+\frac{1}{3^9}+\frac{1}{4^9}+\&c. \;.\;.\; = \frac{\pi^9}{29749,35} \quad \text{prox.}$$

&c.

153. Ex hoc fonte ſeries numerorum Bernoullianorum

1	2	3	4	5	6	7	8	9	
𝔄,	𝔅,	ℭ,	𝔇,	𝔈,	𝔉,	𝔊,	ℌ,	ℑ,	&c.

quantumuis irregularis videatur interpolari, ſeu termini in medio binorum quorumcunque conſtituti aſſignari poterunt: ſi enim terminus medium interiacens inter primum 𝔄 & ſecundum 𝔅, ſeu indici $1\frac{1}{2}$ reſpondens fuerit $=p$; erit vtique:

1 +

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\&c. = \frac{2^2 p}{1.2.3}\pi^3$$

ideoque $p=\frac{3}{2\pi^3}\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\&c.\right)=0,05815227$

Simili modo ſi terminus inter 𝔅 & ℭ medium interiacens ſeu indicem habens $2\frac{1}{2}$ ponatur $=q$, quia erit:

$$1+\frac{1}{2^5}+\frac{1}{3^5}+\&c. = \frac{2^4 q}{1.2.3.4.5}\pi^5$$

fiet $q=\frac{15}{2\pi^5}\left(1+\frac{1}{2^5}+\frac{1}{3^5}+\&c.\right)=0,02541327$

Si ergo iſtarum ſerierum, in quibus exponentes poteſtatum ſunt numeri impares, ſummae exhiberi poſſent, tum quoque ſeries numerorum Bernoullianorum interpolari poſſet.

154. Ponamus nunc $z=\frac{1}{nn+xx}$, & quaeratur ſumma huius ſeriei:

$$s=\frac{1}{nn+1}+\frac{1}{nn+4}+\frac{1}{nn+9}+\ldots+\frac{1}{nn+xx}.$$

Quia eſt $\int z\,dx=\int\frac{dx}{nn+xx}$; erit $\int z\,dx=\frac{1}{n}\,\mathrm{A\,tang}\,\frac{x}{n}$.

Ponatur $\mathrm{A\,cot}\,\frac{x}{n}=u$, erit $\int z\,dx=\frac{1}{n}\left(\frac{\pi}{2}-u\right)$, &

$\frac{x}{n}=\cot u=\frac{\cos u}{\sin u}$, & $\frac{nn+xx}{nn}=\frac{1}{\sin u^2}$, & $z=\frac{\sin u^2}{nn}$,

& $\frac{dx}{n}=-\frac{du}{\sin u^2}$, vnde fit $du=-\frac{dx\sin u^2}{n}$.

 Hinc

Hinc differentialia ipſius z inuenientur hoc modo:

$$dz = \frac{2\,du\,\text{ſin}\,u.\,\text{coſ}\,u}{nn} = -\frac{dx\,\text{ſin}\,u^2.\,\text{ſin}\,2u}{n^3}$$

$$\&\quad \frac{dz}{dx} = -\frac{\text{ſin}\,u^2.\,\text{ſin}\,2u}{n^3}$$

$$\frac{ddz}{2\,dx} = -\frac{du(\text{ſin}\,u\,\text{coſ}\,u\,\text{ſin}\,2u + \text{ſin}\,u^2.\text{coſ}\,2u)}{n^3} = \frac{dx\,\text{ſin}\,u^3.\,\text{ſin}\,3u}{n^4}$$

$$\&\quad \frac{ddz}{2\,dx^2} = \frac{\text{ſin}\,u^3.\,\text{ſin}\,3u}{n^4}.$$

Simili modo erit, vti iam ſupra pro eodem caſu inuenimus:

$$\frac{d^3z}{2.3\,dx^3} = -\frac{\text{ſin}\,u^4.\,\text{ſin}\,4u}{n^5};\quad \frac{d^4z}{2.3.4\,dx^4} = \frac{\text{ſin}\,u^5.\,\text{ſin}\,5u}{n^6}\ \&c.$$

ex quibus formabitur ſumma quaeſita:

$$s = \frac{\pi}{2n} - \frac{u}{n} + \frac{\text{ſin}\,u.\text{ſin}\,u}{2nn} - \frac{\mathfrak{A}}{2}.\frac{\text{ſin}\,u^2.\text{ſin}\,2u}{n^3} + \frac{\mathfrak{B}}{4}.\frac{\text{ſin}\,u^4.\,\text{ſin}\,4u}{n^5}$$
$$- \frac{\mathfrak{C}}{6}.\frac{\text{ſin}\,u^6.\,\text{ſin}\,6u}{n^7} + \frac{\mathfrak{D}}{8}.\frac{\text{ſin}\,u^8.\,\text{ſin}\,8u}{n^9} - \&c.$$
$$+ \text{Conſt.}$$

Si hic ad conſtantem determinandam ponatur $x = 0$, quo fiat $s = 0$, erit $\cot u = 0$, ideoque u angulus $90°$, ac propterea $\text{ſin}\,u = 1$, $\text{ſin}\,2u = 0$, $\text{ſin}\,4u = 0$, $\text{ſin}\,6u = 0$, &c. videtur ergo fore $0 = \frac{\pi}{2n} - \frac{\pi}{2n} + \frac{1}{2nn} + C$, hinc $C = -\frac{1}{2nn}$: at vero notandum eſt, etiamſi reliqui termini euaneſcant, tamen quia coefficientes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, &c.

tan-

tandem in infinitum excreſcunt, eorum ſummam poſſe eſſe finitam.

155. Ad hanc ergo conſtantem rite determinandam ponamus eſſe $x = \infty$, ſummam enim huius ſeriei in infinitum excurrentis ſupra iam in introductione definiuimus, oſtendimusque eſſe eam $= -\frac{1}{2nn} + \frac{\pi}{2n} + \frac{\pi}{n(e^{2n\pi}-1)}$. Poſito autem $x = \infty$, fiet $u = 0$, ideoque $\sin u = 0$, ſimulque ſinus omnium arcuum multiplorum euaneſcent. Cum autem in hac ſerie poteſtates ipſius $\sin u$ creſcant, diuergentia ſeriei impedire nequit, quominus valor ſeriei hoc caſu euaneſcat. Fiet ergo $s = \frac{\pi}{2n} + C$; vnde erit

$$\frac{\pi}{2n} + C = -\frac{1}{2nn} + \frac{\pi}{2n} + \frac{\pi}{n(e^{2n\pi}-1)}, \quad \&$$

$C = -\frac{1}{2nn} + \frac{\pi}{n(e^{2n\pi}-1)}$. Quare ſumma ſeriei quaeſita erit

$$s = \frac{\pi}{2n} - \frac{u}{n} - \frac{\sin u^2}{2nn} - \frac{\mathfrak{A}}{2}\cdot\frac{\sin u^2 . \sin 2u}{n^3} + \frac{\mathfrak{B}}{4}\cdot\frac{\sin u^4 . \sin 4u}{n^5} - \frac{\mathfrak{C}}{6}\cdot\frac{\sin u^6 . \sin 6u}{n^7} + \&c. + \frac{\pi}{n(e^{2n\pi}-1)}.$$

Vbi notandum eſt, ſi n fuerit numerus mediocriter magnus, vltimum terminum $\frac{\pi}{n(e^{2n\pi}-1)}$ tantopere fieri exiguum, vt negligi queat.

156. Ponamus eſſe $x = n$, ita vt denotet:

$$s = \frac{1}{nn+1} + \frac{1}{nn+4} + \frac{1}{nn+1} + \dots + \frac{1}{nn+nn}.$$

Tum vero erit $\cot u = 1$, & $u = 45^\circ = \frac{\pi}{4}$. Quamobrem habebitur $\sin u = \frac{1}{\sqrt{2}}$; $\sin 2u = 1$; $\sin 4u = 0$; $\sin 6u = -1$; $\sin 8u = 0$; $\sin 10u = 1$; &c.

Hancobrem erit:

$$s = \frac{\pi}{4n} - \frac{1}{2nn} + \frac{1}{4nn} - \frac{\mathfrak{A}}{2.2n^3} + \frac{\mathfrak{C}}{6.8n^7} - \frac{\mathfrak{E}}{10.2^5 n^{11}} + \frac{\mathfrak{G}}{14.2^7 n^{15}} - \&c. + \frac{\pi}{n(e^{2n\pi}-1)},$$

in qua expreſſione tantum numeri alterni ex Bernoullianis occurrunt. Si igitur valor ipſius s per computum actu inſtitutum iam fuerit inuentus, hinc quantitas π definiri poterit, erit enim:

$$\pi = 4ns + \frac{1}{n} + \frac{\mathfrak{A}}{1.n^2} - \frac{\mathfrak{C}}{3.2^2 n^6} + \frac{\mathfrak{E}}{5.2^4 n^{10}} - \frac{\mathfrak{G}}{7.2^6.n^{14}} + \&c. - \frac{\pi}{e^{2n\pi}-1}.$$

Etſi enim in termino vltimo ineſt π, tamen quia is tantopere eſt paruus, ſufficit valorem ipſius π proxime noſſe.

EXEMPLUM: Sit $n = 5$; erit:

$$s = \frac{1}{26} + \frac{1}{29} + \frac{1}{34} + \frac{1}{41} + \frac{1}{50}$$

qui

qui termini actu additi dabunt:

$$s = 0,14674630569054949 4$$

vnde erunt termini illi:

$$4ns = 2,9349261138109898 8$$

$$\frac{1}{n} = 0,2$$

$$\frac{\mathfrak{A}}{nn} = \underline{0,0066666666666666 6}$$

$$3,1415927804776565 4$$

$$\frac{\mathfrak{C}}{3.2^2.n^6} = \underline{0,0000001269841269 8}$$

$$3,1415926534935295 6$$

$$\frac{\mathfrak{E}}{5.2^4.n^{10}} = \underline{0,0000000000969696 9}$$

$$3,1415926535904992 5$$

$$\frac{\mathfrak{G}}{7.2^6.n^{14}} = \underline{0,0000000000004266 6}$$

$$3,1415926535900725 9$$

$$\frac{\mathfrak{I}}{9.2^8.n^{18}} = \underline{625}$$

$$3,1415926535900788 4$$

Hic valor iam tam prope ad veritatem accedit, vt mirandum ſit tam leui calculo eousque perueniri poſſe. Eſt vero haec expreſſio aliquantillum iuſto maior, ſubtrahi enim adhuc inde debet $\frac{4\pi}{e^{2n\pi}-1}$, cuius valor, dummodo π prope ſit cognitum, exhiberi poteſt; quod per logarithmos ita expedietur.

Quia eſt $\pi le = 1,3643763538$

erit $le^{2n\pi} = 10\pi le = 13,6437635.$

Cum

Cum iam ſit $\frac{4\pi}{e^{2n\pi}-1} = \frac{4\pi}{e^{2m\pi}} + \frac{4\pi}{e^{4n\pi}} + \&c.$ facile intelligitur ad noſtrum calculum ſufficere primum terminorum ſumſiſſe. Augeamus ergo characteriſticam numero 17, quia habemus totidem figuras decimales, erit:

$$
\begin{array}{rrl}
 & l\pi = & 17,4971498 \\
 & l4 = & 0,6020600 \\
\hline
 & & 18,0992098 \\
\text{ſubtr.} & le^{2n\pi} = & 13,6437635 \\
\hline
 & & 4,4554463
\end{array}
$$

$$
\begin{array}{lrrl}
\text{Ergo} & \frac{4\pi}{e^{2n\pi}} = & 28539 & \text{ſubtrahatur} \\
\text{ab} & & 3,14159265359007884 & \text{erit} \\
\hline
 & \pi = & 3,14159265358979345 &
\end{array}
$$

quae expreſſio in figura demum penultima a veritate recedit; quod mirandum non eſt, cum adhuc terminum $\frac{\mathfrak{L}}{11.2^{10}.n^{22}}$, qui dat 22, ſubtrahere debuiſſemus, ſicque ne vltima quidem figura aberraſſet. Ceterum intelligitur, ſi pro n maiorem numerum vti 10 aſſumſiſſemus, tum facili negotio peripheriam π ad 25 pluresque figuras inueniri potuiſſe.

157. Ponamus nunc quoque pro z functiones transcendentes ipſius x, ſitque $z = lx$, ſumendo logarithmos hyperbolicos, quoniam vulgares facile eo reuocantur; ſit-

ſitque: $s = l1 + l2 + l3 + l4 + \ldots + lx.$
Quia igitur eſt $z = lx$, erit $\int z\,dx = xlx - x$, huius enim differentiale dat $dxlx$. Deinde eſt

$$\frac{dz}{dx} = \frac{1}{x};\ \frac{ddz}{dx^2} = -\frac{1}{x^2};\ \frac{d^3z}{1.2\,dx^3} = \frac{1}{x^3};\ \frac{d^5z}{1.2.3.4\,dx^5} = \frac{1}{x^5};$$
&c.

Hinc itaque concluditur fore:

$$s = xlx - x + \tfrac{1}{2}lx + \frac{\mathfrak{A}}{1.2x} - \frac{\mathfrak{B}}{3.4x^3} + \frac{\mathfrak{C}}{5.6x^5} - \frac{\mathfrak{D}}{7.8x^7} + \&c. + \text{Conſt.}$$

Haec autem conſtans ponendo $x = 1$, quia fit $s = l1 = 0$ ita definietur, vt ſit:

$$C = 1 - \frac{\mathfrak{A}}{1.2} + \frac{\mathfrak{B}}{3.4} - \frac{\mathfrak{C}}{5.6} + \frac{\mathfrak{D}}{7.8} - \&c.$$

quae ſeries ob nimiam diuergentiam eſt inepta ad valorem ipſius C ſaltem proxime eruendum.

158. Non ſolum autem proximum ſed etiam ipſum verum valorem ipſius C inueniemus, ſi conſideremus expreſſionem Walliſianam pro valore ipſius π inuentam, atque in introductione demonſtratam: quae erat:

$$\frac{\pi}{2} = \frac{2.\ 2.\ 4.\ 4.\ 6.\ 6.\ 8.\ 8.\ 10.\ 10.\ 12.\ \&c.}{1.\ 3.\ 3.\ 5.\ 5.\ 7.\ 7.\ 9.\ 9.\ 11.\ 11.\ \&c.}$$

hinc enim logarithmis ſumendis erit:

$$l\pi - l2 = 2l2 + 2l4 + 2l6 + 2l8 + 2l10 + \&c.$$
$$- l1 - 2l3 - 2l5 - 2l7 - 2l9 - 2l11 - \&c.$$

Ponamus ergo in ſerie aſſumta $x = \infty$, & cum ſit:

$$l1 + l2 + l3 + l4 + \ldots + lx = C + (x + \tfrac{1}{2})\, lx - x$$

erit $l1 + l2 + l3 + l4 + \ldots + l2x = C + (2x + \tfrac{1}{2})\, l2x - 2x$

& $l2 + l4 + l6 + l8 + \ldots + l2x = C + (x + \tfrac{1}{2}) lx + xl2 - x$

hinc $l1 + l3 + l5 + l7 + \ldots + l(2x-1) = xlx + (x + \tfrac{1}{2})\, l2 - x$

Cum igitur ſit:

$$l\frac{\pi}{2} = 2l2 + 2l4 + 2l6 + \ldots + 2l2x - l2x$$
$$- 2l1 - 2l3 - 2l5 - \ldots - 2l(2x-1)$$

poſito $x = \infty$, erit:

$$l\frac{\pi}{2} = 2C + (2x+1)\, lx + 2xl2 - 2x - l2 - lx$$
$$- 2xlx - (2x+1)\, l2 + 2x,$$

ideoque $l\frac{\pi}{2} = 2C - 2l2$, ergo $2C = l2\pi$, & $C = \tfrac{1}{2} l2\pi$, vnde in fractionibus decimalibus reperitur:

$$C = 0{,}91893853320467274178032 97$$

atque ſimul ſequens ſeries ſummatur:

$$1 - \frac{\mathfrak{A}}{1.2} + \frac{\mathfrak{B}}{3.4} - \frac{\mathfrak{C}}{5.6} + \frac{\mathfrak{D}}{7.8} - \frac{\mathfrak{E}}{9.10} + \&c. = \tfrac{1}{2}\, l2\pi.$$

159. Cognita nunc iſta conſtante $C = \tfrac{1}{2} l2\pi$, ſumma quotcunque logarithmorum ex hac ſerie $l1 + l2 + l3 + \&c.$ exhiberi poteſt. Si enim ponatur:

$s = l1 + l2 + l3 + l4 + \ldots + lx$, erit

$$s = \tfrac{1}{2} l2\pi + (x + \tfrac{1}{2}) lx - x + \frac{\mathfrak{A}}{1.2x} - \frac{\mathfrak{B}}{3.4x^3} + \frac{\mathfrak{C}}{5.6x^5} - \frac{\mathfrak{D}}{7.8x^7} + \&c.$$

ſiqui-

ſi quidem logarithmi propoſiti fuerit hyperbolici; ſin autem proponantur logarithmi vulgares, tum in terminis $\frac{1}{2}l2\pi+(x+\frac{1}{2})lx$ pro $l2\pi$ & lx ſumi debebunt logarithmi vulgares, reliqui autem ſeriei termini $-x+\frac{\mathfrak{A}}{1.2x}-\frac{\mathfrak{B}}{3.4x^3}+\&c.$ multiplicari debent per $0,4342944819032518 27 = n$.

Erit igitur hoc caſu pro logarithmis vulgaribus:

$$l\pi = 0,49714987269413385435126 8$$
$$l2 = 0,30102999566398119521373 8$$
$$l2\pi = 0,79817986835811504956500 6$$
$$\tfrac{1}{2}l2\pi = 0,39908993417905752478250 3$$

EXEMPLUM.

Quaeratur aggregatum mille logarithmorum tabularium

$$s = l1 + l2 + l3 + \ldots + l1000.$$

Erit ergo $x = 1000$, & $lx = 3,0000000000000$

vnde fit $xlx = 3000,0000000000000$

$\frac{1}{2}lx = 1,5000000000000$

$\frac{1}{2}l2\pi = 0,3990899341790$

$3001,8990899341790$

ſubtr. $nx = 434,2944819032518$

$2567,6046080309272$

Deinde eſt $\frac{n\mathfrak{A}}{1.2x} = 0,0000361912068$

ſubtr. $\frac{n\mathfrak{B}}{3.4x^3} = 0,0000000000012$

$0,0000361912056$

addatur $2567,6046080309272$

ſumma quaeſita $s = 2567,6046442221328.$

 Cum

Cum igitur s ſit logarithmus producti numerorum:

1. 2. 3. 4. 5. 6. 1000

patet hoc productum, ſi actu multiplicetur, conſtare ex 2568 figuris, atque notas a leua initiales fore 4023872 quas inſuper 2561 figurae ſequentur.

160. Ope ergo huius logarithmorum ſummationis producta ex quocunque factoribus, qui ſecundum numeros naturales procedunt, proxime aſſignari poterunt. Huc potiſſimum referri poteſt problema, quo quaeritur vncia media ſeu maxima in poteſtate binomii quâcunque $(a+b)^m$; vbiquidem notandum eſt, ſi m ſit numerus impar, binas dari medias inter ſe aequales, quae iunctim ſumtae praebeant vnciam mediam in poteſtate ſequente pari. Quare cum vncia maxima in quaque poteſtate pari ſit duplo maior quam vncia media in poteſtate praecedente impari, ſufficiet pro poteſtatibus paribus vnciam mediam maximam determinaſſe. Sit igitur $m = 2n$, & vncia media ita exprimetur vt ſit:

$$\frac{2n(2n-1)(2n-2)(2n-3)\;.\;.\;.\;.\;(n+1)}{1.\;2.\;3.\;4\;.\;.\;.\;.\;.\;.\;.\;.\;.\;n}$$

Vocetur iſta vncia media quae quaeritur $= u$, atque ea hoc modo repraeſentari poterit, vt ſit:

$$u = \frac{1.\;2.\;3.\;4.\;5.\;.\;.\;.\;.\;.\;.\;2n}{(1.\;2.\;3.\;4.\;.\;.\;n)^2}$$

ſumtisque logarithmis erit:

$$lu = l1 + l2 + l3 + l4 + l5 + \;.\;.\;. + l2n$$
$$-2l1 - 2l2 - 2l3 - 2l4 - 2l5 \;.\;.\;.\; - 2ln.$$

161.

161. Iam vero ſumendis his logarithmis hyperbolicis erit:

$$l1 + l2 + l3 + l4 + \ldots\ldots + l2n =$$

$$= \tfrac{1}{2} l2\pi + (2n+\tfrac{1}{2})\, ln + (2n+\tfrac{1}{2})\, l2 - 2n$$

$$+ \frac{\mathfrak{A}}{1.2.2n} - \frac{\mathfrak{B}}{3.4.2^3n^3} + \frac{\mathfrak{C}}{5.6.2^5n^5} - \&c.$$

& $2l1 + 2l2 + 2l3 + 2l4 + \ldots + 2ln =$

$$l2\pi + (2n+1)ln - 2n + \frac{2\mathfrak{A}}{1.2n} - \frac{2\mathfrak{B}}{3.4n^3} + \frac{2\mathfrak{C}}{5.6n^5} - \&c.$$

qua expreſſione ab illa ſublata relinquetur:

$$lu = -\tfrac{1}{2}l\pi - \tfrac{1}{2}ln + 2nl2 + \frac{\mathfrak{A}}{1.2.2n} - \frac{\mathfrak{B}}{3.4.2^3n^3} + \frac{\mathfrak{C}}{5.6.2^5n^5} - \&c.$$

$$- \frac{2\mathfrak{A}}{1.2n} + \frac{2\mathfrak{B}}{3.4n^3} - \frac{2\mathfrak{C}}{5.6n^5} + \&c.$$

his vero binis terminis colligendis, erit:

$$lu = l\frac{2^{2n}}{\sqrt{n\pi}} - \frac{3\mathfrak{A}}{1.2.2n} + \frac{15\mathfrak{B}}{3.4.2^3n^3} - \frac{63\mathfrak{C}}{5.6.2^5n^5} + \frac{255\mathfrak{D}}{7.8.2^7n^7} - \&c.$$

Sit $\frac{3\mathfrak{A}}{1.2.2^2n^2} - \frac{15\mathfrak{B}}{3.4.2^4n^4} + \frac{63\mathfrak{C}}{5.6.2^6n^6} - \frac{255\mathfrak{D}}{7.8.2^8n^8} + \&c.$

$= l\left(1 + \frac{A}{2^2n^2} + \frac{B}{2^4n^4} + \frac{C}{2^6n^6} + \frac{D}{2^8n^8} + \&c.\right)$; erit

$$lu = l\frac{2^{2n}}{\sqrt{n\pi}} - 2nl\left(1 + \frac{A}{2^2n^2} + \frac{B}{2^4n^4} + \frac{C}{2^6n^6} + \&c.\right);$$

ideoque $u = \dfrac{2^{2n}}{\left(1 + \frac{A}{2^2n^2} + \frac{B}{2^4n^4} + \frac{C}{2^6n^6} + \&c.\right)^{2n}\sqrt{n\pi}}$.

Erit vero poſito $2n = m$

$$l\left(1 + \frac{A}{2^2 n^2} + \frac{B}{2^4 n^4} + \frac{C}{2^6 n^6} + \frac{D}{2^8 n^8} + \&c.\right) =$$

$$\frac{A}{m^3} + \frac{B}{m^4} + \frac{C}{m^6} + \frac{D}{m^8} + \frac{E}{m^{10}} + \&c.$$

$$- \frac{A^2}{2m^4} - \frac{AB}{m^6} - \frac{AC}{m^8} - \frac{AD}{m^{10}} - \&c.$$

$$- \frac{BB}{2m^8} - \frac{BC}{m^{10}} - \&c.$$

$$+ \frac{A^3}{3m^6} + \frac{A^2B}{m^8} + \frac{A^2C}{m^{10}} + \&c.$$

$$+ \frac{AB^2}{m^{10}} + \&c.$$

$$- \frac{A^4}{4m^8} - \frac{A^3B}{m^{10}} - \&c.$$

$$+ \frac{A^5}{5m^{10}} + \&c.$$

quae expreſſio cum aequalis eſſe debeat huic:

$$\frac{3\mathfrak{A}}{1.2m^2} - \frac{15\mathfrak{B}}{3.4m^4} + \frac{63\mathfrak{C}}{5.6m^6} - \frac{255\mathfrak{D}}{7.8m^8} + \&c.$$

fiet:

$$A = \frac{3\mathfrak{A}}{1.2}$$

$$B = \frac{A^2}{2} - \frac{15\mathfrak{B}}{3.4}$$

$$C = AB - \tfrac{1}{3}A^3 + \frac{63\mathfrak{C}}{5.6}$$

D =

$$D = AC + \tfrac{1}{2}B^2 - A^2B + \tfrac{1}{4}A^4 - \frac{255\,\mathfrak{D}}{7.8}$$

$$E = AD + BC - A^2C - AB^2 + A^3B - \tfrac{1}{5}A^5 + \frac{1023\,\mathfrak{E}}{9.10}$$

&c.

162. Cum iam ſit $\mathfrak{A} = \frac{1}{6}$; $\mathfrak{B} = \frac{1}{30}$; $\mathfrak{C} = \frac{1}{42}$; $\mathfrak{D} = \frac{1}{30}$; $\mathfrak{E} = \frac{5}{66}$; erit:

$$A = \frac{1}{4}$$

$$B = -\frac{1}{96}$$

$$C = \frac{27}{640}$$

$$D = -\frac{90031}{2^{11}.3^2.5.7} \quad \&c.$$

Hinc efficitur:

$$u = \frac{2^{2n}}{\left(1 + \frac{1}{2^4 n^2} - \frac{1}{2^9.3n^4} + \frac{27}{2^{15}.5n^6} - \frac{90031}{2^{19}.3^2.5.7n^8} + \&c.\right)^{2n} \sqrt{n\pi}}$$

$$\text{ſeu } u = \frac{2^{2n}\left(1 - \frac{1}{2^4 n^2} + \frac{7}{2^9.3n^4} - \frac{121}{2^{13}.3.5n^6} + \frac{104969}{2^{19}.3^2.5.7n^8} - \&c.\right)^{2n}}{\sqrt{n\pi}}$$

vel ſi iſta ſeriei eleuatio actu inſtituatur, erit proxime:

$$u = \frac{2^{2n}}{\sqrt{n\pi}\left(1 + \frac{1}{4n} + \frac{1}{32n^2} - \frac{1}{128n^3} - \frac{5}{16.128n^4}\ \&c.\right)}$$

hinc

hinc terminus medius in $(1+1)^{2n}$, erit ad ſummam omnium terminorum 2^{2n}

vti 1 ad $\sqrt{n\pi}\left(1+\frac{1}{4n}+\frac{1}{32n^2}-\frac{1}{128n^3}-\frac{5}{16.128n^4}-\&c.\right)$

vel poſito breuitatis gratia $4n=\nu$, erit iſta ratio:

vt 1 ad $\sqrt{n\pi}\left(1+\frac{1}{\nu}+\frac{1}{2\nu^2}-\frac{1}{2\nu^3}-\frac{5}{8\nu^4}+\frac{23}{8\nu^5}+\frac{53}{16\nu^6}-\&c.\right)$

EXEMPLUM I.

Quaeratur vncia media in binomio $(a+b)^{10}$ *euoluto, quam conſtat eſſe* $=\frac{10.\ 9.\ 8.\ 7.\ 6}{1.\ 2.\ 3.\ 4.\ 5}=252.$

Adhibendo vltimam formulam pro u inuentam erit $n=5$

	$\frac{1}{4n}$ =	0,0500000
	$\frac{1}{32n^2}$ =	0,0012500
		0,0512500
ſubtr.	$\frac{1}{128n^3}$ =	625
		0,0511875
ſubtr.	$\frac{5}{16.128n^4}$ =	39
Ergo	$1+\frac{1}{4n}+\&c.$ =	1,0511836
Huius log.	=	0,0216784
	ln =	0,6989700
	$l\pi$ =	0,4971498
		1,2177982

$l\sqrt{n}$

$lV n\pi(1 + \&c.) = 0,6088991$

a $l2^{2n} = 3,0102999$

$lu = 2,4014008$

vnde ſit $u = 252$.

EXEMPLUM II.

Inueſtigetur ratio, quam in poteſtate centeſima binomii 1+1 *terminus medius ad ſummam omnium* 2^{100} *tenet.*

Vtamur ad hoc formula primum inuenta:

$$lu = l\frac{2^{2n}}{V n\pi} - \frac{3\mathfrak{A}}{1.2.2n} + \frac{15\mathfrak{B}}{3.4.2^3 n^3} - \frac{63\mathfrak{C}}{5.6.2^5 n^5} + \&c.$$

in qua poſito $2n = m$ vt habeatur iſta poteſtas $(1+1)^m$ & loco $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, ſubſtitutis valoribus, fiet:

$$lu = l\frac{2^m}{V\frac{1}{2}m\pi} - \frac{1}{4m} + \frac{1}{24m^3} - \frac{1}{20m^5} + \frac{17}{112m^7} - \frac{31}{36m^9} + \frac{691}{88m^{11}} - \&c.$$

qui logarithmi cum ſint hyperbolici, multiplicentur ii per

$k = 0,4342944819032 51,$

vt transmutentur in tabulares, eritque

$$lu = l\frac{2^m}{V\frac{1}{2}m\pi} - \frac{k}{4m} + \frac{k}{24m^3} - \frac{k}{20m^5} + \frac{17k}{112m^7} - \frac{31k}{36m^9} + \&c.$$

vnde cum vncia media ſit u, erit $2^m : u$ ratio quaeſita, ideoque

$$l\frac{2^m}{u} = lV\tfrac{1}{2}m\pi + \frac{k}{4m} - \frac{k}{24m^3} + \frac{k}{20m^5} - \frac{17k}{112m^7} + \frac{31k}{36m^9} - \frac{691k}{88m^{11}} + \&c.$$

Quare cum ſit ob exponentem $m = 100$

$\frac{k}{m} = 0,0043429448$: $\frac{k}{m^3} = 0,0000004343$;

$\frac{k}{m^5} = 0,0000000000$:

erit :

$$\frac{k}{4m} = 0,0010857362$$

$$\frac{k}{24m^3} = 0,0000000181$$

$$0,0010857181$$

Tum eſt $l\pi = 0,4971498726$

$$l\tfrac{1}{2}m = 1,6989700043$$

$$l\tfrac{1}{2}m\pi = 2,1961198769$$

$$l\sqrt{\tfrac{1}{2}m\pi} = 1,0980599384$$

$$\frac{k}{4m} - \frac{k}{24m^3} + \&c. = 0,0010857181$$

$$1,0991456565 = l\frac{2^{100}}{u}.$$

Erit ergo $\frac{2^{100}}{u} = 12,56451$, atque adeo in poteſtate $(1+1)^{100}$ euoluta terminus medius ſe habebit ad ſummam omnium 2^{100} vti 1 ad 12, 56451.

163. Denotet nunc terminus generalis z functionem exponentialem a^x, ita vt ſummari debeat haec ſeries geometrica:

$$s = a + a^2 + a^3 + a^4 + \quad . \quad . \quad . \quad . \quad + a^x$$

quae

quae cum ſit geometrica, eius ſumma iam conſtat, erit enim $s = \frac{(a^x - 1)a}{a-1}$. Modo autem hic expoſito hanc ſummam inueſtigemus. Quia eſt $z = a^x$, erit $\int z\,dx = \frac{a^x}{la}$, huius enim differentiale eſt $a^x dx$, tum vero erit:

$$\frac{dz}{dx} = a^x la\,; \quad \frac{ddz}{dx^2} = a^x (la)^2\,; \quad \frac{d^3 z}{dx^3} = a^x (la)^3\,; \quad \&c.$$

vnde ſequitur fore:

$$s = a^x\left(\frac{1}{la} + \frac{1}{2} + \frac{\mathfrak{A}}{1.2} la - \frac{\mathfrak{B}}{1.2.3.4}(la)^3 + \frac{\mathfrak{C}}{1.2.3\ldots6}(la)^5 - \&c.\right) + C.$$

Ad conſtantem C definiendam ponatur $x = 0$, & ob $s = 0$, erit $C = -\frac{1}{la} - \frac{1}{2} - \frac{\mathfrak{A}}{1.2} la + \frac{\mathfrak{B}}{1.2.3.4}(la)^3 - \&c.$

ideoque fiet:

$$s = (a^x - 1)\left(\frac{1}{la} + \frac{1}{2} + \frac{\mathfrak{A}}{1.2} la - \frac{\mathfrak{B}}{1.2.3.4}(la)^3 + \frac{\mathfrak{C}}{1.2.3\ldots6}(la)^5 - \&c.\right)$$

Cum igitur ſumma ſit $\frac{(a^x - 1)a}{a - 1}$, erit:

$$\frac{a}{a-1} = \frac{1}{la} + \frac{1}{2} + \frac{\mathfrak{A}}{1.2} la - \frac{\mathfrak{B}}{1.2.3.4}(la)^3 + \frac{\mathfrak{C}(la)^5}{1.2\ldots6} - \&c.$$

vbi la denotat logarithmum hyperbolicum ipſius a: hinc fit $\frac{(a+1)la}{2(a-1)} = 1 + \frac{\mathfrak{A}(la)^2}{1.2} - \frac{\mathfrak{B}(la)^4}{1.2.3.4} + \frac{\mathfrak{C}(la)^6}{1.2\ldots6} - \&c.$ ſicque iſtius ſeriei ſumma exhiberi poterit.

164. Sit terminus generalis $z = \text{ſin}\, ax$, & $s = \text{ſin}\, a + \text{ſin}\, 2a + \text{ſin}\, 3a + \ldots + \text{ſin}\, ax$ quae ſeries cum ſit recurrens quoque ſummari poteſt; erit enim

$$s = \frac{\text{ſin}\, a + \text{ſin}\, ax - \text{ſin}(ax+a)}{1 - 2\,\text{coſ}\, a + 1} = \frac{\text{ſin}\, a + (1-\text{coſ}\, a)\,\text{ſin}\, ax - \text{ſin}\, a\,\text{coſ}\, ax}{2(1-\text{coſ}\, a)}$$

Erit vero $\int z\,dx = \int dx\,\text{ſin}\, ax = -\frac{1}{a}\text{coſ}\, ax$, & $\frac{dz}{dx} = a\,\text{coſ}\, ax$; $\frac{ddz}{dx^2} = -aa\,\text{ſin}\, ax$; $\frac{d^3z}{dx^3} = a^3\,\text{coſ}\, ax$; $\frac{d^5z}{dx^5} = a^5\,\text{coſ}\, ax$ &c.

$$s = C - \frac{1}{a}\text{coſ}\, ax + \tfrac{1}{2}\text{ſin}\, ax + \frac{\mathfrak{A}a\,\text{coſ}\, ax}{1.2} + \frac{\mathfrak{B}a^3\,\text{coſ}\, ax}{1.2.3.4} + \frac{\mathfrak{C}a^5\,\text{coſ}\, ax}{1.2.3.4.5.6} + \frac{\mathfrak{D}a^7\,\text{coſ}\, ax}{1.2\ldots 8} + \&c.$$

Ponatur $x = 0$ vt fiat $s = 0$; eritque:

$$C = \frac{1}{a} - \frac{\mathfrak{A}a}{1.2} - \frac{\mathfrak{B}a^3}{1.2.3.4} - \frac{\mathfrak{C}a^5}{1.2\ldots 6} - \&c. \quad \text{ergo}$$

$$s = \tfrac{1}{2}\text{ſin}\, ax + (1-\text{cſ}\, ax)\left(\frac{1}{a} - \frac{\mathfrak{A}u}{1.2} - \frac{\mathfrak{B}a^3}{1.2.3\,4} - \frac{\mathfrak{C}a^5}{1.2\ldots 6} - \&c.\right)$$

At cum ſit $s = \tfrac{1}{2}\text{ſin}\, ax + \frac{(1-\text{coſ}\, ax)\,\text{ſin}\, a}{2(1-\text{cſ}\, a)}$, fiet:

$$\frac{\text{ſin}\, a}{2(1-\text{cſ}\, a)} = \tfrac{1}{2}\cot\tfrac{1}{2}a = \frac{1}{a} - \frac{\mathfrak{A}a}{1.2} - \frac{\mathfrak{B}a^3}{1.2.3.4} - \frac{\mathfrak{C}a^5}{1.2\ldots 6} - \&c.$$

quam eandem ſeriem iam ſupra §. 127. habuimus.

165. Sit nunc $z = \text{coſ}\, ax$, ac ſeries ſummanda: $s = \text{coſ}\, a + \text{coſ}\, 2a + \text{coſ}\, 3a + \ldots + \text{coſ}\, ax$

cu-

cuius ſeriei, quia eſt recurrens, erit ſumma:

$$s = \frac{\operatorname{coſ} a - 1 + \operatorname{coſ} ax - \operatorname{coſ}(ax+a)}{1 - 2\operatorname{coſ} a + 1} = -\tfrac{1}{2} + \tfrac{1}{2}\operatorname{coſ} ax + \tfrac{1}{2}\cot\tfrac{1}{2}a . \operatorname{ſin} ax.$$

At vero ad ſummam noſtra methodo exprimendam, erit:

$$\int z\,dx = \int dx \operatorname{coſ} ax = \frac{1}{a}\operatorname{ſin} ax, \quad \& \quad \frac{dz}{dx} = -a \operatorname{ſin} ax;$$

$$\frac{d^3 z}{dx^3} = a^3 \operatorname{ſin} ax; \quad \frac{d^5 z}{dx^5} = -a^5 \operatorname{ſin} ax; \quad \&c. \quad \text{Ergo}$$

$$s = C + \frac{1}{a}\operatorname{ſin} ax + \tfrac{1}{2}\operatorname{coſ} ax - \frac{\mathfrak{A} a \operatorname{ſin} ax}{1.2} - \frac{\mathfrak{B} a^3 \operatorname{ſin} ax}{1.2.3.4} - \&c.$$

Sit $x = 0$, erit $s = 0$, & $C = -\frac{1}{2}$, hincque erit:

$$s = -\tfrac{1}{2} + \tfrac{1}{2}\operatorname{coſ} ax + \frac{1}{a}\operatorname{ſin} ax - \frac{\mathfrak{A} a \operatorname{ſin} ax}{1.2} - \frac{\mathfrak{B} a^3 \operatorname{ſin} ax}{1.2.3.4} - \&c.$$

Quare cum ſit $s = -\frac{1}{2} + \frac{1}{2}\operatorname{coſ} ax + \frac{1}{2}\cot\frac{1}{2}a . \operatorname{ſin} ax$,

erit vti iam modo inuenimus:

$$\tfrac{1}{2}\cot\tfrac{1}{2}a = \frac{1}{a} - \frac{\mathfrak{A} a}{1.2} - \frac{\mathfrak{B} a^3}{1.2.3.4} - \frac{\mathfrak{C} a^5}{1.2.3.4.5.6} - \&c.$$

166. Quoniam ſupra inuenimus, ſi a denotet arcum quemcunque, eſſe

$$\frac{\pi}{2} = \frac{a}{2} + \operatorname{ſin} a + \tfrac{1}{2}\operatorname{ſin} 2a + \tfrac{1}{3}\operatorname{ſin} 3a + \tfrac{1}{4}\operatorname{ſin} 4a + \&c.$$

conſideremus hanc ſeriem, ſitque $z = \frac{1}{x}\operatorname{ſin} ax$, vt ſit

$$s = \operatorname{ſin} a + \tfrac{1}{2}\operatorname{ſin} 2a + \tfrac{1}{3}\operatorname{ſin} 3a + \quad . \quad . \quad . \quad + \frac{1}{x}\operatorname{ſin} ax.$$

Hoc autem caſu fit $\int z\,dx = \int \frac{dx}{x}\operatorname{ſin} ax$, quod integrale exhiberi nequit.

Erit vero $\frac{dz}{dx} = \frac{a}{x} \operatorname{coſ} ax - \frac{1}{xx} \operatorname{ſin} ax$;

$$\frac{ddz}{dx^2} = -\frac{a^2}{x} \operatorname{ſin} ax - \frac{2a}{xx} \operatorname{coſ} ax + \frac{2}{x^3} \operatorname{ſin} ax;$$

$$\frac{d^3 z}{dx^3} = -\frac{a^3}{x} \operatorname{coſ} ax + \frac{3a^2}{x^2} \operatorname{ſin} ax + \frac{6a}{x^3} \operatorname{coſ} ax - \frac{6}{x^4} \operatorname{ſin} ax;$$

$$\frac{d^4 z}{dx^4} = \frac{a^4}{x} \operatorname{ſin} ax + \frac{4a^3}{xx} \operatorname{coſ} ax - \frac{12a^2}{x^3} \operatorname{ſin} ax - \frac{24a}{x^4} \operatorname{coſ} ax + \frac{24}{x^5} \operatorname{ſin} ax.$$

Quia igitur neque formulam integralem $\int z dx$ exhibere, neque haec differentialia ſatis commode exprimere licet, ſummam huius ſeriei per hanc methodum definire non poſſumus, ita vt quicquam inde concludi poſſet. Idem incommodum in multis aliis ſeriebus occurrit, quoties terminus generalis non ſatis eſt ſimplex, vt eius differentialia ad commodam legem exprimi queant. Quamobrem in ſequenti Capite alias expreſſiones generales pro ſummis ſerierum, quarum termini generales vel nimis ſunt compoſiti vel prorſus dari nequeunt, eliciemus; quae feliciori ſucceſſu in vſum vocari poterunt. Imprimis autem inſufficientia methodi hic traditae elucet, ſi ſigna terminorum ſeriei propoſitae alternentur, tum enim quantumuis termini generales ſint ſimplices, tamen termini ſummatorii hac methodo exhiberi commode nequeunt.

CA-

CAPUT VII.

METHODUS SUMMANDI SUPERIOR VLTERIUS PROMOTA.

167.

Vt defectum methodi ſummandi ante traditae ſuppleamus, in hoc Capite eiusmodi ſeries conſiderabimus, quarum termini generales magis ſint complexi. Cum igitur expreſſio ante inuenta in progreſſionibus geometricis, etſi aliis methodis facillime ſummari poſſunt, veram ſummam finita formula contentam non praebeat, hic primum eiusmodi ſeries contemplabimur, quarum termini ſint producta ex terminis ſeriei geometricae & alius cuiuscunque. Sit igitur propoſita haec ſeries:

$$\begin{matrix} & 1 & 2 & 3 & 4 & & x \\ s = & ap + & bp^2 + & cp^3 + & dp^4 + & \ldots & + yp^x \end{matrix}$$

quae eſt compoſita ex geometrica p, p^2, p^3, &c. & alia quacunque ſerie $a + b + c + d +$ &c. cuius terminus generalis ſeu indici x reſpondens ſit $= y$, atque expreſſionem generalem inueſtigemus pro valore eius ſummae $s = S.yp^x$.

168. Inſtituamus ratiocinium eodem modo, quo ſupra vſi ſumus, ſitque v terminus antecedens ipſi y in ſerie $a + b + c + d +$ &c. atque A praecedens ipſi a ſeu is qui indici o reſpondet, eritque vp^{x-1} terminus generalis huius ſeriei:

1 A

$$\overset{1}{A} + \overset{2}{ap} + \overset{3}{bp^2} + \overset{4}{cp^3} + \;.\;.\;.\;.\; + \overset{x}{vp^{x-1}}$$

cuius ſumma, ſi indicetur per $S.vp^{x-1}$ erit:

$$S.vp^{x-1} = \frac{1}{p} S.vp^x = S.yp^x - yp^x + A.$$

Cum autem ſit:

$$v = y - \frac{dy}{dx} + \frac{ddy}{2dx^2} - \frac{d^3y}{6dx^3} + \frac{d^4y}{24dx^4} - \frac{d^5y}{120dx^5} + \&c.$$

erit:

$$S.yp^x - yp^x + A = \frac{1}{p} S.yp^x - \frac{1}{p} S\frac{dy}{dx}p^x + \frac{1}{2p} S\frac{ddy}{dx^2}p^x$$
$$- \frac{1}{6p} S\frac{d^3y}{dx^3}p^x + \frac{1}{24p} S\frac{d^4y}{dx^4}p^x - \&c.$$

Ex qua fit:

$$S.yp^x = \frac{1}{p-1}\left(yp^{x+1} - Ap - S\frac{dy}{dx}p^x + S\frac{ddy}{2dx^2}p^x - S\frac{d^3y}{6dx^3}p^x + \&c.\right)$$

Si ergo habeantur termini ſummatorii ſerierum, quarum termini generales ſunt $\frac{dy}{dx}p^x$; $\frac{ddy}{dx^2}p^x$; $\frac{d^3y}{dx^3}p^x$; &c. ex iis definiri poterit terminus ſummatorius Syp^x.

169. Hinc iam ſummae inueniri poterunt ſerierum, quarum termini generales in hac forma $x^n p^x$ continentur. Sit enim $y = x^n$, erit $A = 0$, niſi ſit $n = 0$, quo caſu foret $A = 1$, & quia eſt:

$$\frac{dy}{dx} = nx^{n-1}; \frac{ddy}{2dx^2} = \frac{n(n-1)}{1.\ 2} x^{n-2}; \frac{d^3y}{dx^3} = \frac{n(n-1)(n-2)}{1.\ 2.\ 3} x^{n-3};$$

&c.

erit:

erit:

$$S.x^n p^x = \frac{1}{p-1}\Big(x^n p^{x+1} - Ap - nS.x^{n-1}p^x + \frac{n(n-1)}{1.\,2}S.x^{n-2}p^x$$
$$- \frac{n(n-1)(n-2)}{1.\,2.\,3}S.x^{n-3}p^x + \frac{n(n-1)(n-2)(n-3)}{1.\,2.\,3.\,4}S.x^{n-4}p^x - \&c.\Big)$$

Ex hac forma nunc ſucceſſiue pro n ſubſtituendo numeros $0, 1, 2, 3$, &c. obtinebuntur ſequentes ſummationes; ac primo quidem ſi $n = 0$, fit $A = 1$, in reliquis autem caſibus erit $A = 0$:

$$S.x^0 p^x = S.p^x = \frac{1}{p-1}(p^{x+1} - p) = \frac{p^{x+1} - p}{p-1} = \frac{p(p^x - 1)}{p-1},$$

quae eſt ſumma progreſſionis geometricae cognita:

$$S.xp^x = \frac{1}{p-1}(xp^{x+1} - S.p^x) = \frac{xp^{x+1}}{p-1} - \frac{(p^{x+1} - p)}{(p-1)^2}$$

ſeu $\quad S.xp^x = \frac{pxp^x}{p-1} - \frac{p(p^x - 1)}{(p-1)^2},$

$$S.x^2 p^x = \frac{1}{p-1}(x^2 p^{x+1} - 2S.xp^x + S.p^x) \qquad \text{ſeu}$$

$$S.x^2 p^x = \frac{x^2 p^{x+1}}{p-1} - \frac{2xp^{x+1}}{(p-1)^2} + \frac{p(p+1)(p^x - 1)}{(p-1)^3}.$$

Porro eſt

$$S.x^3 p^x = \frac{1}{p-1}(x^3 p^{x+1} - S.x^2 p^x + 3S.xp^x - S.p^x)$$

ſeu

$$S.x^3 p^x = \frac{x^3 p^{x+1}}{p-1} - \frac{3x^2 p^{x+1}}{(p-1)^2} + \frac{3(p+1)xp^{x+1}}{(p-1)^3} - \frac{p(pp+4p+1)(p^x - 1)}{(p-1)^4}$$

ſicque vlterius progrediendo ſuperiorum poteſtatum x^4p^x; x^5p^x; x^6p^x; &c. ſummae definiri poterunt, hoc vero commodius praeſtabitur ope expreſſionis generalis, quam nunc inueſtigabimus.

170. Quoniam inuenimus eſſe:

$$S.yp^x = \frac{1}{p-1}\left(yp^{x+1} - Ap - S.\frac{dy}{dx}p^x + S.\frac{ddy}{2dx^2}p^x - S.\frac{d^3y}{6dx^3}p^x + \&c.\right)$$

vbi A eſt eiusmodi conſtans, vt ſumma fiat $=0$, ſi ponatur $x=0$: namque hoc caſu fit $y=A$, & $yp^{x+1}=Ap$; hanc conſtantem omittere poterimus, dummodo perpetuo meminerimus ad ſummam quamque ſemper eiusmodi conſtantem adiici oportere, vt facto $x=0$, euaneſcat, ſeu vt alii cuipiam caſui ſatisfiat. Statuamus ergo z loco y, eritque

$$S.p^xz = \frac{p^{x+1}z}{p-1} - \frac{1}{p-1}S.p^x\frac{dz}{dx} + \frac{1}{2(p-1)}S.p^x\frac{ddz}{dx^2}$$
$$-\frac{1}{6(p-1)}S.p^x\frac{d^3z}{dx^3} + \frac{1}{24(p-1)}S.p^x\frac{d^4z}{dx^4} - \frac{1}{120(p-1)}S.p^x\frac{d^5z}{dx^5}$$
$$+ \&c.$$

Deinde ſtatuamus ſucceſſiue $\frac{dz}{dx}$; $\frac{ddz}{dx^2}$; $\frac{d^3z}{dx^3}$; &c. in locum y eritque:

$$S.\frac{p^xdz}{dx} = \frac{p^{x+1}}{p-1}.\frac{dz}{dx} - \frac{1}{p-1}S.\frac{p^xddz}{dx^2} + \frac{1}{2(p-1)}S.\frac{p^xd^3z}{dx^3} - \&c.$$

$$S.\frac{p^x ddz}{dx^2} = \frac{p^{x+1}}{p-1}.\frac{ddz}{dx^2} - \frac{1}{p-1}S.\frac{p^x d^3z}{dx^3} + \frac{1}{2(p-1)}S.\frac{p^x d^4z}{dx^4} - \&c.$$

$$S.\frac{p^x d^3z}{dx^3} = \frac{p^{x+1}}{p-1}.\frac{d^3z}{dx^3} - \frac{1}{p-1}S.\frac{p^x d^4z}{dx^4} + \frac{1}{2(p-1)}S.\frac{p^x d^5z}{dx^5} - \&c.$$

&c.

Si igitur hi valores ſucceſſiue ſubſtituantur, $S.p^x z$ huiusmodi forma exprimetur:

$$S.p^x z = \frac{p^{x+1}z}{p-1} - \frac{\alpha p^{x+1}}{(p-1)}.\frac{dz}{dx} + \frac{\beta p^{x+1}}{(p-1)}\frac{ddz}{dx^2} - \frac{\gamma p^{x+1}}{(p-1)}.\frac{ddz}{dx^3}$$
$$+ \frac{\delta p^{x+1}}{(p-1)}\frac{d^4z}{dx^4} - \frac{\varepsilon p^{x+1}}{(p-1)}.\frac{d^5z}{dx^5} + \&c.$$

171. Ad valores litterarum $\alpha, \beta, \gamma, \delta, \varepsilon$, &c. definiendos, ſubſtituantur pro quouis termino ſeries ante inuentae nempe:

$$\frac{p^{x+1}z}{p-1} = S.p^x z + \frac{1}{p-1}S.\frac{p^x dz}{dx} - \frac{1}{2(p-1)}S\frac{p^x ddz}{dx^2} + \frac{1}{6(p-1)}S\frac{p^x d^3z}{dx^3} - \&c.$$

$$\frac{p^{x+1}dz}{(p-1)dx} = S\frac{p^x dz}{dx} + \frac{1}{p-1}S\frac{p^x ddz}{dx^2} - \frac{1}{2(p-1)}S\frac{p^x d^3dz}{dx^3} + \&c.$$

$$\frac{p^{x+1}ddz}{(p-1)dx^2} = S\frac{p^x ddz}{dx^2} + \frac{1}{(p-1)}S\frac{p^x d^3z}{dx^3} - \&c.$$

$$\frac{p^{x+1}d^3z}{(p-1)dx^3} = S\frac{p^x d^3z}{dx^3} + \&c.$$

Habebimus ergo: $S.p^x z =\!=$

$$\begin{array}{llllll}
S.p^x z + \frac{1}{p-1} S\frac{p^x dz}{dx} & - \frac{1}{2(p-1)} S.\frac{p^x ddz}{dx^2} & + \frac{1}{6(p-1)} S\frac{p^x d^3 z}{dx^3} & - \frac{1}{24(p-1)} S\frac{p^x d^4 z}{dx^4} & + \&c. \\
-\alpha & -\frac{\alpha}{p-1} & +\frac{\alpha}{2(p-1)} & -\frac{\alpha}{6(p-1)} & \\
 & +\beta & +\frac{\beta}{p-1} & -\frac{\beta}{2(p-1)} & \\
 & & -\gamma & -\frac{\gamma}{p-1} & \\
 & & & +\delta &
\end{array}$$

vnde coefficientium $\alpha, \beta, \gamma, \delta$, &c. valores ſequentes obtinebuntur:

$$\alpha = \frac{1}{p-1}$$

$$\beta = \frac{1}{p-1}\left(\alpha + \frac{1}{2}\right)$$

$$\gamma = \frac{1}{p-1}\left(\beta + \frac{\alpha}{2} + \frac{1}{6}\right)$$

$$\delta = \frac{1}{p-1}\left(\gamma + \frac{\beta}{2} + \frac{\alpha}{6} + \frac{1}{24}\right)$$

$$\varepsilon = \frac{1}{p-1}\left(\delta + \frac{\gamma}{2} + \frac{\beta}{6} + \frac{\alpha}{24} + \frac{1}{120}\right) \quad \&c.$$

172. Sit breuitatis gratia $\frac{1}{p-1} = q$, erit:

$$\alpha = q$$

$$\beta = \alpha q + \tfrac{1}{2} q = qq + \tfrac{1}{2} q$$

$$\gamma = \beta q + \tfrac{1}{2}\alpha q + \tfrac{1}{6} q = q^3 + qq + \tfrac{1}{6} q$$

$$\delta = \gamma q + \tfrac{1}{2}\beta q + \tfrac{1}{6}\alpha q + \tfrac{1}{24} q = q^4 + \tfrac{3}{2} q^3 + \tfrac{7}{12} q^2 + \tfrac{1}{24} q$$

$$\varepsilon = \delta q + \tfrac{1}{2}\gamma q + \tfrac{1}{6}\beta q + \tfrac{1}{24}\alpha q + \tfrac{1}{120} q$$

ſeu

ſeu $\varepsilon = q^5 + 2q^4 + \frac{5}{4}q^3 + \frac{1}{4}q^2 + \frac{1}{120}q$ &

$\zeta = q^6 + \frac{5}{2}q^5 + \frac{13}{6}q^4 + \frac{3}{4}q^3 + \frac{31}{360}q^2 + \frac{1}{720}q$ &c.

ſeu hoc modo exprimantur:

$$\alpha = \frac{q}{1}$$

$$\beta = \frac{2qq + q}{1.\,2}$$

$$\gamma = \frac{6q^3 + 6q^2 + q}{1.\,2.\,3}$$

$$\delta = \frac{24q^4 + 36q^3 + 14q^2 + q}{1.\,2.\,3.\,4}$$

$$\varepsilon = \frac{120q^5 + 240q^4 + 150q^3 + 30q^2 + q}{1.\,2.\,3.\,4.\,5}$$

$$\zeta = \frac{720q^6 + 1800q^5 + 1560q^4 + 540q^3 + 62q^2 + q}{1.\,2.\,3.\,4.\,5.\,6}$$

$$\eta = \frac{5040q^7 + 15120q^6 + 16800q^5 + 8400q^4 + 1806q^3 + 126q^2 + q}{1.\,2.\,3.\,4.\,5.\,6.\,7} \quad \text{\&c.}$$

vbi quilibet coefficiens 16800 oritur, ſi ſumma binorum ſuperiorum $1560 + 1800$ per exponentem ipſius q, qui hic eſt 5, multiplicetur.

173. Reſtituamus autem loco q valorem $\frac{1}{p-1}$,

$$\alpha = \frac{1}{1(p-1)}$$

$$\beta = \frac{p+1}{1.2\,(p-1)^2}$$

$$\gamma = \frac{pp + 4p + 1}{1.2.3\,(p-1)^3}$$

 $\delta =$

$$\delta = \frac{p^3 + 11p^2 + 11p + 1}{1.\,2.\,3.\,4\,(p-1)^4}$$

$$\varepsilon = \frac{p^4 + 26p^3 + 66p^2 + 26p + 1}{1.\,2.\,3.\,4.\,5\,(p-1)^5}$$

$$\zeta = \frac{p^5 + 57p^4 + 302p^3 + 302p^2 + 57p + 1}{1.\,2.\,3.\,4.\,5.\,6\,(p-1)^6}$$

$$\eta = \frac{p^6 + 120p^5 + 1191p^4 + 2416p^3 + 1191p^2 + 120p + 1}{1.\,2.\,3.\,4.\,5.\,6.\,7\,(p-1)^7}$$

&c.

Lex harum quantitatum ita ſe habet, vt ſi ponatur terminus quicunque:

$$\frac{p^{n-2} + Ap^{n-3} + Bp^{n-4} + Cp^{n-5} + Dp^{n-6} + \&c.}{1.\,2.\,3\,\ldots\ldots\,(n-1)(p-1)^{n-1}}$$

futurum ſit:

$$A = 2^{n-1} - n$$

$$B = 3^{n-1} - n.2^{n-1} + \frac{n(n-1)}{1.\,2}$$

$$C = 4^{n-1} - n.3^{n-1} + \frac{n(n-1)}{1.\,2}2^{n-1} - \frac{n(n-1)(n-2)}{1.\,2.\,3}$$

$$D = 5^{n-1} - n.4^{n-1} + \frac{n(n-1)}{1.\,2}3^{n-1} - \frac{n(n-1)(n-2)}{1.\,2.\,3}2^{n-1} + \frac{n(n-1)(n-2)(n-3)}{1.\,2.\,3.\,4}$$

&c.

vnde iſti coefficientes α, β, γ, δ, &c. quousque libuerit, continuari poſſunt.

174. Quodſi vero legem, qua hi coefficientes inter ſe cohaerent, conſideremus, facile patet, eos ſeriem re-

recurrentem conſtituere, atque prodire ſi haec fractio euoluatur:

$$\frac{1}{1-\frac{u}{p-1}-\frac{u^2}{2(p-1)}-\frac{u^3}{6(p-1)}-\frac{u^4}{24(p-1)}-\&c.}$$

prodibit enim haec ſeries:

$$1+\alpha u+\beta u^2+\gamma u^3+\delta u^4+\varepsilon u^5+\zeta u^6+\&c.$$

Ponatur illa fractio $=V$, & cum ſit:

$$V=\frac{p-1}{p-1-u-\frac{u^2}{2}-\frac{u^3}{6}-\frac{u^4}{24}-\&c.}$$

erit $V=\frac{p-1}{p-e^u}$; vbi e eſt numerus cuius logarithmus hyperbolicus eſt $=1$.

Atque ſi valor ipſius V per ſeriem exprimatur ſecundum poteſtates ipſius u, orietur:

$$V=1+\alpha u+\beta u^2+\gamma u^3+\delta u^4+\varepsilon u^5+\zeta u^6+\&c.$$

cuius coefficientes α, β, γ, δ, &c. erunt ii ipſi, quorum in praeſenti negotio opus habemus. Iis igitur inventis erit:

$$S.p^x z=\frac{p^{x+1}}{p-1}\left(z-\frac{\alpha dz}{dx}+\frac{\beta ddz}{dx^2}-\frac{\gamma d^3z}{dx^3}+\frac{\delta d^4z}{dx^4}-\&c.\right)$$
$$\pm \text{Conſt.}$$

quae ergo expreſſio eſt terminus ſummatorius ſeriei huius:

$$ap+bp^2+cp^3+\ .\ .\ .\ .\ .\ +p^x z$$

cuius terminus generalis eſt $=p^x z$.

175. Quoniam inuenimus eſſe $V = \frac{p-1}{p-e^u}$, erit $e^u = \frac{pV-p+1}{V}$, & logarithmis ſumendis fiet $u = l(pV-p+1) - lV$, hincq; differentiando $du = \frac{(p-1)dV}{pV^2-(p-1)V}$ quocirca erit $pV^2 = (p-1)V + \frac{(p-1)dV}{du}$. Quoniam ergo eſt $V = 1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \&c.$ erit:

$$pV^2 = p + 2\alpha pu + 2\beta pu^2 + 2\gamma pu^3 + 2\delta pu^4 + 2\varepsilon pu^5 + \&c.$$
$$+ \alpha^2 pu^2 + 2\alpha\beta pu^3 + 2\alpha\gamma pu^4 + 2\alpha\delta pu^5 + \&c.$$
$$+ \beta\beta pu^4 + 2\beta\gamma pu^5 + \&c.$$

$$(p-1)V = (p-1) + \alpha(p-1)u + \beta(p-1)u^2 + \gamma(p-1)u^3 + \delta(p-1)u^4 + \varepsilon(p-1)u^5 + \&c.$$
$$\frac{(p-1)dV}{du} = (p-1)\alpha + 2(p-1)\beta u + 3(p-1)\gamma u^2 + 4(p-1)\delta u^3 + 5(p-1)\varepsilon u^4 + 6(p-1)\zeta u^5 + \&c.$$

quibus expreſſionibus inter ſe coaequatis reperietur:

$$(p-1)\alpha = 1$$
$$2(p-1)\beta = \alpha(p+1)$$
$$3(p-1)\gamma = \beta(p+1) + \alpha^2 p$$
$$4(p-1)\delta = \gamma(p+1) + 2\alpha\beta p$$
$$5(p-1)\varepsilon = \delta(p+1) + 2\alpha\gamma p + \beta\beta p$$
$$6(p-1)\zeta = \varepsilon(p+1) + 2\alpha\delta p + 2\beta\gamma p$$
$$7(p-1)\eta = \zeta(p+1) + 2\alpha\varepsilon p + 2\beta\delta p + \gamma\gamma p$$
&c.

ex

ex quibus formulis, ſi pro p datus numerus aſſumatur, valores coefficientium α, β, γ, δ, &c. facilius determinari poſſunt, quam ex lege primum inuenta.

176. Antequam ad caſus ſpeciales ratione valoris ipſius p deſcendamus, ponamus eſſe $z = x^n$, ita vt haec ſeries ſummari debeat:

$$s = p + 2^n p^2 + 3^n p^3 + 4^n p^4 + \ldots + x^n p^x$$

eritque per expreſſionem ante inuentam:

$$s = p^x \left(\frac{p}{p-1} . x^n - \frac{p}{(p-1)^2} n x^{n-1} + \frac{pp+p}{(p-1)^3} . \frac{n(n-1)}{1.\ 2} x^{x-2} \right.$$
$$\left. - \frac{(p^3+4p^2+p)}{(p-1)^4} . \frac{n(n-1)(n-2)}{1.\ 2.\ 3} x^{n-3} + \text{\&c.} \right)$$
$$\pm C, \text{ quae reddat } s = 0 \text{ ſi ponatur } x = 0.$$

Hinc ponendo pro n ſucceſſiue numeros 0, 1, 2, 3, 4, &c. erit:

$$S.x^0 p^x = p^x . \frac{p}{p-1} - \frac{p}{p-1}$$

$$S.x^1 p^x = p^x \left(\frac{px}{p-1} - \frac{p}{(p-1)^2} \right) + \frac{p}{(p-1)^2}$$

$$S.x^2 p^x = p^x \left(\frac{px^2}{p-1} - \frac{2px}{(p-1)^2} + \frac{p(p+1)}{(p-1)^3} \right) - \frac{p(p+1)}{(p-1)^3}$$

$$S.x^3 p^x = p^x \left(\frac{px^3}{p-1} - \frac{3px^2}{(p-1)^2} + \frac{3p(p+1)x}{(p-1)^3} - \frac{p(p^2+4p+1)}{(p-1)^4} \right)$$
$$+ \frac{p(p^2+4p+1)}{(p-1)^4}$$

$$S.x^4p^x = p^x\left(\frac{px^4}{p-1} - \frac{4px^3}{(p-1)^2} + \frac{6p(p+1)x^2}{(p-1)^3} - \frac{4p(p^2+4p+1)x}{(p-1)^4}\right.$$
$$\left. + \frac{p(p^3+11p^2+11p+1)}{(p-1)^5}\right) - \frac{p(p^3+11p^2+11p+1)}{(p-1)^5}$$

$$S.x^5p^x = \frac{p^{x+1}x^5}{p-1} - \frac{5p^{x+1}x^4}{(p-1)^2} + \frac{10(p+1)p^{x+1}x^3}{(p-1)^3}$$
$$- \frac{10(p^2+4p+1)p^{x+1}x^2}{(p-1)^4} + \frac{5(p^3+11p^2+11p+1)p^{x+1}x}{(p-1)^5}$$
$$- \frac{(p^4+26p^3+66p^2+26p+1)(p^{x+1}-p)}{(p-1)^6}$$

$$S.x^6p^x = \frac{p^{x+1}x^6}{p-1} - \frac{6p^{x+1}x^5}{(p-1)^2} + \frac{15(p+1)p^{x+1}x^4}{(p-1)^3}$$
$$- \frac{20(p^2+4p+1)p^{x+1}x^3}{(p-1)^4} + \frac{15(p^3+11p^2+11p+1)p^{x+1}x^2}{(p-1)^5}$$
$$- \frac{6(p^4+26p^3+66p^2+26p+1)p^{x+1}x}{(p-1)^6}$$
$$+ \frac{(p^5+57p^4+302p^3+302p^2+57p+1)(p^{x+1}-p)}{(p-1)^7}$$

&c.

177. Hinc intelligitur, quoties z fuerit functio rationalis integra ipſius x, toties ſeriei, cuius terminus generalis eſt $p^x z$, ſummam exhiberi poſſe; propterea quod differentialia ipſius z ſumendo, tandem ad euaneſcentia perueniatur. Ita ſi proponatur haec ſeries:

$$p + 3p^2 + 6p^3 + 10p^4 + \ldots + \frac{(xx+x)}{2}p^x,$$

ob

ob $z = \frac{xx+x}{2}$, & $\frac{dz}{dx} = x + \frac{1}{2}$; atque $\frac{ddz}{dx^2} = 1$;

erit terminus ſummatorius:

$$s = \frac{p^{x+1}}{p-1}\left(\tfrac{1}{2}xx + \tfrac{1}{2}x - \frac{(2x+1)}{2(p-1)} + \frac{(p+1)}{2(p-1)^2}\right) - \frac{p}{p-1}\left(\frac{p+1}{2(p-1)^2} - \frac{1}{2(p-1)}\right)$$

ſeu $s = p^{x+1}\left(\frac{xx}{2(p-1)} + \frac{(p-3)x}{2(p-1)^2} + \frac{1}{(p-1)^3}\right) - \frac{p}{(p-1)^3}$.

Sin autem z fuerit functio non rationalis integra, tum iſta termini ſummatorii expreſſio in infinitum excurret. Ita ſi ſit $z = \frac{1}{x}$, vt ſummanda ſit haec ſeries:

$$s = p + \frac{1}{2}p^2 + \frac{1}{3}p^3 + \frac{1}{4}p^4 + \ldots\ldots + \frac{1}{x}p^x,$$

ob

$$\frac{dz}{dx} = -\frac{1}{xx};\ \frac{ddz}{dx^2} = \frac{2}{x^3};\ \frac{d^3z}{dx^3} = -\frac{2.3}{x^4};\ \frac{d^4z}{dx^4} = \frac{2.3.4}{x^5};\ \text{\&c.}$$

prodibit terminus ſummatorius:

$$s = \frac{p^{x+1}}{p-1}\left(\frac{1}{x} + \frac{1}{(p-1)x^2} + \frac{p+1}{(p-1)^2x^3} + \frac{pp+4p+1}{(p-1)^3x^4} + \frac{p^3+11p^2+11p+1}{(p-1)^4x^5} + \text{\&c.}\right)$$

$$+ C.$$

Hoc ergo caſu conſtans C non ex caſu $x = 0$ definiri poteſt: ad eam igitur definiendam ponatur $x = 1$, & quia ſit $s = p$, erit:

$$C = p - \frac{pp}{p-1}\left(1 + \frac{1}{p-1} + \frac{p+1}{(p-1)^2} + \frac{pp+4p+1}{(p-1)^3} + \text{\&c.}\right)$$

178. Ex his perſpicuum eſt, niſi p determinatum numerum ſignificet, partem vtilitatis hinc ad ſummas ſerierum proxime exhibendas redundare. Primum autem patet pro p non poſſe ſcribi 1, propterea, quod omnes coefficientes α, β, γ, δ, &c. fierent infinite magni. Quare cum ſeries, quam nunc tractamus, abeat in eam quam ante iam ſumus contemplati, ſi ponatur $p = 1$, mirum eſt, quod ille caſus tanquam facillimus ex hoc erui nequeat. Tum vero quoque notabile eſt, quod caſu $p = 1$ ſummatio requirat integrale $\int z\,dx$, cum tamen generaliter ſumma ſine vllo integrali exhiberi queat. Sic igitur fit, vt dum omnes coefficientes α, β, γ, δ, &c. in infinitum excreſcunt, ſimul formula illa integralis invehatur. Hicque adeo caſus, quo $p = 1$, eſt ſolus, ad quem generalis expreſſio hic inuenta applicari nequeat. Neque vero hoc caſu generalis forma a vero recedere cenſenda eſt; nam etſi ſinguli termini fiunt infiniti, tamen reuera omnia infinita ſe deſtruunt, reſtatque quantitas finita ſummae aequalis, & congruens cum ea, quae per priorem methodum inuenitur, quod infra fuſius ſumus declaraturi.

179. Sit igitur $p = -1$, atque ſigna in ſerie ſummanda alternatim ſe excipient:

$$\begin{matrix} 1 & 2 & 3 & 4 & & x \\ -a & +b & -c & +d & \ldots\ldots & \pm z \end{matrix}$$

vbi z erit affirmatiuum ſi x fuerit numerus par, negatiuum autem, ſi x ſit numerus impar. Poſito ergo

—

$$-a+b-c+d \ldots \pm z = s, \text{ erit}$$

$$s = \frac{\pm 1}{2}\left(z - \frac{\alpha dz}{dx} + \frac{\beta ddz}{dx^2} - \frac{\gamma d^3 z}{dx^3} + \frac{\delta d^4 z}{dx^4} - \&c.\right) + C.$$

vbi ſignorum ambiguorum ſuperius valet, ſi x ſit numerus par, contra vero ſi x ſit numerus impar. Mutandis ergo ſignis erit:

$$a - b + c - d + e - f + \ldots \mp z =$$

$$\mp \frac{1}{2}\left(z - \frac{\alpha dz}{dx} + \frac{\beta ddz}{dx^2} - \frac{\gamma d^3 z}{dx^3} + \frac{\delta d^4 z}{dx^4} - \&c.\right) + C.$$

vbi ſignorum ambiguitas eandem ſequitur legem.

180. Hoc caſu coefficientes α, β, γ, δ, ε, ζ, &c. inueniri poſſunt ex valoribus ante traditis ponendo vbique $p = -1$. Facilius autem eruentur ex formulis generalibus §. 175. datis, ex quibus ſimul perſpicietur alternos iſtos coefficientes euaneſcere. Facto enim $p = -1$ iſtae formulae abibunt in

$$\begin{aligned}
-\ \alpha &= 1 \\
-\ 4\beta &= 0 \\
-\ 6\gamma &= 0 - \alpha^2 \\
-\ 8\delta &= 0 - 2\alpha\beta \\
-\ 10\varepsilon &= 0 - 2\alpha\gamma - \beta\beta \\
-\ 12\zeta &= 0 - 2\alpha\delta - 2\beta\gamma \\
&\&c.
\end{aligned}$$

 vnde

vnde cum ſit $\beta = 0$, erit quoque $\delta = 0$, porroque $\zeta = 0$, $\theta = 0$, &c. & reliquae litterae ita determinabuntur, vt ſit:

$$\alpha = -\frac{1}{2}$$

$$\gamma = \frac{\alpha^2}{6}$$

$$\varepsilon = \frac{2\alpha\gamma}{10}$$

$$\eta = \frac{2\alpha\varepsilon + \gamma\gamma}{14}$$

$$\iota = \frac{2\alpha\eta + 2\gamma\varepsilon}{18} \quad \&c.$$

181. Quo iſte calculus commodius abſolui poſſit introducamus nouas litteras ſitque:

$$\alpha = -\frac{A}{1.2}$$

$$\gamma = \frac{B}{1.2.3.4}$$

$$\varepsilon = -\frac{C}{1.2.3.4.5.6}$$

$$\eta = \frac{D}{1.2.3\ldots8}$$

$$\iota = -\frac{E}{1.2.3\ldots10} \quad \&c.$$

Eritque ſumma ante exhibita:

$$= \frac{1}{2}\left(z + \frac{A\,dz}{1.2\,dx} - \frac{B\,d^3z}{1.2.3.4\,dx^3} + \frac{C\,d^5z}{1.2\ldots6\,dx^5} - \frac{D\,d^7z}{1.2\ldots8\,dx^7} + \&c.\right) + C.$$

Co-

Coefficientes vero ex ſequentibus formulis definientur:

$$A = 1$$
$$3B = \frac{4.3}{1.2}\frac{AA}{2}$$
$$5C = \frac{6.5}{1.2} AB$$
$$7D = \frac{8.7}{1.2} AC + \frac{8.7.6.5}{1.2.3.4}\frac{BB}{2}$$
$$9E = \frac{10.9}{1.2} AD + \frac{10.9.8.7}{1.2.3.4}.BC$$
$$11F = \frac{12.11}{1.2} AE + \frac{12.11.10.9}{1.2.3.4}.BD + \frac{12.11.10.9.8.7}{1.2.3.4.5.6}\frac{CC}{2}$$
&c.

quae hoc modo facilius atque ad calculum accommodatius repraeſentari poſſunt:

$$A = 1$$
$$B = 2.\frac{AA}{2}$$
$$C = 3.AB$$
$$D = 4.AC + 4.\frac{6.5}{3.4}.\frac{BB}{2}$$
$$E = 5.AD + 5.\frac{8.7}{3.4}.BC$$
$$F = 6.AE + 6.\frac{10.9}{3.4}.BD + 6.\frac{10.9.8.7}{3.4.5.6}.\frac{CC}{2}$$
$$G = 7.AF + 7.\frac{12.11}{3.4}.BE + 7.\frac{12.11.10.9}{3.4.5.6}.CD$$
&c.

Hinc

Hinc igitur calculo inſtituto reperietur:

$$
\begin{aligned}
A &= 1 \\
B &= 1 \\
C &= 3 \\
D &= 17 \\
E &= 155 = 5 \cdot 31 \\
F &= 2073 = 691 \cdot 3 \\
G &= 38227 = 7 \cdot 5461 = 7 \cdot \frac{127 \cdot 129}{3} \\
H &= 929569 = 3617 \cdot 257 \\
I &= 28820619 = 43867 \cdot 9 \cdot 73 \quad \&c.
\end{aligned}
$$

182. Si hos numeros attentius perpendamus, ex factoribus 691, 3617, 43867, facile concludere licet, hos numeros cum ſupra exhibitis Bernoullianis nexum habere, indeque determinari poſſe. Hanc igitur relationem inueſtiganti mox patebit hos numeros ex Bernoullianis $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, &c. ſequenti modo formari poſſe:

$$
\begin{aligned}
A &= 2 \cdot 1 \cdot 3\, \mathfrak{A} = 2(2^2-1)\,\mathfrak{A} \\
B &= 2 \cdot 3 \cdot 5\, \mathfrak{B} = 2(2^4-1)\,\mathfrak{B} \\
C &= 2 \cdot 7 \cdot 9\, \mathfrak{C} = 2(2^6-1)\,\mathfrak{C} \\
D &= 2 \cdot 15 \cdot 17\, \mathfrak{D} = 2(2^8-1)\,\mathfrak{D} \\
E &= 2 \cdot 31 \cdot 33\, \mathfrak{E} = 2(2^{10}-1)\,\mathfrak{E} \\
F &= 2 \cdot 63 \cdot 65\, \mathfrak{F} = 2(2^{12}-1)\,\mathfrak{F} \\
G &= 2 \cdot 127 \cdot 129\, \mathfrak{G} = 2(2^{14}-1)\,\mathfrak{G} \\
H &= 2 \cdot 255 \cdot 257\, \mathfrak{H} = 2(2^{16}-1)\,\mathfrak{H} \\
&\&c.
\end{aligned}
$$

Cum

Cum igitur numeri Bernoulliani ſint fracti, coefficientes vero noſtri integri, patet hos factores ſemper tollere fractiones; eruntque ergo:

$A = 1$
$B = 1$
$C = 3$
$D = 17$
$E = 5.31 = 155$
$F = 3.691 = 2073$
$G = 7.43.127 = 38227$
$H = 257.3617 = 929569$
$I = 9.73.43867 = 28820619$
$K = 5.31.41.174611 = 1109652905$
$L = 89.683.854513 = 51943281731$
$M = 3.4097.236364091 = 2905151042481$
$N = 2731.8191.8553103 = 191329672483963$
&c.

Ex his ergo numeris integris viciſſim numeri Bernoulliani inueniri poterunt.

183. Adhibendo igitur numeros Bernoullianos ſeriei propoſitae:

$$\overset{1}{a} - \overset{2}{b} + \overset{3}{c} - \overset{4}{d} + \overset{5}{e} - \ldots \mp \overset{x}{z},$$

ſumma erit:

$$\mp\left(\frac{1}{2}z + \frac{(2^2-1)\mathfrak{A}dz}{1.2\,dx} - \frac{(2^4-1)\mathfrak{B}d^3z}{1.2.3.4\,dx^3} + \frac{(2^6-1)\mathfrak{C}d^5z}{1.2\ldots6\,dx^5} - \frac{(2^8-1)\mathfrak{D}d^7z}{1.2\ldots8\,dx^7} + \&c.\right)$$

$$+ \text{Conſt.}$$

Hinc autem perſpicitur iſtos numeros non caſu in hanc expreſſionem ingredi; quemadmodum enim ſeries propoſita oritur, ſi ab iſta:

$$a + b + c + d + \ldots\ldots + z,$$

vbi omnes termini ſignum habent + ſubtrahatur ſumma alternorum $b + d + f +$ &c. bis ſumta; ita quoque expreſſio inuenta in duas reſolui poteſt partes, quarum altera eſt ſumma omnium terminorum ſigno + affectorum, quae erit:

$$\int z\,dx + \frac{1}{2}z + \frac{\mathfrak{A}\,dz}{1.2\,dx} - \frac{\mathfrak{B}\,d^3z}{1.2.3.4\,dx^3} + \frac{\mathfrak{C}\,d^5z}{1.2\ldots6\,dx^5} - \text{\&c.}$$

Summa vero alternorum pari modo inuenietur, quo ſupra vſi ſumus. Cum enim vltimus terminus ſit z indici x reſpondens, antecedens indici $x-2$ reſpondens erit:

$$z - \frac{2\,dz}{dx} + \frac{2^2\,ddz}{1.2\,dx^2} - \frac{2^3\,d^3z}{1.2.3\,dx^3} + \frac{2^4\,d^4z}{1.2.3.4\,dx^4} - \text{\&c.}$$

quae forma ex illa, qua ante terminus antecedens exprimebatur; oritur, ſi loco x ſcribatur $\frac{x}{2}$. Habebitur ergo ſumma alternorum, ſi in ſumma omnium vbique loco x ſcribatur $\frac{x}{2}$, quae propterea erit:

$$\frac{1}{2}\int z\,dx + \frac{1}{2}z + \frac{2\,\mathfrak{A}\,dz}{1.2\,dx} - \frac{2^3\,\mathfrak{B}\,d^3z}{1.2.3.4\,dx^3} + \frac{2^5\,\mathfrak{C}\,d^5z}{1.2\ldots6\,dx^5} - \text{\&c.}$$

cuius duplum ſi a ſumma praecedente ſubtrahatur, exiſtente x numero pari, vel ſi praecedens ſumma a duplo huius ſi x eſt numerus impar ſubtrahatur, reſiduum oſtendet ſummam ſeriei:

$a -$

$$\overset{1}{a}-\overset{2}{b}+\overset{3}{c}-\overset{4}{d}+\overset{5}{e}\;.\;.\;.\;.\;.\;.\;.\;\mp\overset{x}{z}$$

quae ergo erit:

$$\mp\left(\frac{1}{2}z+\frac{(2^2-1)\mathfrak{A}dz}{1.2dx}-\frac{(2^4-1)\mathfrak{B}d^3z}{1.2.3.4dx^3}+\&c.\right)+C.$$

quae eſt eadem expreſſio, quam modo inueneramus.

184. Sumatur pro z poteſtas ipſius x, nempe x^n, vt reperiatur ſumma ſeriei:

$$1-2^n+3^n-4^n+\;.\;.\;.\;.\;.\;\mp x^n.$$

Ob $\frac{dz}{1dx}=\frac{n}{1}x^{n-1}$; $\frac{d^3z}{1.2.3dx^3}=\frac{n(n-1)(n-2)}{1.\;2.\;3}x^{n-3}$; &c.

erit adhibendis coefficientibus A, B, C, D, &c. ſumma quaeſita:

$$\mp\frac{1}{2}\left(x^n+\frac{A}{2}nx^{n-1}-\frac{B}{4}\frac{n(n-1)(n-2)}{1.\;2.\;3}x^{n-3}+\frac{C}{6}\frac{n(n-1)(n-2)(n-3)(n-4)}{1.\;2.\;3.\;4.\;5}x^{n-5}\right.$$
$$\left.-\frac{D}{8}\frac{n(n-1)\ldots.(n-6)}{1.\;2\ldots\ldots 7}x^{n-7}+\&c.\right)+\text{Conſt.}$$

vbi ſignum ſuperius valet ſi ſit x numerus par, inferius vero ſi impar. Conſtans autem ita definiri debet, vt ſumma euaneſcat, ſi $x=0$, quo caſu ſignum ſuperius valet. Pro n ergo ſucceſſiue numeros 0, 1, 2, 3, &c. ſubſtituendo ſequentes prodibunt ſummationes:

I. $1-1+1-1+\;.\;.\;.\;.\;.\;\mp 1=\mp\frac{1}{2}(1)+\frac{1}{2}$

ſcilicet ſi numerus terminorum fuerit par, ſumma erit $=0$, ſin impar erit $=+1$.

II. $1 - 2 + 3 - 4 + \ldots \mp x = \mp \frac{1}{2}(x + \frac{1}{2}) + \frac{1}{4}$
ſcilicet ſi numerus terminorum ſit par, ſumma erit $= -\frac{1}{2}x$
& pro numero terminorum impari $= + \frac{1}{2}x + \frac{1}{2}$.

III. $1 - 2^2 + 3^2 - 4^2 + \ldots \mp x^2 = \mp \frac{1}{2}(x^2 + x)$
ſcilicet pro pari numero $= -\frac{1}{2}xx - \frac{1}{2}x$
& pro impari numero $= + \frac{1}{2}xx + \frac{1}{2}x$

IV. $1 - 2^3 + 3^3 - 4^3 + \ldots \mp x^3 = \mp \frac{1}{2}(x^3 + \frac{3}{2}xx - \frac{1}{4}) - \frac{1}{8}$
ſcilicet pro pari $= -\frac{1}{2}x^3 - \frac{3}{4}x^2$
& pro impari $= \frac{1}{2}x^3 + \frac{3}{4}x^2 - \frac{1}{4}$.

V. $1 - 2^4 + 3^4 - 4^4 + \ldots \mp x^4 = \mp \frac{1}{2}(x^4 + 2x^3 - x)$
ſcilicet pro numero pari $= -\frac{1}{2}x^4 - x^3 + \frac{1}{2}x$
& pro numero impari $= \frac{1}{2}x^4 + x^3 - \frac{1}{2}x$
&c.

185. Apparet ergo in poteſtatibus paribus praeter $n = 0$, conſtantem adiiciendam euaneſcere, hisque caſibus ſummam terminorum numero ſiue parium ſiue imparium tantum ratione ſigni diſcrepare. Quodſi ergo x fuerit numerus infinitus, quoniam is eſt neque par neque impar, haec conſideratio ceſſare debet, ac propterea in ſumma termini ambigui ſunt reiiciendi: vnde ſequitur huiusmodi ſerierum in infinitum continuatarum ſummam exprimi per ſolam quantitatem conſtantem adiiciendam.

Hanc-

Hancobrem erit:

$$1 - 1 + 1 - 1 + \&c. \text{ in infinitum} = \frac{1}{2}$$

$$1 - 2 + 3 - 4 + \&c. \quad . \quad . \quad . = \frac{A}{4} = + \frac{(2^2-1)\mathfrak{A}}{2}$$

$$1 - 2^2 + 3^2 - 4^2 + \&c. \quad . \quad . \quad . = 0$$

$$1 - 2^3 + 3^3 - 4^3 + \&c. \quad . \quad . \quad . = - \frac{B}{8} = - \frac{(2^4-1)\mathfrak{B}}{4}$$

$$1 - 2^4 + 3^4 - 4^4 + \&c. \quad . \quad . \quad . = 0$$

$$1 - 2^5 + 3^5 - 4^5 + \&c. \quad . \quad . \quad . = \frac{C}{12} = + \frac{(2^6-1)\mathfrak{C}}{6}$$

$$1 - 2^6 + 3^6 - 4^6 + \&c. \quad . \quad . \quad . = 0$$

$$1 - 2^7 + 3^7 - 4^7 + \&c. \quad . \quad . \quad . = - \frac{D}{16} = - \frac{(2^8-1)\mathfrak{D}}{8}$$

&c.

Quae eaedem ſummae per methodum ſupra traditam ſeries, in quibus ſigna + & — alternantur, ſummandi inueniuntur.

186. Si pro n ſtatuantur numeri negatiui, expreſſio ſummae in infinitum excurret. Sit $n = -1$, erit ſumma ſeriei:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \quad . \quad . \quad . \quad . \mp \frac{1}{x} =$$

$$\mp \frac{1}{2}\left(\frac{1}{x} - \frac{A}{2x^2} + \frac{B}{4x^4} - \frac{C}{6x^6} + \frac{D}{8x^8} - \&c.\right) + \text{Conſt.}$$

Hic autem quia conſtans non ex caſu $x = 0$ definiri poteſt, ex alio caſu erit definienda. Ponatur $x = 1$,

 at-

atque ob ſummam $= 1$ & ſignum inferius erit:

$$\text{Conſt.} = 1 - \frac{1}{2}\left(\frac{1}{1} - \frac{A}{2} + \frac{B}{4} - \frac{C}{6} + \&c.\right) \quad \text{ſeu}$$

$$\text{Conſt.} = \frac{1}{2} + \frac{A}{4} - \frac{B}{8} + \frac{C}{12} - \frac{D}{16} + \&c.$$

Vel ponatur $x = 2$, ob ſummam $= \frac{1}{2}$, & ſignum ſuperius reperietur:

$$\text{Conſt.} = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} - \frac{A}{2.2^2} + \frac{B}{4.2^4} - \frac{C}{6.2^6} + \&c.\right)$$

$$\text{ſeu Conſt.} = \frac{3}{4} - \frac{A}{4.2^2} + \frac{B}{8.2^4} - \frac{C}{12.2^6} + \frac{D}{16.2^8} - \&c.$$

ſin autem ponatur $x = 4$, erit:

$$\text{Conſt.} = \frac{17}{24} - \frac{A}{4.4^2} + \frac{B}{8.4^4} - \frac{C}{12.4^6} + \frac{D}{16.4^8} - \&c.$$

Vtcunque autem conſtans definiatur, idem prodibit valor, qui ſimul ſummam ſeriei in infinitum continuatae, quae eſt $= l2$, indicabit.

187. Ceterum ex his nouis numeris A, B, C, D, E, &c. ſummae ſerierum poteſtatum reciprocarum parium, in quibus tantum numeri impares occurrunt, commode ſummari poterunt. Si enim ponatur:

$$1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} + \&c. = s \quad \text{erit}$$

$$\frac{1}{2^{2n}} + \frac{1}{4^{2n}} + \frac{1}{6^{2n}} + \&c. = \frac{s}{2^{2n}},$$

quae ab illa ſubtracta relinquet:

$$1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \&c. = \frac{(2^{2n}-1)s}{2^{2n}}.$$

Cum

Cum igitur valores ipſius s pro ſingulis numeris n iam ſupra exhibuerimus: (125), erit:

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \&c. = \frac{A}{1.2} \cdot \frac{\pi^2}{4}$$

$$1 + \frac{1}{3^4} + \frac{1}{3^4} + \frac{1}{7^4} + \&c. = \frac{B}{1.2.3.4} \cdot \frac{\pi^4}{4}$$

$$1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \&c. = \frac{C}{1.2.3 \ldots 6} \cdot \frac{\pi^6}{4}$$

$$1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \&c. = \frac{D}{1.2.3 \ldots 8} \cdot \frac{\pi^8}{4}$$

$$1 + \frac{1}{3^{10}} + \frac{1}{5^{10}} + \frac{1}{7^{10}} + \&c. = \frac{E}{1.2.3 \ldots 10} \cdot \frac{\pi^{10}}{4}$$

&c.

Sin autem omnes numeri ingrediantur, ſignaque alternentur quia erit:

$$1 - \frac{1}{2^{2n}} + \frac{1}{3^{2n}} - \frac{1}{4^{2n}} + \&c. = \frac{(2^{2n}-1)\,s - s}{2^{2n}}$$

habebitur:

$$1 + \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \&c. = \frac{(A-2\mathfrak{A})}{1.2} \cdot \frac{\pi^2}{4} = \frac{(2-1)\mathfrak{A}}{1.2} \cdot \pi^2$$

$$1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \&c. = \frac{(B-2\mathfrak{B})}{1.2.3.4} \cdot \frac{\pi^4}{4} = \frac{(2^3-1)\mathfrak{B}}{1.2.3.4} \cdot \pi^4$$

$$1 + \frac{1}{2^6} + \frac{1}{3^6} - \frac{1}{4^6} + \frac{1}{5^6} - \&c. = \frac{(C-2\mathfrak{C})}{1.2 \ldots 6} \cdot \frac{\pi^6}{4} = \frac{(2^5-1)\mathfrak{C}}{1.2 \ldots 6} \cdot \pi^6$$

$$1 + \frac{1}{2^8} + \frac{1}{3^8} - \frac{1}{4^8} + \frac{1}{5^8} - \&c. = \frac{(D-2\mathfrak{D})}{1.2 \ldots 8} \cdot \frac{\pi^8}{4} = \frac{(2^7-1)\mathfrak{D}}{1.2 \ldots 8} \cdot \pi^8$$

$$1 - \frac{1}{2^{10}} + \frac{1}{3^{10}} - \frac{1}{4^{10}} + \frac{1}{5^{10}} - \&c. = \frac{(E-2\mathfrak{E})}{1.2 \ldots 10} \cdot \frac{\pi^{10}}{4} = \frac{(2^9-1)\mathfrak{E}}{1.2 \ldots 10} \cdot \pi^{10}$$

&c.

188.

188. Quemadmodum hactenus ſeriem ſumus contemplati, cuius termini erant producta ex terminis progreſſionis geometricae p, p^2, p^3, &c. & ex terminis ſeriei cuiuscunque a, b, c, &c. ita poterimus ſimili ratione proſequi ſeriem, cuius termini ſint producta ex terminis duarum quarumcunque ſerierum, quarum altera tanquam cognita aſſumatur. Sit ſeries cognita:

$$\overset{1}{A} + \overset{2}{B} + \overset{3}{C} + \ldots + \overset{x}{Z}$$ altera vero incognita

$$a + b + c + \ldots + z$$

atque quaeratur ſumma huius ſeriei:

$$Aa + Bb + Cc + \ldots + Zz$$

quae ponatur $= Zs$. Sit in ſerie cognita terminus penultimus $= Y$, atque poſito $x - 1$ loco x expreſſio ſummae $S.Zs$ abibit in

$$Y\left(s - \frac{ds}{dx} + \frac{dds}{2dx^2} - \frac{d^3s}{6dx^3} + \frac{d^4s}{24dx^4} - \text{\&c.}\right)$$

Quae cum exprimat ſummam ſeriei Zs termino vltimo Zz multatae, erit:

$$Zs - Zz = Ys - \frac{Yds}{dx} + \frac{Ydds}{2dx^2} - \frac{Y^3d^3s}{6dx^3} + \text{\&c.}$$

quae aequatio continet relationem, qua ſumma Zs pendet ab Y, Z & z.

189. Ad hanc aequationem reſoluendam negligantur primum termini differentiales, eritque $s = \frac{Zz}{Z-Y}$, ponatur iſte valor $\frac{Zz}{Z-Y} = P^1$, ſit que reuera $s = P^1 + p$, quo

quo valore in aequatione ſubſtituto fiet:

$$(Z-Y)p = -\frac{Y\,dP^{I}}{dx} + \frac{Y\,ddP^{I}}{2\,dx^2} - \&c.$$
$$\qquad\qquad - \frac{Y\,dp}{dx} + \frac{Y\,ddp}{2\,dx^2} - \&c.$$

addatur vtrinque YP^{I}, & cum $P^{I} - \frac{dP^{I}}{dx} + \frac{ddP^{I}}{2\,dP^{I}} - \&c.$ ſit valor ipſius P^{I}, qui prodit ſi loco x ponatur $x-1$, ſit iſte valor $= P$, eritque

$$(Z-Y)p + YP^{I} = YP - \frac{Y\,dp}{dx} + \frac{Y\,ddp}{2\,dx^2} - \&c.$$

vnde neglectis differentialibus erit: $p = \frac{Y(P-P^{I})}{Z-Y}$.

Ponatur $\frac{Y(P-P^{I})}{Z-Y} = Q^{I}$, ſitque $p = Q^{I} + q$; fiet

$$(Z-Y)q = -\frac{Y(dQ^{I}+dq)}{dx} + \frac{Y(ddQ^{I}+ddq)}{2\,dx^2} - \&c.$$

poſitoque Q pro valore ipſius Q^{I}, quem induit ſi loco x ſcribatur $x-1$, erit:

$$(Z-Y)q + YQ^{I} = YQ - \frac{Y\,dq}{dx} + \frac{Y\,ddq}{2\,dx^2} - \&c.$$

vnde neglectis differentialibus fit $q = \frac{Y(Q-Q^{I})}{Z-Y}$.

Ponatur $\frac{Y(Q-Q^{I})}{Z-Y} = R^{I}$, ſitque reuera $q = R^{I} + r$; ac ſimili modo reperitur $r = \frac{Y(R-R^{I})}{Z-Y}$; ſicque procedendo erit ſumma quaeſita:

$$Zs = Z(P^{I} + Q^{I} + R^{I} + \&c.).$$

190. Propoſita ergo ſerie quacunque:

$$Aa + Bb + Cc + \ldots\ldots + Yy + Zz$$

eius ſumma ſequenti modo definietur:

Ponatur poſito $x - 1$ loco x

$\frac{Zz}{Z-Y} = P^{I}$; abeatque P^{I} in P

$\frac{Y(P-P^{I})}{Z-Y} = Q^{I}$; abeatque Q^{I} in Q

$\frac{Y(Q-Q^{I})}{Z-Y} = R^{I}$; abeatque R^{I} in R

$\frac{Y(R-R^{I})}{Z-Y} = S^{I}$; abeatque S^{I} in S

&c.

His valoribus inuentis erit ſumma ſeriei $=$

$$ZP^{I} + ZQ^{I} + ZR^{I} + ZS^{I} + \&c.$$

$+$ Conſtante, quae reddat ſummam $= 0$, ſi ponatur $x = 0$, ſeu quod eodem redit, quae efficiat, vt cuipiam caſui ſatisfiat.

191. Formula haec, quia nullis differentialibus eſt implicata, in plurimis caſibus facillime adhibetur, atque etiam veram ſummam ſaepenumero exhibet. Sic ſi proponatur haec ſeries:

$$p + 4p^2 + 9p^3 + 16p^4 + \ldots\ldots + x^2p^x$$

fiat $Z = p^x$ & $z = x^2$, erit $Y = p^{x-1}$, atque $\frac{Z}{Z-Y} = \frac{p}{p-1}$, & $\frac{Y}{Z-Y} = \frac{1}{p-1}$. Hinc fiet

$P^{I} =$

$$P^I = \frac{px^2}{p-1}; \quad P = \frac{pxx - 2px + p}{p-1}$$

$$Q^I = -\frac{2px+p}{(p-1)^2}; \quad Q = -\frac{2px+3p}{(p-1)^2}$$

$$R^I = \frac{2p}{(p-1)^3}; \quad R = \frac{2p}{(p-1)^3}$$

$S^I = 0$, & reliqui euaneſcunt omnes: vnde erit ſumma =

$$p^x\left(\frac{px^2}{p-1} - \frac{2px+p}{(p-1)^2} + \frac{2p}{(p-1)^3}\right) - \frac{p}{(p-1)^2} - \frac{2p}{(p-1)^3}$$

$$= p^{x+1}\left(\frac{x^2}{p-1} - \frac{2x}{(p-1)^2} + \frac{p+1}{(p-1)^3}\right) - \frac{p-1}{(p-1)^3},$$

quemadmodum iam ſupra inuenimus.

192. Simili modo, quo ad hanc ſummae expreſſionem peruenimus, aliam inuenire poterimus expreſſionem, ſi ſeries propoſita non ex duabus aliis ſit compoſita: quae illis potiſſimum caſibus in vſum vocari poterit, cum in praecedente expreſſione ad denominatores euaneſcentes peruenitur. Sit igitur propoſita haec ſeries:

$$s = \overset{1}{a} + \overset{2}{b} + \overset{3}{c} + \overset{4}{d} + \cdots + \overset{x}{z}$$

quoniam poſito $x-1$ loco x, ſumma vltimo termino truncatur, erit:

$$s - z = s - \frac{ds}{dx} + \frac{dds}{2dx^2} - \frac{d^3s}{6dx^3} + \frac{d^4s}{24dx^4} - \&c.$$

ſeu $$z = \frac{ds}{dx} - \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} - \frac{d^4s}{24dx^4} + \&c.$$

 Quia

Quia hic ipſa ſumma s non occurrit, negligantur differentialia altiora, fietque $s = \int z\,dx$, ponatur $\int z\,dx = P^{I}$, cuius valor abeat in P ſi pro x ſcribatur $x - 1$: ſitque reuera $s = P^{I} + p$, erit:

$$z = \frac{dP^{I}}{dx} - \frac{ddP^{I}}{2\,dx^2} + \&c. + \frac{dp}{dx} - \frac{ddp}{2\,dx^2} + \&c.$$

quia eſt $P = P^{I} - \frac{dP^{I}}{dx} + \frac{ddP^{I}}{2\,dx^2} - \&c.$

erit $z - P^{I} + P = \frac{dp}{dx} - \frac{ddp}{2\,dx^2} + \&c.$ vnde fit $p = \int(z - P^{I} + P)dx$. Si porro ponatur $\int(z - P^{I} + P)dx = Q^{I}$, hicque valor abeat in Q poſito $x - 1$ loco x, fit $\int(z - P^{I} + P - Q^{I} + Q)\,dx = R^{I} = Q^{I} - \int(Q^{I} - Q)\,dx$ porro $R^{I} - \int(R^{I} - R)\,dx = S^{I}$; &c. erit ſumma quaeſita:

$$s = P^{I} + Q^{I} + R^{I} + S^{I} + \&c. + Conſt.$$

qua vni caſui ſatisfiat.

193. Mutatis aliquantum litteris iſta ſummatio huc redit. Propoſita ſerie ſummanda:

$$s = \overset{1}{a} + \overset{2}{b} + \overset{3}{c} + \overset{4}{d} + \quad . \quad . \quad . \quad . \quad + \overset{x}{z}$$

ponatur	*poſito* $x - 1$	*loco* x		
$\int z\,dx = P$	abeatque P	in	p	
$P - \int(P - p)\,dx = Q$	abeatque Q	in	q	
$Q - \int(Q - q)\,dx = R$	abeatque R	in	r	&c.

quibus valoribus inuentis erit ſumma quaeſita:

$$s = P + Q + R + S + \&c.$$

hac-

haecque expreſſio expedite oſtendit ſummam, ſi formulae iſtae integrales exhiberi queant. Sit, vt vſum eius exemplo illuſtremus, $z = xx + x$ eritque

$$P = \tfrac{1}{3}x^3 + \tfrac{1}{2}xx \;;\quad p = \tfrac{1}{3}x^3 - \tfrac{1}{2}xx + \tfrac{1}{6}$$

$$P - p = xx - \tfrac{1}{6} \;\&\; \int(P - p)\,dx = \tfrac{1}{3}x^3 - \tfrac{1}{6}x$$

$$Q = \tfrac{1}{2}xx + \tfrac{1}{6}x \;;\quad q = \tfrac{1}{2}xx - \tfrac{5}{6}x + \tfrac{1}{3}$$

$$Q - q = x - \tfrac{1}{3} \;\&\; \int(Q - q)\,dx = \tfrac{1}{2}xx - \tfrac{1}{3}x$$

$$R = \tfrac{1}{2}x \;;\quad r = \tfrac{1}{2}x - \tfrac{1}{2}$$

$$R - r = \tfrac{1}{2} \;;\quad \int(R - r)\,dx = \tfrac{1}{2}x$$

$S = 0$, reliquique valores euaneſcunt. Quare ſumma quaeſita erit:

$$\tfrac{1}{3}x^3 + \tfrac{1}{2}xx + \tfrac{1}{2}xx + \tfrac{1}{6}x = \tfrac{1}{3}x^3 + xx + \tfrac{2}{3}x = \tfrac{1}{3}x(x+1)(x+2)$$
$$+ \tfrac{1}{2}x.$$

Hocque ergo modo omnium ſerierum, quarum termini generales ſunt functiones rationales integrae ipſius x, ſummae ope integrationum continuarum inueniri poſſunt. Ex quibus facile perſpicitur, quam amplum occupet campum doctrina de ſummatione ſerierum, neque omnibus methodis, quae tum habentur tum adhuc excogitari poſſunt, capiendis plura volumina ſufficere.

194. Hactenus ſummas ſerierum inueſtigauimus a termino primo vsque ad eum cuius index eſt x, quibus cognitis ponendo $x = \infty$ ipſius ſeriei in infinitum continuatae ſumma innoteſcet. Saepenumero autem hoc expeditius praeſtatur, ſi non ſumma terminorum a primo

vsque ad eum cuius index eſt x, ſed ſumma omnium terminorum ab iſto, cuius index eſt x, in infinitum vsque quaeratur, hocque caſu imprimis expreſſiones vltimae fiunt tractabiliores. Sit igitur propoſita ſeries cuius terminus generalis ſeu indici x reſpondens ſit $= z$, ſequens indici $x+1$ reſpondens ſit $= z^{I}$, huncque vltra ſequentes ſint z^{II}, z^{III}, &c. quaeraturque ſumma huius ſeriei infinitae:

$$x,\quad x+1,\quad x+2,\quad x+3,\quad \&c.$$

$$s = z + z^{I} + z^{II} + z^{III} + \&c. \text{ in infinitum.}$$

Haec igitur ſumma s erit functio ipſius x, in qua ſi ponatur $x+1$ loco x, orietur ſumma prior termino z truncata. Cum ergo hac mutatione s abeat in $s + \frac{ds}{dx} + \frac{dds}{2dx^2} + \&c.$ erit:

$$s - z = s + \frac{ds}{dx} + \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} + \frac{d^4s}{24dx^4} + \frac{d^5s}{120dx^5} + \&c.$$

$$\text{ſeu } 0 = z + \frac{ds}{dx} + \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} + \frac{d^4s}{24dx^4} + \frac{d^5s}{120dx^5} + \&c.$$

195. Si nunc vt ante ratiocinium inſtituamus, fiet neglectis differentialibus ſuperioribus, $s = C - \int z\,dx$. Ponatur ergo $\int z\,dx = P$, ſitque reuera $s = C - P + p$, erit

$$0 = z - \frac{dP}{dx} - \frac{ddP}{2dx^2} - \frac{d^3P}{6dx^3} - \&c.$$
$$+ \frac{dp}{dx} + \frac{ddp}{2dx^2} + \frac{d^3p}{6dx^3} + \&c.$$

Abe-

Abeat P in P^{I}, ſi loco x ponatur $x+1$, eritque:

$$0 = z + P - P^{I} + \frac{dp}{dx} + \frac{ddp}{2\,dx^2} + \frac{d^3p}{6\,dx^3} + \&c.$$

Hinc neglectis differentialibus altioribus fiet:

$p = \int(P^{I} - P)\,dx - P$. Statuatur $\int(P^{I} - P)\,dx - P = -Q$, ſitque $p = -Q + q$, erit:

$$0 = z + P - P^{I} - \frac{dQ}{dx} - \frac{ddQ}{2\,dx^2} - \&c. + \frac{dq}{dx} + \frac{ddq}{2\,dx^2} + \&c.$$

Abeat Q in Q^{I} ſi loco x ponatur $x+1$ eritque

$$0 = z + P - P^{I} + Q - Q^{I} + \frac{dq}{dx} + \frac{ddq}{2\,dx^2} + \&c.$$

vnde ſequitur $q = \int(Q^{I} - Q)\,dx - Q$. Quamobrem ſi comma cuique quantitati infixum denotet eius valorem, quem induit poſito $x+1$ loco x, ponaturque

$$\int z\,dx = P$$
$$P - \int(P^{I} - P)\,dx = Q$$
$$Q - \int(Q^{I} - Q)\,dx = R$$
$$R - \int(R^{I} - R)\,dx = S \quad \&c.$$

erit ſeriei propoſitae $z + z^{I} + z^{II} + z^{III} + z^{IV} + \&c.$ ſumma $= C - P - Q - R - S - \&c.$ vbi conſtans C ita debet definiri, vt poſito $x = \infty$ tota ſumma euaneſcat. Quia autem applicatio huius expreſſionis integrationes requirit, hoc loco eius vſum declarare non licet.

196. Vt autem formulas integrales euitemus, ſtatuamus ſummam ſeriei $= ys$, exiſtente y functione ipſius

ſius x quacunque cognita, cuius valores y^{I}, y^{II}, &c. qui prodeunt ponendo $x+1$, $x+2$, &c. loco x, erunt noti. Si iam ponatur $x+1$ loco x prodibit ſuperior ſeries termino primo mulctata, cuius ſumma propterea erit $y'\left(s+\frac{ds}{dx}+\frac{dds}{2dx^2}+\frac{d^3s}{6dx^3}+\&c.\right)=ys-z$

ſeu $$z+\frac{y'ds}{dx}+\frac{y'dds}{2dx^2}+\frac{y'd^3s}{6dx^3}+\&c.=(y-y')s$$

vnde neglectis differentialibus oritur $s=\frac{z}{y-y'}$. Statuatur $\frac{z}{y'-y}=P$, ſitque reuera $s=-P+p$, erit:

$$\left.\begin{aligned}&-\frac{y'dP}{dx}-\frac{y'ddP}{2dx^2}-\frac{y'd^3P}{6dx^3}-\&c.\\&+\frac{y'dp}{dx}+\frac{y'ddp}{2dx^2}+\frac{y'd^3p}{6dx^3}+\&c.\end{aligned}\right\}=(y-y')p$$

ideoque $\frac{y'dp}{dx}+\frac{y'ddp}{2dx^2}+\frac{y'd^3p}{6dx^3}+\&c.=y'(P'-P)-(y'-y)p.$

Statuatur $Q=\frac{y'(P'-P)}{y'-y}$, ſitque $p=Q+q$; erit:

$$y'(Q'-Q)+y'\left(\frac{dq}{dx}+\frac{ddq}{2dx^2}+\&c.\right)=-(y'-y)q.$$

Statuatur $R=\frac{y'(Q'-Q)}{y'-y}$, ſitque $q=-R+r$.

Hocque modo ſi vlterius progrediamur. Seriei propoſitae: $z+z^{I}+z^{II}+z^{III}+z^{IV}+\&c.$ ſumma ſequenti modo inuenietur.

Sum-

Sumta pro lubitu functione ipſius x, quae ſit $= y$, ſtatuatur:

$$P = \frac{z}{y' - y} = \frac{z}{\Delta y}$$

$$Q = \frac{y'(P' - P)}{y' - y} = \frac{y \Delta P}{\Delta y} + \Delta P$$

$$R = \frac{y'(Q' - Q)}{y' - y} = \frac{y \Delta Q}{\Delta y} + \Delta Q$$

$$S = \frac{y'(R' - R)}{y' - y} = \frac{y \Delta R}{\Delta y} + \Delta R$$

&c.

Hincque erit ſumma quaeſita:

$$= C - Py + Qy - Ry + Sy - \&c.$$

Sumta pro C eiusmodi conſtante, vt poſito $x = \infty$ ſumma euaneſcat.

192. Sumatur $y = a^x$, ob $y' = a^{x+1}$, erit: $y' - y = a^x(a - 1)$, vnde fiet:

$$P = \frac{z}{a^x(a-1)} \;;\quad P' = \frac{z'}{a^{x+1}(a-1)}$$

$$Q = \frac{a(P' - P)}{a - 1} = \frac{(z' - az)}{a^x(a-1)^2} \;;\quad Q' = \frac{(z'' - az')}{a^{x+1}(a-1)^2}$$

$$R = \frac{a(Q' - Q)}{a - 1} = \frac{z'' - 2az' + aaz}{a^x(a-1)^3}$$

$$S = \frac{a(R' - R)}{a - 1} = \frac{z''' - 3az'' + 3a^2z' - a^3z}{a^x(a-1)^4}$$

&c.

Quocirca ſumma ſeriei propoſita erit:

$$C - \frac{z}{a-1} + \frac{z' - az}{(a-1)^2} - \frac{z'' + 2az' - a^2 z}{(a-1)^3}$$
$$+ \frac{z''' - 3az'' + 3a^2 z' - a^3 z}{(a-1)^4}$$
&c.

Haec vero eadem ſummae expreſſio iam ſupra eſt inventa Capite primo. Hinc autem aliis pro y valoribus accipiendis infinitae aliae expreſſiones erui poterunt; vnde ea, quae cuique caſui maxime ſit accommodata, eligi poteſt.

CAPUT VIII.

DE VSU CALCULI DIFFERENTIALIS IN FORMANDIS SERIEBUS.

198.

Vnum adhuc calculi differentialis vſum in doctrina ſerierum commemorabimus, qui in ipſa formatione ſerierum conſiſtit, & ad quem iam ſupra prouocauimus, cum quaeſtio eſſet de fractione, cuius denominator ſit poteſtas quaecunque functionis cuiuſpiam, in ſeriem euoluenda. Iſta methodus autem ſimilis eſt ei, qua iam aliquoties ſumus vſi, dum functio in ſeriem convertenda aequalis fingitur cuipiam ſeriei, in ſingulis terminis coefficientes indeterminatos habenti, qui deinceps aequalitate conſtituta determinentur. Haec autem determinatio ſaepenumero mirifice adiuuatur, ſi antequam ea ſuſcipiatur ad differentialia cum prima, tum nonnunquam quoque ad ſecunda aequatio perducatur. Quae methodus cum in calculo integrali ampliſſimi ſit vſus, eam hic diligentius exponemus.

199. Primum igitur breuiter repetamus, quae ſupra de euolutione fractionum in ſeries ſine calculi differentialis ſubſidio attulimus. Sit fractio quaecunque propoſita:

$$\frac{A + Bx + Cx^2 + Dx^3 + \&c.}{\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \&c.} = s$$

quam in ſeriem ſecundum poteſtates ipſius x procedentem conuerti oporteat. Fingatur pro s ſeries indeterminata:

$$s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \mathfrak{F}x^5 + \mathfrak{G}x^6 + \&c.$$

Cum igitur fractione per multiplicationem ſublata ſit:

$$A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6 + \&c.$$
$$= s(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \zeta x^5 + \eta x^6 + \&c.)$$

ſi pro s ſeries ficta ſubſtituatur prodibit ſequens aequatio:

$$A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c. =$$

$$\begin{array}{llllll}
\mathfrak{A}\alpha & + \mathfrak{B}\alpha x & + \mathfrak{C}\alpha x^2 & + \mathfrak{D}\alpha x^3 & + \mathfrak{E}\alpha x^4 & + \mathfrak{F}\alpha x^5 + \&c. \\
 & + \mathfrak{A}\beta & + \mathfrak{B}\beta & + \mathfrak{C}\beta & + \mathfrak{D}\beta & + \mathfrak{E}\beta + \&c. \\
 & & + \mathfrak{A}\gamma & + \mathfrak{B}\gamma & + \mathfrak{C}\gamma & + \mathfrak{D}\gamma + \&c. \\
 & & & + \mathfrak{A}\delta & + \mathfrak{B}\delta & + \mathfrak{C}\delta + \&c. \\
 & & & & + \mathfrak{A}\varepsilon & + \mathfrak{B}\varepsilon + \&c. \\
 & & & & & + \mathfrak{A}\zeta + \&c.
\end{array}$$

Aequalitate ergo inter ſingulos terminos, qui easdem ipſius x poteſtates continent, conſtituta fiet:

$$\mathfrak{A}\alpha - A = 0$$
$$\mathfrak{B}\alpha + \mathfrak{A}\beta - B = 0$$
$$\mathfrak{C}\alpha + \mathfrak{B}\beta + \mathfrak{A}\gamma - C = 0$$
$$\mathfrak{D}\alpha + \mathfrak{C}\beta + \mathfrak{B}\gamma + \mathfrak{A}\delta - D = 0$$
$$\mathfrak{E}\alpha + \mathfrak{D}\beta + \mathfrak{C}\gamma + \mathfrak{B}\delta + \mathfrak{A}\varepsilon - E = 0 \quad \&c.$$

ex

ex quibus aequationibus coefficientes ficti $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$, &c. determinantur, ſicque ſeries infinita inuenitur:

$$\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \&c.$$

fractioni propoſitae s aequalis. Atque in hac forma ſi tam numerator quam denominator fractionis s finito terminorum numero conſtent, omnes ſeries recurrentes comprehenduntur, de quibus iam ſupra fuſius eſt tractatum.

200. Quodſi autem vel numerator vel denominator vel vterque ad dignitatem quamcunque fuerit eleuatus, tum hoc modo ſeries difficulter obtinetur; propterea quod negotium, niſi functio eleuata ſit binomium, perquam fit laborioſum. Calculo autem differentiali iſte labor euitari poteſt. Adſit primum ſolus numerator, ſitque:

$$s = (A + Bx + Cxx)^n,$$

vnde quaeratur ſeries ſecundum poteſtates ipſius x procedens huic trinomii dignitati aequalis; quam quidem finitam fore conſtat, ſi exponens n fuerit numerus integer affirmatiuus. Fingatur iterum pro s ſeries indefinita:

$$s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \mathfrak{F}x^5 + \mathfrak{G}x^6 + \&c.$$

cuius terminum primum $\mathfrak{A}$ conſtat eſſe $= A^n$: ſi enim ponatur $x = 0$, ex priori forma propoſita fit $s = A^n$, ex ſerie autem ficta $s = \mathfrak{A}$. Haec autem primi termini determinatio ex ipſa rei natura eſt petenda, ſi ad differentialia deſcendere velimus, quia hinc primus terminus non determinatur, vti mox patebit.

201. Cum ſit $s = (A + Bx + Cx^2)^n$, erit logarithmis ſumendis $ls = nl\,(A + Bx + Cx^2)$, hincque ſumtis differentialibus habebitur:

$$\frac{ds}{s} = \frac{nB\,dx + 2nC\,x\,dx}{A + Bx + Cx^2}, \quad \text{ſeu}$$

$$(A + Bx + Cx^2)\frac{ds}{dx} = ns\,(B + 2Cx).$$

Ex ſerie autem ficta eſt:

$$\frac{ds}{dx} = \mathfrak{B} + 2\mathfrak{C}x + 3\mathfrak{D}x^2 + 4\mathfrak{E}x^3 + 5\mathfrak{F}x^4 + \&c.$$

ſi igitur haec ſeries loco $\frac{ds}{dx}$, & pro s ipſa ſeries ficta ſubſtituatur, prodibit ſequens aequatio:

$$\begin{array}{l} A\mathfrak{B} + 2A\mathfrak{C}x + 3A\mathfrak{D}x^2 + 4A\mathfrak{E}x^3 + 5A\mathfrak{F}x^4 + \&c. \\ \quad\quad + B\mathfrak{B} \quad + 2B\mathfrak{C} \quad + 3B\mathfrak{D} \quad + 4B\mathfrak{E} \quad + \&c. \\ \quad\quad\quad\quad + C\mathfrak{B} \quad + 2C\mathfrak{C} \quad + 3C\mathfrak{D} \quad + \&c. \\ \hline nB\mathfrak{A} + nB\mathfrak{B} + nB\mathfrak{C} + nB\mathfrak{D} + nB\mathfrak{E} + \&c. \\ \quad\quad + 2nC\mathfrak{A} + 2nC\mathfrak{B} + 2nC\mathfrak{C} + 2nC\mathfrak{D} + \&c. \end{array} =$$

Aequalitate ergo hic inter terminos eiusdem ipſius x poteſtatis conſtituta erit:

$$\mathfrak{B} = \frac{nB\mathfrak{A}}{A}$$

$$\mathfrak{C} = \frac{(n-1)B\mathfrak{B} + 2nCA}{2A}$$

$$\mathfrak{D} = \frac{(n-2)B\mathfrak{C} + (2n-1)C\mathfrak{B}}{3A}$$

$\mathfrak{E} =$

$$\mathfrak{E} = \frac{(n-3)\,B\mathfrak{D} + (2n-2)\,C\mathfrak{C}}{4A}$$

$$\mathfrak{F} = \frac{(n-4)\,B\mathfrak{E} + (2n-3)\,C\mathfrak{D}}{5A}$$

&c.

Cum igitur vt ante vidimus ſit $\mathfrak{A} = A^n$, erit $\mathfrak{B} = nA^{n-1}B$, hincque reliqui coefficientes omnes ſucceſſiue determinabuntur. Lex autem, quam ipſi ſequuntur facillime ex his formulis patet, quae vehementer obſcura manſiſſet, ſi trinomium actu eleuare voluiſſemus.

202. Haec eadem methodus ſuccedit, ſi polynomium quodcunque ad quampiam dignitatem eleuari debeat. Sit

$$s = (A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.)^n$$

fingaturque:

$$s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \&c.$$

erit $\mathfrak{A} = A^n$, qui valor colligitur, ſi ponatur $x = 0$. Sumtis iam vt ante logarithmis, eorumque differentialibus reperietur:

$$\frac{ds}{s} = \frac{nBdx + 2nCxdx + 3nDx^2dx + 4nEx^3dx + \&c.}{A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.}$$

ſeu $(A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.)\frac{ds}{dx} =$

$$s\,(nB + 2nCx + 3nDx^2 + 4nEx^3 + \&c.)$$

Cum

Cum igitur ſit:

$$\frac{ds}{dx} = \mathfrak{B} + 2\mathfrak{C}x + 3\mathfrak{D}x^2 + 4\mathfrak{E}x^3 + 5\mathfrak{F}x^4 + \&c.$$

Erit his ſeriebus pro s & $\frac{ds}{dx}$ ſubſtitutis:

$$\begin{array}{llllll}
A\mathfrak{B} & + 2A\mathfrak{C}x & + 3A\mathfrak{D}x^2 & + 4A\mathfrak{E}x^3 & + 5A\mathfrak{F}x^4 & + \&c. \\
 & + B\mathfrak{B} & + 2B\mathfrak{C} & + 3B\mathfrak{D} & + 4B\mathfrak{E} & + \&c. \\
 & & + C\mathfrak{B} & + 2C\mathfrak{C} & + 3C\mathfrak{D} & + \&c. \\
 & & & + D\mathfrak{B} & + 2D\mathfrak{C} & + \&c. \\
 & & & & + E\mathfrak{B} & + \&c. = \\
\hline
nB\mathfrak{A} & + nB\mathfrak{B} & + nB\mathfrak{C} & + nB\mathfrak{D} & + nB\mathfrak{E} & + \&c. \\
 & + 2nC\mathfrak{A} & + 2nC\mathfrak{B} & + 2nC\mathfrak{C} & + 2nC\mathfrak{D} & + \&c. \\
 & & + 3nD\mathfrak{A} & + 3nD\mathfrak{B} & + 3nD\mathfrak{C} & + \&c. \\
 & & & + 4nE\mathfrak{A} & + 4nE\mathfrak{B} & + \&c. \\
 & & & & + 5nF\mathfrak{A} & + \&c.
\end{array}$$

Vnde deriuantur ſequentes determinationes:

$$A\mathfrak{B} = n\,B\mathfrak{A}$$
$$2A\mathfrak{C} = (n-1)B\mathfrak{B} + 2nC\mathfrak{A}$$
$$3A\mathfrak{D} = (n-2)B\mathfrak{C} + (2n-1)C\mathfrak{B} + 3nD\mathfrak{A}$$
$$4A\mathfrak{E} = (n-3)B\mathfrak{D} + (2n-2)C\mathfrak{C} + (3n-1)D\mathfrak{B} + 4nE\mathfrak{A}$$
$$5A\mathfrak{F} = (n-4)B\mathfrak{E} + (2n-3)C\mathfrak{D} + (3n-2)C\mathfrak{C} + (4n-1)E\mathfrak{B} + 5nF\mathfrak{A}$$
$$\&c.$$

vnde quemadmodum coefficientes ficti $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. a ſe inuicem pendeant, hincque determinentur, cum ſit $\mathfrak{A} = A^n$, luculentiſſime apparet.

203. Quoniam, ſi quantitas $A+Bx+Cx^2+Dx^3+\&c.$ ex finito terminorum numero conſtat, numerusque n fuerit integer affirmatiuus, quaecunque poteſtas finito etiam terminorum numero conſtare debet: manifeſtum eſt hoc caſu, formulas modo inuentas tandem euaneſcere debere, atque cum omnes termini adeſſe debeant, vt primum vnus euanuerit, ſimul omnes ſequentes euaneſcere debere. Ponamus formulam propoſitam $A+Bx+Cx^2$ eſſe trinomium, eiusque cubum quaeri, vt ſit $n=3$, erit

$$\begin{array}{llcl}
\mathfrak{A} = A^3 & \text{ideoque} & ; & \mathfrak{A} = A^3 \\
A\mathfrak{B} = 3B\mathfrak{A} & & ; & \mathfrak{B} = 3A^2B \\
2A\mathfrak{C} = 2B\mathfrak{B} + 6C\mathfrak{A} & & ; & \mathfrak{C} = 3AB^2 + 3A^2C \\
3A\mathfrak{D} = 1B\mathfrak{C} + 5C\mathfrak{B} & & ; & \mathfrak{D} = B^3 + 6ABC \\
4A\mathfrak{E} = 0 + 4C\mathfrak{C} & & ; & \mathfrak{E} = 3B^2C + 3AC^2 \\
5A\mathfrak{F} = -B\mathfrak{E} + 3C\mathfrak{D} & & ; & \mathfrak{F} = 3BC^2 \\
6A\mathfrak{G} = -2B\mathfrak{F} + 2C\mathfrak{E} & & ; & \mathfrak{G} = C^3 \\
7A\mathfrak{H} = -3B\mathfrak{G} + 1C\mathfrak{F} & & ; & \mathfrak{H} = 0 \\
8A\mathfrak{I} = -4B\mathfrak{H} + 0 & & ; & \mathfrak{I} = 0.
\end{array}$$

Quoniam igitur iam bini ſunt $=0$, ſequentiumque quilibet a duobus praecedentibus pendet, patet, omnes ſequentes pariter euaneſcere debere. Hancque ob cauſam lex, qua hi coefficientes a ſe inuicem pendere ſunt inventi, eo magis eſt notatu digna.

204. Si n fuerit numerus negatiuus, ita vt s aequale fiat fractioni, ſeries in infinitum excurret. Sit igitur

 $c =$

$$s = \frac{1}{(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \&c.)^n}$$

fingatur pro eius valore haec ſeries:

$$s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \mathfrak{F}x^5 + \&c.$$

Atque ſi in ſuperioribus formulis pro litteris A, B, C, D, &c. ponantur α, β, γ, δ, &c. ſimulque fiat n negatiuum, ſequentes determinationes coefficientium $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. prodibunt:

$$\mathfrak{A} = \alpha^{-n} = \frac{1}{\alpha^n}$$
$$\alpha\mathfrak{B} + n\beta\mathfrak{A} = 0$$
$$2\alpha\mathfrak{C} + (n+1)\beta\mathfrak{B} + 2n\gamma\mathfrak{A} = 0$$
$$3\alpha\mathfrak{D} + (n+2)\beta\mathfrak{C} + (2n+1)\gamma\mathfrak{B} + 3n\delta\mathfrak{A} = 0$$
$$4\alpha\mathfrak{E} + (n+3)\beta\mathfrak{D} + (2n+2)\gamma\mathfrak{C} + (3n+1)\delta\mathfrak{B} + 4n\varepsilon\mathfrak{A} = 0$$
$$5\alpha\mathfrak{F} + (n+4)\beta\mathfrak{E} + (2n+3)\gamma\mathfrak{D} + (3n+2)\delta\mathfrak{C} + (4n+1)\varepsilon\mathfrak{B} + 5n\zeta\mathfrak{A} = 0$$
&c.

Quae formulae eandem continent legem horum coefficientium numerorum, quam iam ſupra obſeruauimus in introductione; cuiusque adeo veritatem nunc demum rigide demonſtrare licuit.

205. Haec ita ſe habent, ſi numerator fractionis fuerit vnitas, vel etiam quaepiam ipſius x poteſtas, puta x^m; poſteriori enim caſu tantum oportebit ſeriem priori inuentam $\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \&c.$ multiplicare per x^m. At ſi numerator conſtet ex duobus pluribusue terminis, tum ſupra quidem legem progreſſionis

nis non obſeruauimus, quamobrem eam hic per differentiationem inueſtigemus. Sit igitur:

$$s = \frac{A + Bx + Cx^2 + Dx^3 + \&c.}{(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \&c.)^n}$$

fingaturque pro valore huius fractionis ſequens ſeries:

$$s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \mathfrak{F}x^5 + \&c.$$

cuius primus terminus $\mathfrak{A}$ vt definiatur, ponatur $x = 0$, eritque ex priori expreſſione $s = \frac{A}{\alpha^n}$, ex ficta vero $s = \mathfrak{A}$, vnde neceſſe eſt, vt ſit $\mathfrak{A} = \frac{A}{\alpha^n}$. Quo termino determinato reliqui per differentiationem innoteſcent.

206. Sumtis logarithmis erit:

$$ls = l(A + Bx + Cx^2 + Dx^3 + \&c.)$$
$$- nl(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \&c.)$$

hincque differentiando orietur:

$$\frac{ds}{s} = \frac{Bdx + 2Cdx + 3Dx^2dx + \&c.}{A + Bx + Cx^2 + Dx^3 + \&c.}$$
$$- \frac{n\beta dx - 2n\gamma xdx - 3n\delta x^2dx - \&c.}{\alpha + \beta x + \gamma x^2 + \delta x^3 + \&c.}$$

Sublatisque per multiplicationem fractionibus erit:

$$\left(\begin{array}{lllll} A\alpha + A\beta x & + A\gamma x^2 & + A\delta x^3 & + \&c. \\ \quad + B\alpha & + B\beta & + B\gamma & + \&c. \\ & + C\alpha & + C\beta & + \&c. \\ & & + D\alpha & + \&c. \end{array}\right\} \frac{ds}{dx} =$$

$$\left(\begin{array}{llllll} B\alpha + B\beta x & + B\gamma x^2 & + B\delta x^3 & + B\varepsilon x^4 & + \&c. \\ \quad + 2C\alpha & + 2C\beta & + 2C\gamma & + 2C\delta & + \&c. \\ & + 3D\alpha & + 3D\beta & + 3D\gamma & + \&c. \\ & & + 4E\alpha & + 4E\beta & + \&c. \\ & & & + 5F\alpha & + \&c. \end{array}\right\} s$$

$$- \left(\begin{array}{llllll} A\beta + 2A\gamma x & + 3A\delta x^2 & + 4A\varepsilon x^3 & + 5A\zeta x^4 & + \&c \\ \quad + B\beta & + 2B\gamma & + 3B\delta & + 4B\varepsilon & + \&c. \\ & + C\beta & + 2C\gamma & + 3C\delta & + \&c. \\ & & + D\beta & + 2D\gamma & + \&c. \\ & & & + E\beta & + \&c. \end{array}\right\} ns.$$

Cum nunc ſit $\frac{ds}{dx} = \mathfrak{B} + 2\mathfrak{C}x + 3\mathfrak{D}x^2 + 4\mathfrak{E}x^3 + \&c.$ erit factis ſubſtitutionibus:

$$A\alpha\mathfrak{B} + nA\beta\mathfrak{A} - B\alpha\mathfrak{A} = 0$$

$$2A\alpha\mathfrak{C} + (n+1)A\beta\mathfrak{B} + 2nA\gamma\mathfrak{A} + (n-1)B\beta\mathfrak{A} - 2C\alpha\mathfrak{A} = 0$$

$$\begin{array}{l} 3A\alpha\mathfrak{D} + (n+2)A\beta\mathfrak{C} + (2n+1)A\gamma\mathfrak{B} + 3n\,A\delta\mathfrak{A} \\ + \quad B\alpha\mathfrak{C} + nB\beta\mathfrak{B} + (2n-1)B\gamma\mathfrak{A} \\ - \quad C\alpha\mathfrak{B} + (n-2)C\beta\mathfrak{A} \\ - \quad 3D\alpha\mathfrak{A} \end{array} \Bigg\} = 0$$

$$\begin{array}{l} 4A\alpha\mathfrak{E} + (n+3)A\beta\mathfrak{D} + (2n+2)A\gamma\mathfrak{C} + (3n+1)A\delta\mathfrak{B} + 4n\,A\varepsilon\mathfrak{A} \\ + \quad 2B\alpha\mathfrak{D} + (n+1)B\beta\mathfrak{C} + 2n\,B\gamma\mathfrak{B} + (3n-1)B\delta\mathfrak{A} \\ + \quad 0\,C\alpha\mathfrak{C} + (n-1)C\beta\mathfrak{B} + (2n-2)C\gamma\mathfrak{A} \\ - \quad 2D\alpha\mathfrak{B} + (n-3)D\beta\mathfrak{A} \\ - \quad 4E\alpha\mathfrak{A} \end{array} \Bigg\} = 0$$

Hinc

Hinc lex, qua iſtae formulae progrediuntur, facile perſpicitur: prima enim cuiusque aequationis linea eandem ſequitur legem, quam §. 284. habuimus. Tum vero coefficientes ſecundarum linearum oriuntur, ſi a coefficientibus ſuperioribus ſubtrahatur $n+1$, ſimilique modo ex linea ſecunda formatur linea tertia & ſequentes, a coefficientibus ſuperioribus continuo ſubtrahendo $n+1$; ipſae autem litterae quemvis terminum componentes per ſolam inſpectionem facillime formantur.

207. Sin autem quoque numerator fractionis fuerit quaepiam poteſtas: ſcilicet

$$s = \frac{(A + Bx + Cx^2 + Dx^3 + \&c.)^m}{(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \&c.)^n}$$

fingaturque $s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \&c.$ erit $\mathfrak{A} = \frac{A^m}{\alpha^n}$; reliqui vero coefficientes ex ſequentibus formulis determinabuntur:

$$\left.\begin{array}{l} A\alpha\mathfrak{B} + nA\beta\mathfrak{A} \\ \quad - mB\alpha\mathfrak{A} \end{array}\right\} = 0$$

$$\left.\begin{array}{l} 2A\alpha\mathfrak{C} + (n+1)A\beta\mathfrak{B} + 2nA\gamma\mathfrak{A} \\ \quad - (m-1)B\alpha\mathfrak{B} + (n-m)B\beta\mathfrak{A} \\ \quad - 2mC\alpha\mathfrak{A} \end{array}\right\} = 0$$

$$\left.\begin{array}{l} 3A\alpha\mathfrak{D} + (n+2)A\beta\mathfrak{C} + (2n+1)A\gamma\mathfrak{B} + 3nA\delta\mathfrak{A} \\ \quad - (m-2)B\alpha\mathfrak{C} + (n-m+1)B\beta\mathfrak{B} + (2n-m)B\gamma\mathfrak{A} \\ \quad - (2m-1)C\alpha\mathfrak{B} + (n-2m)C\beta\mathfrak{A} \\ \quad - 3mD\alpha\mathfrak{A} \end{array}\right\} = 0$$

$$\left.\begin{array}{l} 4A\alpha\mathfrak{E} + (n+3)A\beta\mathfrak{D} + (2n+2)A\gamma\mathfrak{C} + (3n+1)A\delta\mathfrak{B} + 4nA\varepsilon\mathfrak{A} \\ \quad - (m-3)B\alpha\mathfrak{D} + (n-m+2)B\beta\mathfrak{C} + (2n-m+1)B\gamma\mathfrak{B} + (3n-m)B\delta\mathfrak{A} \\ \quad - (2m-2)C\alpha\mathfrak{C} + (n-2m+1)C\beta\mathfrak{B} + (2n-2m)C\gamma\mathfrak{A} \\ \quad - (3m-1)D\alpha\mathfrak{B} + (n-3m)D\beta\mathfrak{A} \\ \quad - 4mE\alpha\mathfrak{A} \end{array}\right\} = 0$$

&c.

Lex

Lex, qua iſtae formulae vlterius continuantur, ex inſpectione facilius apparet, quam verbis deſcribi queat. Descendendo autem coefficientes diminuuntur differentia $n+m$; horizontaliter autem progrediendo augentur continuo differentia $n-1$.

208. Hoc igitur modo doctrina de ſeriebus recurrentibus amplificatur, dum iſtum defectum ſuppleuimus, atque legem coefficientium definiuimus, ſi non ſolum denominator fractionis fuerit poteſtas quaecunque, ſed etiam numerator ex quotlibet terminis conſtet, ad quam legem detegendam ſola inductio non ſufficiebat. Praeter plurimos autem vſus ſerierum recurrentium, quos iam expoſuimus, maximam quoque afferunt vtilitatem ad ſummas quarumuis ſerierum proxime inueniendas: cuius ſpecimen iam in Capite primo huius ſectionis exhibuimus, dum ſeriem ſubſtitutione $x = \frac{y}{1+ny}$ in aliam transmutauimus, quae ſaepenumero terminorum numero finito conſtet. Eaque methodus vlterius extendi potuiſſet, ſi pro x aliae functiones ſubſtitutae fuiſſent. Quoniam vero tum lex progreſſionis ſerierum, quae loco poteſtatum ipſius x poni deberent, non ſatis luculenter conſtabat, in hunc locum iſtam amplificationem reſeruare viſum eſt; cum memorata lex iam penitus eſſet detecta. Interim tamen re diligentius perpenſa comperimus idem negotium ſine hac progreſſionis lege expediri poſſe, in ſubſidium tantum vocando methodum, qua hic ad hanc ipſam legem inueſtigandam ſumus vſi.

209. Sit igitur propoſita ſeries quaecunque

$$s = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c.$$

quam in aliam transformari oporteat, cuius termini ſinguli ſint fractiones, quarum denominatores ſecundum poteſtates formulae huiusmodi $\alpha + \beta x + \gamma x^2 + \delta x^3 + \&c.$ procedant. Quo igitur a ſimplicioribus incipiamus, ponamus eſſe:

$$s = \frac{\mathfrak{A}}{\alpha + \beta x} + \frac{\mathfrak{B}x}{(\alpha + \beta x)^2} + \frac{\mathfrak{C}x^2}{(\alpha + \beta x)^3} + \frac{\mathfrak{D}x^3}{(\alpha + \beta x)^4} + \&c.$$

aequalitate illius ſerei cum hac expreſſione conſtituta, multiplicetur vbique per $\alpha + \beta x$, fietque:

$$\begin{matrix} A\alpha + B\alpha x + C\alpha x^2 + D\alpha x^3 + \&c. \\ + A\beta + B\beta \;\; + C\beta \;\; + \&c. \end{matrix} = \mathfrak{A} + \frac{\mathfrak{B}x}{\alpha + \beta x} + \frac{\mathfrak{C}x^2}{(\alpha + \beta x)^2} + \&c.$$

ſtatuatur $\mathfrak{A} = A\alpha$; fiatque:

$$A\beta + B\alpha = A^{I}$$
$$B\beta + C\alpha = B^{I}$$
$$C\beta + D\alpha = C^{I}$$
$$D\beta + E\alpha = D^{I} \quad \&c.$$

erit diuiſione per x inſtituta:

$$A^{I} + B^{I}x + C^{I}x^2 + D^{I}x^3 + \&c. = \frac{\mathfrak{B}}{\alpha + \beta x} + \frac{\mathfrak{C}x}{(\alpha + \beta x)^2} + \frac{\mathfrak{D}x^2}{(\alpha + \beta x)^3} + \&c.$$

Multiplicetur denuo per $\alpha + \beta x$, poſitoque

$$A^{I}\beta + B^{I}\alpha = A^{II}$$
$$B^{I}\beta + C^{I}\alpha = B^{II}$$
$$C^{I}\beta + D^{I}\alpha = C^{II} \quad \&c. \quad \text{fiet}$$

$$A^{I}\alpha + A^{II}x + B^{II}x^2 + C^{II}x^3 + \&c. = \mathfrak{B} + \frac{\mathfrak{C}x}{\alpha + \beta x} + \frac{\mathfrak{D}x^2}{(\alpha + \beta x)^2} + \&c.$$

Sit

Sit igitur $\mathfrak{B} = A^{I}\alpha$; atque operationem vt ante inſtituendo, ſi fiat:

$$\begin{array}{l|l} A^{II}\beta + B^{II}\alpha = A^{III} & A^{III}\beta + B^{III}\alpha = A^{IV} \\ B^{II}\beta + C^{II}\alpha = B^{III} & B^{III}\beta + C^{III}\alpha = B^{IV} \\ C^{II}\beta + D^{II}\alpha = C^{III} & C^{III}\beta + D^{III}\alpha = C^{IV} \\ \&c. & \&c. \end{array}$$

erit $\mathfrak{C} = A^{II}\alpha$; $\mathfrak{D} = A^{III}\alpha$; $\mathfrak{E} = A^{IV}\alpha$; &c. vnde ſumma ſeriei propoſitae hoc modo exprimetur, vt ſit:

$$s = \frac{A\alpha}{\alpha+\beta x} + \frac{A^{I}\alpha x}{(\alpha+\beta x)^2} + \frac{A^{II}\alpha x^2}{(\alpha+\beta x)^3} + \frac{A^{III}\alpha x^3}{(\alpha+\beta x)^4} + \&c.$$

Quae eadem ſeries orta fuiſſet ex ſubſtitutione $\frac{x}{\alpha+\beta x} = y$ ſeu $x = \frac{\alpha y}{1-\beta y}$.

210. Haec transformatio optimo cum ſucceſſu adhibetur, ſi ſeries propoſita $A + Bx + Cx^2 + \&c.$ ita fuerit comparata, vt tandem confundatur cum ſerie recurrente ſeu potius geometrica ex fractione $\frac{P}{\alpha+\beta x}$ orta. Tum enim valores A^{I}, B^{I}, C^{I}, D^{I}, &c. tandem euaneſcent; hincque multo magis litterae A^{II}, A^{III}, A^{IV}, &c. conſtituent ſeriem maxime conuergentem. Poterimus autem ſimili modo denominatores trinomiales & polynomiales quoscunque adhibere, qui vſum habebunt eximium, ſi ſeries propoſita tandem cum recurrente confundatur. Propoſita ergo ſerie:

$$s = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c.$$

ſta-

ſtatuatur $s = \frac{\mathfrak{A} + \mathfrak{B}x}{\alpha + \beta x + \gamma x^2} + \frac{\mathfrak{A}^{I}x^2 + \mathfrak{B}^{I}x^3}{(\alpha + \beta x + \gamma x^2)^2} +$

$$\frac{\mathfrak{A}^{II}x^4 + \mathfrak{B}^{II}x^5}{(\alpha + \beta x + \gamma x^2)^3} + \frac{\mathfrak{A}^{III}x^6 + \mathfrak{B}^{III}x^7}{(\alpha + \beta x + \gamma x^2)^4} + \text{\&c.}$$

Multiplicetur vbique per $\alpha + \beta x + \gamma x^2$, poſitoque

$$\begin{aligned} A\gamma + B\beta + C\alpha &= A^{I} \\ B\gamma + C\beta + D\alpha &= B^{I} \\ C\gamma + D\beta + E\alpha &= C^{I} \\ &\text{\&c.} \end{aligned} \quad \& \quad \begin{aligned} \mathfrak{A} &= A\alpha \\ \mathfrak{B} &= A\beta + B\alpha \end{aligned}$$

orietur aequatio priori ſimilis, diuiſione per xx inſtituta:

$$A^{I} + B^{I}x + C^{I}x^2 + D^{I}x^3 + E^{I}x^4 + \text{\&c.} =$$

$$\frac{\mathfrak{A}^{I} + \mathfrak{B}^{I}x}{\alpha + \beta x + \gamma xx} + \frac{\mathfrak{A}^{II}x^2 + \mathfrak{B}^{II}x^3}{(\alpha + \beta x + \gamma xx)^2} + \frac{\mathfrak{A}^{III}x^4 + \mathfrak{B}^{III}x^5}{(\alpha + \beta x + \gamma xx)^3} + \text{\&c.}$$

Si igitur vt ante operatio inſtituatur faciendo

$$\begin{aligned} A^{I}\gamma + B^{I}\beta + C^{I}\alpha &= A^{II} \\ B^{I}\gamma + C^{I}\beta + D^{I}\alpha &= B^{II} \\ C^{I}\gamma + D^{I}\beta + E^{I}\alpha &= C^{II} \\ &\text{\&c.} \end{aligned} \quad \begin{aligned} \mathfrak{A}^{I} &= A^{I}\alpha \\ \mathfrak{B}^{I} &= A^{I}\beta + B^{I}\alpha \end{aligned}$$

porroque:

$$\begin{aligned} A^{II}\gamma + B^{II}\beta + C^{II}\alpha &= A^{III} \\ B^{II}\gamma + C^{II}\beta + D^{II}\alpha &= B^{III} \\ C^{II}\gamma + D^{II}\beta + E^{II}\alpha &= C^{III} \\ &\text{\&c.} \end{aligned} \quad \begin{aligned} \mathfrak{A}^{II} &= A^{II}\alpha \\ \mathfrak{B}^{II} &= A^{II}\beta + B^{II}\alpha \end{aligned}$$

ſicque vlterius valores ſimiles inueſtigando erit:

$$s =$$

$$\frac{A\alpha + (A\beta + B\alpha)x}{\alpha + \beta x + \gamma xx} + \frac{(A^{I}\alpha + (A^{I}\beta + B^{I}\alpha)x)x^2}{(\alpha + \beta x + \gamma xx)^2} + \frac{(A^{II} + (A^{II}\beta + B^{II}\alpha)x)x^4}{(\alpha + \beta x + \gamma xx)^3}$$

$$+ \text{\&c.}$$

211. Si ponatur $x = 1$, qua poſitione amplitudini nihil decedit, cum α, β, γ pro lubitu accipi poſſint, fueritque

$$s = A + B + C + D + E + F + G + \&c.$$

Cum putentur ſucceſſiue ſequentes valores:

$$\begin{array}{l|l} A\gamma + B\beta + C\alpha = A' & A'\gamma + B'\beta + C'\alpha = A'' \\ B\gamma + C\beta + D\alpha = B' & B'\gamma + C'\beta + D'\alpha = B'' \text{ ſicque} \\ C\gamma + D\beta + E\alpha = C' & C'\gamma + D'\beta + E'\alpha = C'' \text{ porro} \\ \&c. & \&c. \end{array}$$

inſuper vero brevitatis ergo ponatur:

$$\alpha + \beta + \gamma = m$$

obtinebitur ſumma ſeriei propoſitae hoc modo expreſſa

$$s = \begin{cases} (\alpha + \beta)\left(\frac{A}{m} + \frac{A'}{m^2} + \frac{A''}{m^3} + \frac{A'''}{m^4} + \&c.\right) \\ + \alpha\left(\frac{B}{m} + \frac{B'}{m} + \frac{B''}{m^3} + \frac{B'''}{m^4} + \&c.\right) \end{cases}$$

212. Eodem modo denominatores ex pluribus terminis conſtantes accipi poſſunt; & quoniam operatio ex praecedentibus facile perſpicitur, hic tantum caſum pro quadrinomio euoluamus: Sit ergo

$$s = A + B + C + D + E + F + G + \&c.$$

Quaerantur valores ſequentes:

$$A\delta + B\gamma + C\beta + D\alpha = A'$$
$$B\delta + C\gamma + D\beta + E\alpha = B'$$
$$C\delta + D\gamma + E\beta + F\alpha = C'$$
$$\&c.$$

A'δ

$$A'\delta + B'\gamma + C'\beta + D'\alpha = A''$$
$$B'\delta + C'\gamma + D'\beta + E'\alpha = B''$$
$$C'\delta + D'\gamma + E'\beta + F'\alpha = C''$$
&c.
$$A''\delta + B''\gamma + C''\beta + D''\alpha = A'''$$
$$B''\delta + C''\gamma + D''\beta + E''\alpha = B'''$$
$$C''\delta + D''\gamma + E''\beta + F''\alpha = C'''$$
&c.

Tum vero ſit $\alpha + \beta + \gamma + \delta = m$; eritque

$$s = \begin{cases} (\alpha+\beta+\gamma)\left(\frac{A}{m} + \frac{A'}{m^2} + \frac{A''}{m^3} + \frac{A'''}{m^4} + \&c.\right) \\ + (\alpha+\beta)\left(\frac{B}{m} + \frac{B'}{m^2} + \frac{B''}{m^3} + \frac{B'''}{m^4} + \&c.\right) \\ + \alpha\left(\frac{C}{m} + \frac{C'}{m^2} + \frac{C''}{m^3} + \frac{C'''}{m^4} + \&c.\right) \end{cases}$$

vnde ſimul progreſſio, ſi adhuc plures partes denominatori m tribuantur, clariſſime perſpicitur.

213. Neque vero abſolute opus eſt, vt denominatores fractionum, ad quas ſummam ſeriei reducimus, ſint poteſtates eiusdem formulae $\alpha + \beta x + \gamma x^2 +$ &c. ſed haec ipſa in ſingulis terminis variari poteſt. Quo hoc clarius pateat, ſumamus primo tantum duos terminos, fingaturque ſeries

$$s = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c.$$

in hanc ſeriem fractionum conuerti:

$$s = \frac{\mathfrak{A}}{\alpha + \beta x} + \frac{\mathfrak{A}' x}{(\alpha + \beta x)(\alpha' + \beta' x)} + \frac{\mathfrak{A}'' x^2}{(\alpha + \beta x)(\alpha' + \beta' x)(\alpha'' + \beta'' x)} + \&c.$$

Multiplicetur primum vtrinque per $\alpha + \beta x$, ponaturque

$$A\beta + B\alpha = A'$$
$$B\beta + C\alpha = B' \quad \& \quad \mathfrak{A} = A\alpha$$
$$C\beta + D\alpha = C'$$
$$\&c.$$

fietque per x diuiſo

$$A' + B'x + C'x^2 + D'x^3 + \&c. = \frac{\mathfrak{A}'}{\alpha' + \beta' x} + \frac{\mathfrak{A}'' x}{(\alpha' + \beta' x)(\alpha'' + \beta'' x)} + \&c.$$

Deinde ſimili modo multiplicando per $\alpha' + \beta' x$; tumque per $\alpha'' + \beta'' x$, & ita porro, ſi ſtatuatur:

$A'\beta' + B'\alpha' = A''$	$A''\beta'' + B''\alpha'' = A'''$	$A'''\beta''' + B'''\alpha''' = A''''$
$B'\beta' + C'\alpha' = B''$	$B''\beta'' + C''\alpha'' = B'''$	$B'''\beta''' + C'''\alpha''' = B''''$
$C'\beta' + D'\alpha' = C''$	$C''\beta'' + D''\alpha'' = C'''$	$C'''\beta''' + D'''\alpha''' = C''''$
&c.	&c.	&c.

fiet $\mathfrak{A}' = A'\alpha'$; $\mathfrak{A}'' = A''\alpha''$; $\mathfrak{A}''' = A'''\alpha'''$; &c. atque hinc ſeries propoſita conuertetur in hanc:

$$s = \frac{A\alpha}{\alpha + \beta x} + \frac{A'\alpha' x}{(\alpha + \beta x)(\alpha' + \beta' x)} + \frac{A''\alpha'' x}{(\alpha + \beta x)(\alpha' + \beta' x)(\alpha'' + \beta'' x)} + \&c.$$

vbi valores α, β, α', β', α'', β'', α''', β''', &c. ſunt arbitrarii; quouis autem caſu ita accipi poſſunt, vt iſta noua ſeries maxime conuergat.

214. Applicemus hoc quoque ad factores trinomiales, ſitque propoſita ſerie quacunque:

$s =$

$s = A + B + C + D + E + F + G + \&c.$

$A\gamma + B\beta + C\alpha = A'$	$A'\gamma' + B'\beta' + C'\alpha' = A''$
$B\gamma + C\beta + D\alpha = B'$	$B'\gamma' + C'\beta' + D'\alpha' = B''$
$C\gamma + D\beta + E\alpha = C'$	$C'\gamma' + D'\beta' + E'\alpha' = C''$
&c.	&c.
$A''\gamma'' + B''\beta'' + C''\alpha'' = A'''$	$A'''\gamma''' + B'''\beta''' + C'''\alpha''' = A''''$
$B''\gamma'' + C''\beta'' + D''\alpha'' = B'''$	$B'''\gamma''' + C'''\beta''' + D'''\alpha''' = B''''$
$C''\gamma'' + D''\beta'' + E''\alpha'' = C'''$	$C'''\gamma''' + D'''\beta''' + E'''\alpha''' = C''''$
&c.	&c.

Deinde ſtatuatur breuitatis gratia:

$$\alpha + \beta + \gamma = m$$
$$\alpha' + \beta' + \gamma' = m'$$
$$\alpha'' + \beta'' + \gamma'' = m''$$
$$\alpha''' + \beta''' + \gamma''' = m'''$$
&c.

eritque ſeriei propoſitae ſumma:

$$s = \frac{\alpha(A+B)}{m} + \frac{\alpha'(A'+B')}{m\,m'} + \frac{\alpha''(A''+B'')}{m\,m'm''} + \frac{\alpha'''(A'''+B''')}{m\,m'\,m''\,m'''} + \&c.$$
$$+ \frac{\beta A}{m} + \frac{\beta' A'}{m\,m'} + \frac{\beta'' A''}{m\,m'm''} + \frac{\beta''' A'''}{m\,m'\,m''\,m'''} + \&c.$$

215. Quoniam haec tam late patent, vt vſus minus clare percipi poſſit, reſtringamus transformationem §. 213. traditam, ſitque $x = -1$, vt habeatur haec ſeries:

$s = A - B + C - D + E - F + G - \&c.$

ſtatuaturque:

$B-A=A'$	$B'-2A'=A''$	$B''-3A''=A'''$	$B'''-4A'''=A''''$
$C-B=B'$	$C'-2B'=B''$	$C''-3B''=B'''$	$C'''-4B'''=B''''$
$D-C=C'$	$D'-2C'=C''$	$D''-3C''=C'''$	$D'''-4C'''=C''''$
$E-D=D'$	$E'-2D'=D''$	$E''-3D''=D'''$	$E'''-4D'''=D''''$
&c.	&c.	&c.	&c.

Quibus valoribus inuentis, erit ſumma ſeriei propoſitae aequalis ſequenti ſeriei:

$$s = \frac{A}{2} - \frac{A'}{2.3} + \frac{A''}{2.3.4} - \frac{A'''}{2.3.4.5} + \frac{A''''}{2.3.4.5.6} - \&c.$$

Simili igitur modo ſeries quaecunque propoſita in innumerabiles alias ſibi aequales transmutari poteſt, inter quas ſine dubio ſeries maxime conuergentes reperientur, quarum ope ſumma propoſita vero proxime indagari queat.

216. Reuertamur autem ad inuentionem ſerierum, quarum progreſſionis legem calculus differentialis declarat. Cum igitur hoc in quantitatibus algebraicis iam ſit praeſtitum, progrediamur ad transcendentes, quaeraturque ſeries huic logarithmo aequalis:

$$s = l(1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \&c.)$$

fingatur quaeſito ſatisfacere haec ſeries:

$$s = \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \mathfrak{E}x^5 + \mathfrak{F}x^6 + \&c.$$

Cum igitur ex illius aequationis differentiatione ſequatur

$$\frac{ds}{dx} = \frac{\alpha + 2\beta x + 3\gamma x^2 + 4\delta x^3 + 5\varepsilon x^4 + \&c.}{1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \&c.}$$

erit:

$$(1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \&c.)\frac{ds}{dx} = \alpha + 2\beta x + 3\gamma x^2 + 4\delta x^3 + \&c.$$

Quia

Quia vero ex ficta aequatione eſt:

$$\frac{ds}{dx} = \mathfrak{A} + 2\mathfrak{B}x + 3\mathfrak{C}x^2 + 4\mathfrak{D}x^3 + 5\mathfrak{E}x^4 + \&c.$$

facta hac ſubſtitutione oritur haec aequatio:

$$\begin{array}{llllll}
\mathfrak{A} & + 2\mathfrak{B}x & + 3\mathfrak{C}x^2 & + 4\mathfrak{D}x^3 & + 5\mathfrak{E}x^4 & + \&c. \\
 & + \mathfrak{A}\alpha & + 2\mathfrak{B}\alpha & + 3\mathfrak{C}\alpha & + 4\mathfrak{D}\alpha & + \&c. \\
 & & + \mathfrak{A}\beta & + 2\mathfrak{B}\beta & + 3\mathfrak{C}\beta & + \&c. \\
 & & & + \mathfrak{A}\gamma & + 2\mathfrak{B}\gamma & + \&c. \\
 & & & & + \mathfrak{A}\delta & + \&c. = \\
\hline
\alpha & + 2\beta x & + 3\gamma x^2 & + 4\delta x^3 & + 5\epsilon x^4 & + \&c.
\end{array}$$

Ex qua ſequentes determinationes obtinentur:

$$\mathfrak{A} = \alpha$$
$$\mathfrak{B} = -\tfrac{1}{2}\mathfrak{A}\alpha + \beta$$
$$\mathfrak{C} = -\tfrac{2}{3}\mathfrak{B}\alpha - \tfrac{1}{3}\mathfrak{B}\beta - \tfrac{1}{3}\mathfrak{A}\beta + \gamma$$
$$\mathfrak{D} = -\tfrac{3}{4}\mathfrak{C}\alpha - \tfrac{2}{4}\mathfrak{B}\beta - \tfrac{1}{4}\mathfrak{A}\gamma + \delta$$
$$\mathfrak{E} = -\tfrac{4}{5}\mathfrak{D}\alpha - \tfrac{3}{5}\mathfrak{C}\beta - \tfrac{2}{5}\mathfrak{B}\gamma - \tfrac{1}{5}\mathfrak{A}\delta + \epsilon$$
$$\&c.$$

217. Propoſita nunc ſit quantitas exponentialis:

$$s = e^{\alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \epsilon x^5 + \&c.}$$

in qua e denotet numerum, cuius logarithmus hyperbolicus eſt $= 1$, atque fingatur ſeries quaeſita:

$$s = 1 + \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \mathfrak{E}x^5 + \&c.$$

iam enim ex caſu $x = 0$ patet, primum terminum eſſe debere vnitatem. Cum igitur ſumendis logarithmis ſit

$ls =$

$$ls = \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \zeta x^6 + \&c.$$

erit differentialibus ſumtis:

$$\frac{ds}{dx} = s(\alpha + 2\beta x + 3\gamma x^2 + 4\delta x^3 + 5\varepsilon x^4 + \&c.)$$

At vero ex aequatione fiçta erit:

$$\frac{ds}{dx} = \mathfrak{A} + 2\mathfrak{B}x + 3\mathfrak{C}x^2 + 4\mathfrak{D}x^3 + 5\mathfrak{E}x^4 + \&c. =$$

$$\begin{array}{llllll}
\alpha & + \mathfrak{A}\alpha x & + \mathfrak{B}\alpha x^2 & + \mathfrak{C}\alpha x^3 & + \mathfrak{D}\alpha x^4 & + \&c. \\
 & + 2\beta & + 2\mathfrak{A}\beta & + 2\mathfrak{B}\beta & + 2\mathfrak{C}\beta & + \&c. \\
 & & + 3\gamma & + 3\mathfrak{A}\gamma & + 3\mathfrak{B}\gamma & + \&c. \\
 & & & + 4\delta & + 4\mathfrak{A}\delta & + \&c. \\
 & & & & + 5\varepsilon & + \&c.
\end{array}$$

ex quibus ſequentes prodeunt litterarum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. determinationes:

$$\mathfrak{A} = \alpha$$
$$\mathfrak{B} = \beta + \tfrac{1}{2}\mathfrak{A}\alpha$$
$$\mathfrak{C} = \gamma + \tfrac{2}{3}\mathfrak{A}\beta + \tfrac{1}{3}\mathfrak{B}\alpha$$
$$\mathfrak{D} = \delta + \tfrac{3}{4}\mathfrak{A}\gamma + \tfrac{2}{4}\mathfrak{B}\beta + \tfrac{1}{4}\mathfrak{C}\alpha$$
$$\mathfrak{E} = \varepsilon + \tfrac{4}{5}\mathfrak{A}\delta + \tfrac{3}{5}\mathfrak{B}\gamma + \tfrac{2}{5}\mathfrak{C}\beta + \tfrac{1}{5}\mathfrak{D}\alpha \quad \&c.$$

218. Si quoque arcus, cuius ſinus vel coſinus quaeritur, exprimatur binomio vel polynomio, vel etiam ſerie infinita, hoc modo quoque eius ſinus & coſinus per ſeriem infinitam exprimi poſſunt. At vero quo hoc commodiſſime fiat, non ſufficit ad differentialia prima proceſſiſſe, ſed opus eſt, vt differentialia ſecundi gradus in ſubſidium vocemus. Sit igitur

$s =$

$s = \text{ſin}\,(\alpha + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \&c.)$

fingaturque ſeries quae quaeritur:

$s = \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \mathfrak{E}x^5 + \&c.$

primum enim terminum conſtat euaneſcere: quia vero ad differentialia ſecunda deſcendendum eſt, coefficientem $\mathfrak{A}$ quoque aliunde definiri oportet, quod fiet ſi x ponamus infinite paruum. Tum enim ob arcum $= \alpha x$ ſinus ipſi fiet aequalis, eritque ergo $\mathfrak{A} = \alpha$. Ponamus nunc breuitatis gratia $z = \alpha x + \beta x^2 + \gamma x^3 + \&c.$ vt ſit $s = \text{ſin}\, z$, erit differentiando $ds = dz \,\text{coſ}\, z$, denuoque differentiando $dds = ddz \,\text{coſ}\, z - dz^2 \,\text{ſin}\, z$. Quia igitur eſt $\text{ſin}\, z = s$ & $\text{coſ}\, z = \frac{ds}{dz}$; erit

$dds = \frac{ds\,ddz}{dz} - s\,dz^2$, ſeu $dz\,dds + s\,dz^3 = ds\,ddz$.

219. Ponamus arcum z tantum binomio exprimi eſſeque $z = \alpha x + \beta x^2$; erit $dz = (\alpha + 2\beta x)\,dx$, & poſito dx conſtante, $ddz = 2\beta\,dx^2$; atque $dz^3 = (\alpha^3 + 6\alpha^2\beta x + 12\alpha\beta^2 x^2 + 8\beta^3 x^3)\,dx^3$. Deinde ob $s = \mathfrak{A} + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \&c.$

erit $\frac{ds}{dx} = \mathfrak{A} + 2\mathfrak{B}x + 3\mathfrak{C}x^2 + 4\mathfrak{D}x^3 + \&c.$

& $\frac{dds}{dx^2} = \quad 2\mathfrak{B} + 6\mathfrak{C}x + 12\mathfrak{D}x^2 + \&c.$

Quibus valoribus in aequatione differentio-differentiali ſubſtitutis fiet:

 dz-

$$\frac{dzdds}{dx^3} = 1.2\mathfrak{B}\alpha + 2.3\mathfrak{C}\alpha x + 3.4\mathfrak{D}\alpha x^2 + 4.5\mathfrak{E}\alpha x^3 + \&c.$$
$$+ 2.1.2\mathfrak{B}\beta + 2.2.3\mathfrak{C}\beta + 2.3.4\mathfrak{D}\beta + \&c.$$
$$\frac{+sdz^3}{dx^3} = \quad + \mathfrak{A}\alpha^3 + \mathfrak{B}\alpha^3 + \mathfrak{C}\alpha^3 + \&c.$$
$$+ 6\mathfrak{A}\alpha^2\beta + 6\mathfrak{B}\alpha^2\beta + \&c.$$
$$+ 12\mathfrak{A}\alpha\beta^2 + \&c.$$
$$\frac{dsddz}{dx^3} = 2\mathfrak{A}\beta + 4\mathfrak{B}\beta + 6\mathfrak{C}\beta + 8\mathfrak{D}\beta + \&c.$$

Vnde coefficientes ſequenti modo definientur:

$$\mathfrak{B} = \frac{2\mathfrak{A}\beta}{2\alpha}$$
$$\mathfrak{C} = 0 - \frac{\mathfrak{A}\alpha 2}{2.3}$$
$$\mathfrak{D} = -\frac{2\mathfrak{C}\beta}{4\alpha} - \frac{6\mathfrak{A}\alpha\beta}{3.4} - \frac{\mathfrak{B}\alpha^2}{3.4}$$
$$\mathfrak{E} = -\frac{4\mathfrak{D}\beta}{5\alpha} - \frac{12\mathfrak{A}\beta^2}{4.5} - \frac{6\mathfrak{B}\alpha\beta}{4.5} - \frac{\mathfrak{C}\alpha^2}{4.5}$$
$$\mathfrak{F} = -\frac{6\mathfrak{E}\beta}{6\alpha} - \frac{8\mathfrak{A}\beta^3}{5.6\alpha} - \frac{12\mathfrak{B}\beta\beta}{5.6} - \frac{6\mathfrak{C}\alpha\beta}{5.6} - \frac{\mathfrak{D}\alpha^2}{5.6}$$
$$\mathfrak{G} = -\frac{8\mathfrak{F}\beta}{7\alpha} - \frac{8\mathfrak{B}\beta^3}{6.7\alpha} - \frac{12\mathfrak{C}\beta\beta}{6.7} - \frac{6\mathfrak{D}\alpha\beta}{6.7} - \frac{\mathfrak{E}\alpha 2}{6.7}$$

&c.

Quibus valoribus inuentis erit:

$$\text{ſin}\,(\alpha x + \beta x^2) = \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \&c.$$

exiſtente $\mathfrak{A} = \alpha$,

220. Pari modo coſinus cuiusque anguli in ſeriem conuertitur, quia autem arcus rariſſime per polynomium ex-

exprimitur, oſtendamus vſum differentio-differentialium in invenienda ſerie pro coſinu arcus x. Sit ergo $s = \cos x$, & fingatur:

$$s = 1 - \mathfrak{A}x^2 + \mathfrak{B}x^4 - \mathfrak{C}x^6 + \mathfrak{D}x^8 - \&c.$$

Quia eſt $ds = -dx \sin x$ & $dds = -dx^2 \cos x = -sdx^2$, erit $dds + sdx^2 = 0$; ſubſtitutione ergo facta fiet:

$$\frac{dds}{dx^2} = -1.2\mathfrak{A} + 3.4\mathfrak{B}x^2 - 5.6\mathfrak{C}x^4 + 7.8\mathfrak{D}x^6 - \&c.$$

$$s = 1 - \mathfrak{A}x^2 + \mathfrak{B}x^4 - \mathfrak{C}x^6 + \&c.$$

& ex coaequatione terminorum ſequitur:

$$\mathfrak{A} = \frac{1}{1.2}$$

$$\mathfrak{B} = \frac{\mathfrak{A}}{3.4} = \frac{1}{1.2.3.4}$$

$$\mathfrak{C} = \frac{\mathfrak{B}}{5.6} = \frac{1}{1.2.3\ldots6}$$

$$\mathfrak{D} = \frac{\mathfrak{C}}{7.8} = \frac{1}{1.2.3\ldots8} \quad \&c.$$

Patet ergo quod iam ſupra fuſius demonſtrauimus eſſe:

$$\cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3\ldots6} + \frac{x^8}{1.2.3\ldots8} - \&c.$$

prior vero ſeries pro ſinu poſito $\beta = 0$ & $\alpha = 1$ dabit:

$$\sin x = \frac{x}{1} - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3\ldots7} + \frac{x^9}{1.2.3\ldots9} - \&c.$$

221. Ex his ſeriebus pro ſinu & coſinu notiſſimis deducuntur ſeries pro tangente, cotangente, ſecante

& coſecante cuiusuis anguli. Tangens enim prodit ſi ſinus per coſinum, cotangens ſi coſinus per ſinum, ſecans ſi radius 1 per coſinum, & coſecans ſi radius per ſinum diuidatur. Series autem ex his diuiſionibus ortae maxime videntur irregulares; verum excepta ſerie ſecantem exhibente reliquae per numeros Bernoullianos ſupra definitos $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. ad facilem progreſſionis legem reduci poſſunt. Quoniam enim ſupra §.127 inuenimus eſſe:

$$\frac{\mathfrak{A}u^2}{1.2} + \frac{\mathfrak{B}u^4}{1.2.3.4} + \frac{\mathfrak{C}u^6}{1.2.3\ldots6} + \frac{\mathfrak{D}u^8}{1.2.3\ldots8} + \&c. = 1 - \frac{u}{2}\cot\tfrac{1}{2}u$$

erit poſito $\frac{1}{2}u = x$;

$$\cot x = \frac{1}{x} - \frac{2^2\mathfrak{A}x}{1.2} - \frac{2^4\mathfrak{B}x^3}{1.2.3.4} - \frac{2^6\mathfrak{C}x^5}{1.2.3\ldots6} - \frac{2^8\mathfrak{D}x^7}{1.2\ldots.8} - \&c.$$

atque ſi ponatur $\frac{1}{2}x$ pro x, erit:

$$\cot\tfrac{1}{2}x = \frac{2}{x} - \frac{2\mathfrak{A}x}{1.2} - \frac{2\mathfrak{B}x^3}{1.2.3.4} - \frac{2\mathfrak{C}x^5}{1.2.3\ldots6} - \frac{2\mathfrak{D}x^7}{1.2.3\ldots8} - \&c.$$

222. Hinc autem tangens cuiusuis arcus ſequenti modo per ſeriem exprimetur.

Cum ſit $\tan 2x = \frac{2\,\tan x}{1-\tan x^2}$, erit:

$$\text{cotang. } 2x = \frac{1}{2\tan x} - \frac{\tan x}{2} = \tfrac{1}{2}\cot x - \tfrac{1}{2}\tan x;$$

ideoque $\tan x = \cot x - 2\cot 2x$. Cum igitur ſit

$$\cot x = \frac{1}{x} - \frac{2^2\mathfrak{A}x}{1.2} - \frac{2^4\mathfrak{B}x^3}{1.2.3.4} - \frac{2^2\mathfrak{C}x^5}{1.2\ldots6} - \frac{2^8\mathfrak{D}x^7}{1.2\ldots8} - \&c.$$

2 cot

$$2\cot 2x = \frac{1}{x} - \frac{2^4\mathfrak{A}x}{1.2} - \frac{2^8\mathfrak{B}x^3}{1.2.3.4} - \frac{2^{12}\mathfrak{C}x^5}{1.2\ldots6} - \frac{2^{16}\mathfrak{D}x^7}{1.2\ldots8} - \&c.$$

erit hanc ſeriem ab illa ſubtrahendo:

$$\mathrm{tg}\,x = \frac{2^2(2^2-1)\mathfrak{A}x}{1.2} + \frac{2^4(2^4-1)\mathfrak{B}x^3}{1.2.3.4} + \frac{2^6(2^6-1)\mathfrak{C}x^5}{1.2\ldots\ldots6} + \frac{2^8(2^8-1)\mathfrak{D}x^7}{1.2\ldots\ldots8} + \&c.$$

Si ergo hic introducantur numeri A, B, C, D, &c. §. 182. inuenti, erit:

$$\mathrm{tang}\,x = \frac{2Ax}{1.2} + \frac{2^3Bx^3}{1.2.3.4} + \frac{2^5Cx^5}{1.2\ldots6} + \frac{2^7Dx^7}{1.2\ldots8} + \&c.$$

223. Coſecans autem ſequenti modo inuenietur. Quia eſt $\cot x = \mathrm{tang}\,x + 2\cot 2x = \frac{1}{\cot x} + 2\cot 2x$; erit $\cot x^2 = 2\cot x.\cot 2x + 1$, & radice extracta:

$$\cot x = \cot 2x + \mathrm{coſec.}\,2x, \quad \text{unde fit}$$

$\mathrm{coſec.}\,2x = \cot x - \cot 2x$, & x pro $2x$, poſito $\mathrm{coſec.}\,x = \cot\frac{1}{2}x - \cot x$. Quare cum cotangentes habeamus ſcilicet:

$$\cot\tfrac{1}{2}x = \frac{2}{x} - \frac{2\mathfrak{A}x}{1.2} - \frac{2\mathfrak{B}x^3}{1.2.3.4} - \frac{2\mathfrak{C}x^5}{1.2\ldots6} - \&c.$$

$$\cot x = \frac{1}{x} - \frac{2^2\mathfrak{A}x}{1.2} - \frac{2^4\mathfrak{B}x^3}{1.2.3.4} - \frac{2^6\mathfrak{C}x^5}{1.2\ldots6} - \&c.$$

erit hac ſerie ab illa ſubtracta:

$$\mathrm{coſec.}\,x = \frac{1}{x} + \frac{2(2-1)\mathfrak{A}x}{1.2} + \frac{2(2^3-1)\mathfrak{B}x^3}{1.2.3.4} + \frac{2(2^5-1)\mathfrak{C}x^5}{1.2\ldots\ldots6} + \&c.$$

224. Per hos autem numeros Bernoullianos ſecans exprimi non poteſt, ſed requirit alios numeros, qui in ſummas poteſtatum reciprocarum imparium ingrediuntur. Si enim ponatur:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \&c. = \alpha \cdot \frac{\pi}{2^2}$$

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \&c. = \frac{\beta}{1.2} \cdot \frac{\pi^3}{2^4}$$

$$1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \&c. = \frac{\gamma}{1.2.3.4} \cdot \frac{\pi^5}{2^6}$$

$$1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \frac{1}{9^7} - \&c. = \frac{\delta}{1.2\ldots6} \cdot \frac{\pi^7}{2^8}$$

$$1 - \frac{1}{3^9} + \frac{1}{5^9} - \frac{1}{7^9} + \frac{1}{9^9} - \&c. = \frac{\epsilon}{1.2\ldots8} \cdot \frac{\pi^9}{2^{10}}$$

$$1 - \frac{1}{3^{11}} + \frac{1}{5^{11}} - \frac{1}{7^{11}} + \frac{1}{9^{11}} - \&c. = \frac{\zeta}{1.2\ldots10} \cdot \frac{\pi^{11}}{2^{12}}$$

&c.

erit:

$$\alpha = 1$$
$$\beta = 1$$
$$\gamma = 5$$
$$\delta = 61$$
$$\epsilon = 1385$$
$$\zeta = 50521$$
$$\eta = 2702765$$
$$\theta = 199360981$$
$$\iota = 19391512145$$
$$\kappa = 2404879661671 \quad \&c.$$

ex

ex hisque valoribus obtinebitur:

$$\text{ſec.}\,x = \alpha + \frac{\beta}{1.2}xx + \frac{\gamma}{1.2.3.4}x^4 + \frac{\delta}{1.2\ldots6}x^6 + \frac{\varepsilon}{1.2\ldots8}x^8 + \&c.$$

225. Vt autem nexum huius ſeriei cum numeris α, β, γ, δ, &c. oſtendamus, conſideremus ſeriem ſupra tractatam:

$$\frac{\pi}{n\,\text{ſin}\,\frac{m}{n}\pi} =$$

$$\frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \&c.$$

Ponatur $m = \frac{1}{2}n - k$, eritque

$$\frac{\pi}{2n\,\text{coſ}\,\frac{k}{n}\pi} =$$

$$\frac{1}{n-2k} + \frac{1}{n+2k} - \frac{1}{3n-2k} - \frac{1}{3n+2k} + \frac{1}{5n-2k} + \&c.$$

Sit $\frac{k\pi}{n} = x$, ſeu $k\pi = nx$, erit

$$\frac{\pi}{2n}\,\text{ſec.}\,x = \frac{\pi}{n\pi-2nx} + \frac{\pi}{n\pi+2nx} - \frac{\pi}{3n\pi-2nx} - \&c.$$

ſeu

$$\text{ſec.}\,x = \frac{2}{\pi-2x} + \frac{2}{\pi+2x} - \frac{2}{3\pi-2x} - \frac{2}{3\pi+2x} + \frac{2}{5\pi-2x} + \&c.$$

$$\text{ſec.}\,x = \frac{4\pi}{\pi^2-4x^2} - \frac{4.3\pi}{9\pi^2-4xx} + \frac{4.5\pi}{25\pi^2-4xx} - \frac{4.7\pi}{49\pi^2-4xx} + \&c.$$

ſi

ſi nunc ſinguli termini in ſeries conuertantur, fiet:

$$\text{ſec.}\, x = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \&c.\right)$$

$$+ \frac{2^4 x^2}{\pi^3}\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \&c.\right)$$

$$+ \frac{2^6 x^4}{\pi^5}\left(1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \&c.\right)$$

&c.

quarum ſerierum loco ſi valores ſupra aſſignati ſubſtituantur, prodibit eadem ſeries pro ſecante, quam dedimus.

226. Hinc ſimul patet lex, qua numeri α, β, γ, δ, &c. quibus ſummae poteſtatum imparium conſtituuntur, procedunt. Cum enim ſit

$$\text{ſec.}\, x = \frac{1}{\text{coſ}\, x} = \alpha + \frac{\beta}{1.2} x^2 + \frac{\gamma}{1.2.3.4} x^4 + \frac{\delta}{1.2\ldots6} + \&c.$$

neceſſe eſt vt haec ſeries aequalis ſit fractioni

$$\frac{1}{1 - \frac{xx}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2\ldots6} + \frac{x^8}{1.2\ldots8} - \&c.}$$

aequalitate ergo conſtituta fiet $1 =$

$$\alpha + \frac{\beta}{1.2} x^2 + \frac{\gamma}{1.2.3.4} x^4 + \frac{\delta}{1.2\ldots6} x^6 + \frac{\varepsilon}{1.2\ldots8} x^8 + \&c.$$

$$- \frac{\alpha}{1.2} - \frac{\beta}{1.2.1.2} - \frac{\gamma}{1.2.1\ldots4} - \frac{\delta}{1.2.1\ldots6}\ \&c.$$

+

$$+\frac{\alpha}{1.2.3.4}+\frac{\beta}{1\ldots4.1.2}+\frac{\gamma}{1\ldots4.1\ldots4}-\&c.$$

$$-\frac{\alpha}{1\ldots6}-\frac{\beta}{1\ldots6.1.2}-\&c.$$

$$+\frac{\alpha}{1\ldots8}\quad\&c.$$

vnde ſequuntur hae aequationes:

$$\alpha = 1$$

$$\beta = \frac{2.1}{1.2}\alpha$$

$$\gamma = \frac{4.3}{1.2}\beta - \frac{4.3.2.1}{1.2.3.4}\alpha$$

$$\delta = \frac{6.5}{1.2}\gamma - \frac{6.5.4.3}{1.2.3.4}\beta + \frac{6\ldots1}{1\ldots6}\alpha$$

$$\varepsilon = \frac{8.7}{1.2}\delta - \frac{8.7.6.5}{1.2.3.4}\gamma + \frac{8\ldots3}{3\ldots8}\beta - \frac{8\ldots1}{1\ldots8}\alpha$$

&c.

Ex hisque formulis inuenti ſunt iſtarum litterarum valores, quos in §. 224. exhibuimus; & quorum ope ſummae ſerierum in hac forma contentarum,

$$1-\frac{1}{3^n}+\frac{1}{5^n}-\frac{1}{7^n}+\frac{1}{9^n}-\&c.$$

ſi n fuerit numerus impar, exprimi poſſunt.

CAPUT IX.

DE VSU CALCULI DIFFERENTIALIS IN AEQUATIONIBUS RESOLUENDIS.

227.

Conſtitutionem aequationum ad functionum rationem reduci poſſe ſupra iam ſatis oſtenſum eſt. Denotet enim y functionem quamcunque ipſius x, ſi ponatur $y = 0$, in hac forma omnes omnino aequationes finitae ſiue ſint algabraicae ſiue transcendentes comprehenduntur. Aequatio autem $y = 0$ reſolui dicitur, ſi is ipſius x valor definiatur, qui in functione y ſubſtitutus, eam actu nihilo aequalem reddat. Plerumque autem plures eiusmodi valores pro x dantur, qui aequationis $y = 0$ radices vocantur. Si igitur ponamus numeros f, g, h, i, &c. eſſe radices aequationis $y = 0$, functio y ita erit comparata, vt ſi in ea loco x vel f, vel g, vel h, &c. ſubſtituatur, fiat reuera $y = 0$.

228. Quoniam igitur functio y euaneſcit, ſi in ea loco x ponatur f, ſeu $x + (f - x)$, exiſtente f radice aequationis $y = 0$, erit per ea, quae ſupra de functionibus demonſtrauimus:

$$0 = y + \frac{(f-x)dy}{dx} + \frac{(f-x)^2 ddy}{2dx^2} + \frac{(f-x)^3 d^3y}{6dx^3} + \&c.$$

ex

ex qua aequatione valor radicis f ita definitur, vt quicquid pro x fuerit poſitum, indeque valores quantitatum y, $\frac{dy}{dx}$, $\frac{ddy}{2dx^2}$, &c ſubſtituti, ſemper reſultet aequatio verum valorem ipſius f exhibens. Quo hoc clarius percipiatur, ponamus eſſe $y = x^3 - 2x^2 + 3x - 4$; erit $\frac{dy}{dx} = 3xx - 4x + 3$; $\frac{ddy}{2dx^2} = 3x - 2$; & $\frac{d^3y}{6dx^3} = 1$.

Quibus valoribus ſubſtitutis oritur:

$$0 = x^3 - 2x^2 + 3x - 4 + (f-x)(3xx - 4x + 3)$$
$$+ (f-x)^2(3x-2) + (f-x)^3$$

ſeu multiplicationibus actu inſtitutis:

$$f^3 - 2ff + 3f - 4 = 0$$

oritur ſcilicet aequatio ſimilis ipſi propoſitae, quae propterea easdem continet radices.

229. Quanquam autem hoc modo ad nouam aequationem non peruenitur, ex qua valor radicis f facilius definiri queat; tamen hinc ingentia ſubſidia ad inventionem radicum deduci poſſunt. Si enim pro x aſſumtus fuerit valor iam proxime ad quampiam radicem aequationis accedens, ita vt $f-x$ ſit quantitas valde parva, tum termini aequationis:

$$0 = y + \frac{(f-x)\,dy}{dx} + \frac{(f-x)^2\,ddy}{2\,dx^2} + \frac{(f-x)^3\,d^3y}{6\,dx^3} + \&c.$$

vehementer conuergent, hancque ob cauſam non multum a veritate aberrabitur, ſi praeter binos terminos ini-

tiales reliqui reiiciantur. Erit ergo ſi pro x iam valor cuipiam aequationis $y = 0$ radici prope aequalis fuerit aſſumtus, proxime $0 = y + \frac{(f-x)dy}{dx}$ ſeu $f = x - \frac{ydx}{dy}$, ex qua formula etſi non verus, tamen admodum propinquus radicis f valor reperietur, qui deinceps denuo loco x ſubſtitutus, multo adhuc propiorem valorem pro f ſuppeditabit; ſicque continuo propius ad verum radicis f valorem accedetur.

230. Hinc igitur primum radices omnium dignitatum ex quibuscunque numeris extrahi poſſunt. Sit enim propoſitus numerus $a^n + b$ ex quo radicem poteſtatis n extrahi oporteat. Ponatur $x^n = a^n + b$ ſeu $x^n - a^n - b = 0$, vt ſit $y = x^n - a^n - b$; erit

$$\frac{dy}{dx} = nx^{n-1};\quad \frac{ddy}{2dx^2} = \frac{n(n-1)}{1.2}x^{n-2};\quad \frac{d^3y}{6dx^3} = \frac{n(n-1)(n-2)}{1.\ 2.\ 3}x^{n-3};$$
&c.

Hinc ſi radix quaeſita ponatur $= f$, vt ſit $f = \sqrt[n]{(a^n + b)}$ erit:

$$0 = x^n - a^n - b + n(f-x)x^{n-1} + \frac{n(n-1)}{1.2}(f-x)^2x^{n-2} + \&c.$$

Si igitur pro x iam ſtatuatur numerus ad valorem radicis quaeſitae f prope accedens, quod fiet ponendo $x = a$, ſi quidem b ſit numerus tam paruus, vt $a^n + b < (a+1)^n$; erit $b = na^{n-1}(f-a)$ proxime, ideoque $f = a + \frac{b}{na^{n-1}}$, vnde valor radicis multo propius

pius cognoſcetur. Sin autem adhuc tertium terminum aſſumere velimus, vt ſit

$$b = n a^{n-1}(f-a) + \frac{n(n-1)}{1.2} a^{n-2}(f-a)^2, \text{ fiet}$$

$$(f-a)^2 = -\frac{2a}{n-1}(f-a) + \frac{2b}{n(n-1)a^{n-2}}, \text{ ideoque}$$

$$f = a - \frac{a}{n-1} \pm V\left(\frac{aa}{(n-1)^2} + \frac{2b}{n(n-1)a^{n-2}}\right) \text{ ſeu}$$

$$f = \frac{(n-2)a + V(aa + 2(n-1)b : na^{n-2})}{n-1}.$$

Quare ope extractionis radicis quadratae valor radicis f adhuc propius reperietur.

EXEMPLUM.

Quaeramus radicem quadratam ex numero quocunque c, *ſeu ſit* xx — c = y.

Ponatur ergo numerus radici proximus $= a$, & $b = c - aa$, ob $aa + b = c$, & quia eſt $n = 2$, fiet prior formula $f = a + \frac{c-aa}{2a} = \frac{c+aa}{2a}$, altera vero dat $f = Vc$, quae eſt ipſa radix quaeſita. Cum igitur ſit proxime radix $= \frac{c+aa}{2a}$, hic ipſe valor pro a ſcribatur, eritque propius radix $f = \frac{cc + 6aac + a^4}{4a(c+aa)}$. Sit verbi gratia $c = 5$; erit ex priori formula $f = \frac{5}{2a} + \frac{a}{2}$. Ponatur ergo $a = 2$;

erit $f = 2,25$; nunc ponatur $a = 2,25$; fiet $f = 2,236111$, ſtatuatur porro $a = 2,236111$, erit $f = 2,2360679$, qui valor iam minime a vero diſcrepat.

231. Simili autem modo radix cuiuscunque aequationis inueniri poteſt proxime ope aequationis $f = x - \frac{ydx}{dy}$, poſtquam ſcilicet pro x aſſumtus fuerit valor parum a quapiam aequationis radice diſcrepans. Ad huiusmodi vero valorem pro x inueniendum, ſubſtituantur ſucceſſiue pro x varii valores, inter eosque is eligatur, qui functionis y minimum hoc eſt cyphrae proximum valorem indicat. Sic ſi ſit $y = x^3 - 2xx + 3x - 4$,

poſito $x = 0$ fit $y = -4$

$x = 1 \;...\; y = -2$

$x = 2 \;...\; y = +2$

vnde videmus radicem contineri inter valores 1 & 2, ipſius x. Cum ergo ſit $\frac{dy}{dx} = 3xx - 4x + 3$, habebitur pro radice f aequationis $x^3 - 2xx + 3x - 4 = 0$ inuenienda haec aequatio:

$$f = x - \frac{ydx}{dy} = x - \frac{(x^3 - 2xx + 3x - 4)}{3xx - 4x + 3}.$$

Sit ergo $x = 1$; fiet $f = 1 + \frac{2}{2} = 2$. Nunc ponatur $x = 2$, fiet $f = 2 - \frac{2}{7} = \frac{12}{7}$. Sit ergo $x = \frac{12}{7}$; erit

erit $f = \frac{12}{7} - \frac{104}{1701} = \frac{2812}{1701} = 1,653$. Si vlterius progredi velimus, logarithmis commodius vtemur.

Ponatur ergo $x = 1,653$, eritque

$lx = 0,2182729$	$x = 1,653000$
$lx^2 = 0,4365458$	$x^2 = 2,732409$
$lx^3 = 0,6548187$	$x^3 = 4,516673$

$x^3 = 4,516673$	
$3x = 4,959000$	
$9,475673$	$3xx+3 = 11,197227$
$2xx+4 = 9,464818$	$4x = 6,612000$
num. $0,010855$	den. $= 4,585227$
l num. $= 8,0356298$	
l den. $= 0,6613608$	$x = 1,653000$
l fract. $= 7,3742690$;	fractio $= 0,002367$
	$f = 2,650633$

quì valor iam proxime ad verum accedit.

232. Citiores autem approximationes ex generali expreſſione deducere poterimus. Cum enim poſita functione quacunque $y = 0$, ſi radix huius aequationis fuerit $x = f$, inuenerimus eſſe:

$$0 = y + \frac{(f-x)\,dy}{dx} + \frac{(f-x)^2\,ddy}{2\,dx^2} + \frac{(f-x)^3\,d^3y}{6\,dx^3} + \&c.$$

ſit $f - x = z$, ita vt ſit radix $f = x + z$, atque ponatur

$$\frac{dy}{dx} = p;\quad \frac{dp}{dx} = q;\quad \frac{dq}{dx} = r;\quad \frac{dr}{dx} = s;\quad \text{erit:}$$

$$0 = y + zp + \frac{z^2q}{2} + \frac{z^3r}{6} + \frac{z^4s}{24} + \frac{z^5t}{120} + \&c.$$

in

in qua aequatione ſumto pro x valore quocunque, ex quo ſimul y, p, q, r, s, &c. determinantur, inueniri debet quantitas z, qua inuenta habebitur aequationis propoſitae $y = 0$, radix $f = x + z$. In id ergo eſt incumbendum, vt quam commodiſſime ex hac aequatione valor incognitae z eruatur.

233. Fingatur pro z ſeries conuergens haec:

$$z = A + B + C + D + E + \&c.$$

atque facta ſubſtitutione erit:

$$\begin{aligned}
y &= y \\
pz &= Ap + Bp + Cp + Dp + Ep + \&c. \\
\tfrac{1}{2}qz^2 &= \quad + \tfrac{1}{2}A^2q + ABq + ACq + ADq + \&c. \\
&\qquad\qquad + \tfrac{1}{2}BBq + BCq + \&c. \\
\tfrac{1}{6}rz^3 &= \quad \tfrac{1}{6}A^3r + \tfrac{1}{2}A^2Br + \tfrac{1}{2}A^2Cr + \&c. \\
&\qquad\qquad + \tfrac{1}{2}AB^2r + \&c. \\
\tfrac{1}{24}sz^4 &= \quad \tfrac{1}{24}A^4s + \tfrac{1}{6}A^3Bs + \&c. \\
\tfrac{1}{120}tz^5 &= \quad \tfrac{1}{120}A^5t + \&c.
\end{aligned}$$

Vnde obtinentur ſequentes aequationes:

$$A = -\frac{y}{p}$$

$$B = -\frac{yyq}{2p3}$$

$$C = -\frac{y^3qq}{2p^5} + \frac{y^3r}{6p^4}$$

$$D = -\frac{5y^4q^3}{8p^7} + \frac{5y^4qr}{12p^6} - \frac{y^4s}{24p^5} \quad \&c.$$

ideo-

ideoque erit:

$$z = -\frac{y}{p} - \frac{y^2 q}{2p^3} - \frac{y^3 qq}{2p^5} + \frac{y^3 r}{6p^4} - \frac{5y^4 q^3}{8p^7} + \frac{5y^4 qr}{12p^6} - \frac{y^4 s}{24p^5} - \&c.$$

EXEMPLUM.

Sit propoſita haec aequatio $x^5 + 2x - 2 = 0$.

Erit ergo $y = x^5 + 2x - 2$;

$$\frac{dy}{dx} = p = 5x^4 + 2$$

$$\frac{dp}{dx} = q = 20x^3$$

$$\frac{dq}{dx} = r = 60x^2$$

$$\frac{dr}{dx} = s = 120x \quad \&c.$$

Ponatur autem nunc $x = 1$, quia hic valor parum a radice diſcrepat, erit:

$$y = 1; \quad p = 7; \quad q = 20; \quad r = 60; \quad s = 120.$$

vnde fiet:

$$z = -\frac{1}{7} - \frac{10}{7^3} - \frac{200}{7^5} + \frac{10}{7^4} - \frac{5.1000}{7^7} + \frac{500}{7^6} - \frac{5}{7^5}$$

ſeu $z = -\frac{1}{7} - \frac{10}{7^3} - \frac{130}{7^5} - \frac{1745}{7^7}$ &c. eritque ergo $z = -0{,}18$, & radix $f = 0{,}82$, qui valor ſi denuo loco x ſubſtitueretur, prodiret radix maxime verae propinqua.

234. Inuenimus ergo ſeriem infinitam, quae cuiusvis aequationis radicem exprimit: ea autem hoc laborat incommodo vt tum lex progreſſionis non pateat, tum ipſa nimis ſit perplexa atque ad vſum non ſatis accommodata. Alio igitur modo idem negotium ſuſcipiamus, ſeriemque magis regularem inueſtigemus, cuiuscunque aequationis propoſitae radicem exprimentem.

Sit vt ante propoſita aequatio $y = 0$, exiſtente y functione quacunque ipſius x; & quaeſtio huc redit, vt valor ipſius x definiatur, qui loco x ſubſtitutus functionem y reddat nihilo aequalem. Cum autem y ſit functio ipſius x, viciſſim x tanquam functio ſpectari poterit ipſius y, atque hac conſideratione adhibita quaerendus eſt valor ipſius functionis x, quem induit, cum quantitas y euaneſcit. Si igitur f ponatur deſignare iſtum ipſius x valorem, qui erit radix aequationis $y = 0$, quoniam x abit in f, ſi ſtatuatur $y = 0$, erit per ea quae ſupra ſunt demonſtrata:

$$f = x - \frac{y\,dx}{dy} + \frac{y^2\,ddx}{2dy^2} + \frac{y^3\,d^3x}{6dy^3} + \frac{y^4\,d^4x}{24dy^4} - \&c.$$

in qua aequatione ſtatuitur differentiale dy conſtans. Si igitur ponatur:

$$\frac{dx}{dy} = p;\quad \frac{dp}{dy} = q;\quad \frac{dq}{dy} = r;\quad \frac{dr}{dy} = s;\quad \&c.$$

erit his valoribus introductis, vt conſideratio differentialis conſtantis exuatur:

$$f = x - p + \frac{1}{2}qy^2 - \frac{1}{6}ry^3 + \frac{1}{24}sy^4 - \frac{1}{120}ty^5 + \&c.$$

235.

235. Tributo ergo ipſi x quocunque valore, ſimul valores ipſius y, atque quantitatum p, q, r, s, &c. determinabuntur; hisque inuentis habebitur ſeries infinita valorem radicis f exprimens. Sin autem aequatio $y = 0$ plures admittat radices, tum eae prodibunt, ſi pro x diverſi valores aſſumantur: quia enim y eundem valorem induere poteſt, etiamſi ipſi x diuerſi valores tribuantur, mirum non eſt eandem ſeriem ſaepenumero plures valores ſuppeditare poſſe. Quo igitur his caſibus ambiguitas tollatur, ſimulque ſeries conuergens reddatur, pro x aſſumi debet valor iam prope ad valorem eius radicis, quae quaeritur, accedens. Hoc enim modo valor ipſius y fiet admodum paruus, ſerieique termini vehementer decreſcent, ita vt paucis terminis ſumendis iam ſatis iustus valor pro f inueniatur. Hic igitur valor ſi deinceps loco x ſubſtituatur, quantitas y multo minor euadet, ſeriesque multo magis conuerget; hocque modo ſtatim radix f tam exacte innoteſcet, vt error futurus ſit minimus. Hincque ſumma huius expreſſionis praerogatiua prae ea, quam ante elicueramus, manifeſto perſpicitur.

236. Ponamus extrahendam eſſe radicem poteſtatis n ex numero quocunque N. Sumta igitur proxima poteſtate exponentis n, numerus propoſitus facile reſoluetur in hanc formam $N = a^n + b$. Erit ergo

$$x^n = a^n + b \quad \& \quad y = x^n - a^n - b; \text{ vnde fit:}$$

 $dy =$

$$dy = nx^{n-1}dx \;;\; \& \; \frac{dx}{dy} = p = \frac{1}{nx^{n-1}}$$

$$dp = -\frac{(n-1)dx}{nx^2} \;;\; \& \; \frac{dp}{dy} = q = -\frac{(n-1)}{nnx^{2n-1}}$$

$$dq = \frac{(n-1)(2n-1)dx}{nnx^{2n}} \;;\; \& \; \frac{dq}{dy} = r = \frac{(n-1)(2n-1)}{n^3x^{3n-1}}$$

$$dr = -\frac{(n-1)(2n-1)(3n-1)dx}{n^3x^{3n}} \;;\; \& \; s = -\frac{(n-1)(2n-1)(3n-1)}{n^4x^{4n}}$$

&c.

Ponatur nunc $x = a$, eritque $y = -b$, atque radix quaeſita $f = \sqrt[n]{(a^n + b)}$ hoc modo exprimetur:

$$f = a + \frac{b}{na^{n-1}} - \frac{(n-1)bb}{n.2n.a^{2n-1}} + \frac{(n-1)(2n-1)b^3}{n.2n.3n.a^{3n-1}} - \frac{(n-1)(2n-1)(3n-1)b^4}{n.2n.3n.4n\;a^{4n-1}} + \&c.$$

ſicque prodit eadem ſeries, quae vulgo per euolutionem binomii $(a^n + b)^{\frac{1}{n}}$ erui ſolet.

237. Poſtquam ergo in actuali extractione radix proxime vera a fuerit inuenta, ſimulque reſiduum b fuerit repertum, tum ad radicem inſuper addi oportet valorem fractionis $\frac{b}{na^{n-1}}$, quo propius vera radix obtineatur. Erit autem $a^{n-1} = \frac{N-b}{a}$, ob $N = a^n + b$. At vero hoc modo radix iuſto maior inuenietur, quoniam tertius terminus ſubtrahi debet. Quo igitur per diui-

diuiſionem reſidui b radix multo propius ad verum accedens inueniatur idoneus diuiſor debet inueſtigari, qui fingatur eſſe $na^{n-1} + \alpha b + \beta bb + \gamma b^3 +$ &c.

Cum igitur debeat eſſe:

$$\frac{b}{na^{n-1} + \alpha b + \beta b^2 + \gamma b^3 + \text{\&c.}} =$$

$$\frac{b}{na^{n-1}} - \frac{(n-1)bb}{2n^2a^{2n-1}} + \frac{(n-1)(2n-1)b^3}{6n^2a^{3n-1}} - \frac{(n-1)(2n-1)(3n-1)b^4}{24n^4a^{4n-1}}$$
$$+ \text{\&c.}$$

fiet multiplicatione per $na^{n-1} + \alpha b + \beta b^2 + \gamma b^3 +$ &c. inſtituta:

$$b = b$$

$$- \frac{(n-1)bb}{2na^n} + \frac{(n-1)(2n-1)b^3}{6n^2a^{2n}} - \frac{(n-1)(2n-1)(3n-1)b^4}{24n^3a^{3n}} + \text{\&c.}$$

$$+ \frac{\alpha b^2}{na^{n-1}} - \frac{(n-1)\alpha b^3}{2n^2a^{2n-1}} + \frac{(n-1)(2n-1)\alpha b^4}{6n^3a^{3n-1}}$$

$$+ \frac{\beta b^3}{na^{n-1}} - \frac{(n-1)\beta b^4}{2n^2a^{2n-1}}$$

$$+ \frac{\gamma b^4}{na^{n-1}}$$

Hinc deducuntur ſequentes determinationes:

$$\alpha = \frac{n-1}{2a}$$

$$\beta = \frac{(n-1)\alpha}{2na^n} - \frac{(n-1)(2n-1)}{6na^{n+1}} = - \frac{(n-1)(n+1)}{12na^{n+1}}$$

$$\gamma = \frac{(n-1)\beta}{2na^n} - \frac{(n-1)(2n-1)\alpha}{6nna^{2n}} + \frac{(n-1)(2n-1)(3n-1)}{24n^2a^{2n+1}}$$

ſeu $\gamma = \frac{(n-1)(n+1)}{24na^{2n+1}}$. Frac-

Fractio ergo ad radicem iam inuentam a inſuper addenda erit:

$$\frac{b}{na^{n-1}+\frac{(n-1)b}{2a}-\frac{(nn-1)bb}{12na^{n+1}}+\frac{(nn-1)b^3}{24na^{2n+1}}}.$$

238. Quod ſi ergo radix quadrata extrahi debeat ex numero N, atque inuenta iam ſit radix proxima $=a$, cum reſiduo $=b$, ad radicem inuentam inſuper addi debet quotus, qui oritur, ſi reſiduum b diuidatur per $2a+\frac{b}{2a}-\frac{bb}{8a^3}+\frac{b^3}{16a^5}-$ &c. Sin autem radix cubica extrahi debeat, tum reſiduum b diuidi debet per $3a^2+\frac{b}{a}-\frac{2bb}{9a^4}+\frac{b^3}{9a^7}-$ &c. quarum formularum vſum in his exemplis declarabimus.

EXEMPLUM I.

Extrahatur radix quadrata ex numero 200.

Ponatur $N=200$, & cum proximum quadratum ſit 196, erit $a=14$, & reſiduum $b=4$, quod propterea diuidi debebit per $28+\frac{1}{7}-\frac{1}{7}\cdot\frac{1}{196}+\frac{1}{7\cdot196\cdot98}$, eritque ergo diuiſor $=28,142135$, per quem ſi 4 diuidatur, obtinebitur fractio decimalis ad 14 addenda, quae iuſta erit ad 10 figuras & vltra.

EXEMPLUM II.

Extrahatur radix cubica ex numero $N=10$.

Proximus cubus eſt 8, & reſiduum $=2$, vnde $a=2$

&

& $b = 2$, atque diuiſor $= 12 + 1 - \frac{1}{18} = 12{,}9444$. Quare radix cubica quaeſita erit proxime $=$

$$2\frac{2}{12{,}9444} = 2\frac{10000}{64722}.$$

239. Series pro radice inuenta etiam conſiderari poteſt tanquam recurrens orta ex quapiam fractione, hoc enim modo plures termini ſeriei ad multo pauciores, qui numeratorem & denominatorem fractionis conſtituant, reuocabuntur. Sic leui attentione adhibita perſpicietur fore proxime:

$$(a+b)^n = a^n.\frac{a + \frac{(n+1)}{2}b}{a + \frac{(n-1)}{2}b}$$ atque adhuc propius

$$(a+b)^n = a^n.\frac{aa + \frac{(n+2)}{2}ab + \frac{(n+1)(n+2)}{12}bb}{aa - \frac{(n-2)}{2}ab + \frac{(n-1)(n-2)}{12}bb}.$$

Simili modo plures terminos introducendo fractiones adhuc accuratiores obtineri poſſunt:

$$(a+b)^n = a^n.$$

$$\frac{a^3 + \frac{(n+3)}{2}a^2b + \frac{(n+3)(n+2)}{10}ab^2 + \frac{(n+3)(n+2)(n+1)b^3}{120}}{a^3 - \frac{(n-3)}{2}a^2b + \frac{(n-3)(n-2)}{10}ab^2 - \frac{(n-3)(n-2)(n-1)b^3}{120}}.$$

Quin etiam huiusmodi forma generalis exhiberi poteſt, ad quam commode exprimendam ſit:

A =

$$A = \frac{m(n+m)}{1.\ 2m} \qquad \mathfrak{A} = \frac{m(n-m)}{1.\ 2m}$$

$$B = \frac{(m-1)(n+m-1)}{2\ (2m-1)} A \qquad \mathfrak{B} = \frac{(m-1)(n-m+1)}{2\ (2m-1)} \mathfrak{A}$$

$$C = \frac{(m-2)(n+m-2)}{3\ (2m-2)} B \qquad \mathfrak{C} = \frac{(m-2)(n-m+2)}{3\ (2m-2)} \mathfrak{B}$$

$$D = \frac{(m-3)(n+m-3)}{4\ (2m-3)} C \qquad \mathfrak{D} = \frac{(m-3)(n-m+3)}{4\ (2m-3)} \mathfrak{C}$$

&c. &c.

His autem valoribus determinatis erit:

$$(a+b)^n = a^n . \frac{a^m + Aa^{m-1}b + Ba^{m-2}b^2 + Ca^{m-3}b^3 + \&c.}{a^m - \mathfrak{A}a^{m-1}b + \mathfrak{B}a^{m-2}b^2 - \mathfrak{C}a^{m-3}b^3 + \&c.}$$

240. Si igitur hic pro n ſubſtituatur numerus fractus, iſtae formulae ad extractionem radicum apprime erunt accommodatae. Sic ſi radix quaecunque poteſtatis n extrahi debeat ex forma $a^n + b$, ſequentes formulae in vſum vocari poſſunt:

$$(a^n+b)^{\frac{1}{n}} = a . \frac{2na^n + (n+1)b}{2na^n + (n-1)b}$$

$$(a^n+b)^{\frac{1}{n}} = a . \frac{12n^2a^{2n} + 6n(2n+1)a^nb + (2n+1)(n+1)bb}{12n^2a^{2n} + 6n(2n-1)a^nb + (2n-1)(n-1)bb}.$$

Sin autem ponatur $a^n + b = N$, vt ſit $a^n = N - b$; erit:

$$(a^n+b)^{\frac{1}{n}} = a . \frac{2nN - (n-1)b}{2nN - (n+1)b}$$

$$(a^n+b)^{\frac{1}{n}} = a . \frac{12n^2N^2 - 6n(2n-1)Nb + (2n-1)(n-1)bb}{12n^2N^2 - 6n(2n+1)Nb + (2n+1)(n+1)bb}.$$

241.

241. Formula igitur generalis pro radice cuiusque aequationis inuenienda in aequationibus, quae ex pluribus terminis conſtant, eundem praeſtat vſum, quem ſolita regula binomii ad reſolutionem aequationum purarum $x^n = c$ afferre ſolet, atque adeo hoc caſu in regulam illam ipſam abit. Sin autem aequatio fuerit affecta vel etiam transcendens, expreſſio noſtra generalis ſemper aequali ſucceſſu in vſum vocatur, ſeriemque praebet infinitam, quae valorem radicis exhibet. Quamobrem cum in hoc negotio ſumma vis iſtius formulae generalis conſiſtat, eius vſum hic aliquanto fuſius oſtendamus. Sit igitur propoſita haec aequatio affecta tribus terminis conſtans:

$$x^n + cx = \mathrm{N},$$

denotantibus c & N quantitates quascunque datas. Ponatur $x^n + cx - \mathrm{N} = y$, erit $dy = (nx^{n-1} + c)\,dx$, hincque fiet $p = \frac{1}{nx^{n-1} + c}$, tum eſt

$$dp = -\frac{n(n-1)x^{n-2}\,dx}{(nx^{n-1}+c)^2}; \quad \& \quad q = -\frac{n(n-1)x^{n-2}}{(nx^{n-1}+c)^3}.$$

Simili modo ob $r = \frac{dq}{dy}$; $s = \frac{dr}{dy}$ &c. reperietur:

$$r = \frac{n^2(n-1)(2n-1)x^{2n-4} - n(n-1)(n-2)cx^{n-3}}{(nx^{n-1}+c)^5}$$

$$s = -\frac{n^3(n-1)(2n-1)(3n-1)x^{3n-6} + 4n^2(n-1)(n-2)(2n-1)cx^{2n-5} - n(n-1)(n-2)(n-3)c^2x^{n-4}}{(nx^{n-1}+c)^7}$$

$$t = +\frac{\begin{array}{l} n^4(n-1)(2n-1)(3n-1)(4n-1)x^{4n-8} - n^3(n-1)(n-2)(2n-1)(29n-11)cx^{3n-7} \\ + n^2(n-1)(n-2)(2n-1)(nn-29)c^2x^{2n-6} - n(n-1)(n-2)(n-3)(n-4)c^3x^{n-5} \end{array}}{(nx^{n-1}+c)^9}$$

&c.

 Qui-

Quibus valoribus inuentis, erit aequationis propoſitae radix

$$f = x - py + \frac{1}{2}qyy - \frac{1}{6}ry^3 + \frac{1}{24}sy^4 - \frac{1}{120}ty^5 + \&c.$$

quicquid enim pro x ſubſtituatur, vnde ſimul litterae y, p, q, r, &c. valores determinatos induunt, ſumma ſeriei aequabitur valori vnius radicis.

EXEMPLUM I.

Sit propoſita haec aequatio $x^3 + 2x = 2$.
Erit $c = 2$, $N = 2$, & $n = 3$, atque $y = x^3 + 2x - 2$.
Ponatur $x = 1$, erit $y = 1$, & $p = \frac{1}{5}$; $q = -\frac{6}{5^3}$; $r = \frac{84}{5^5}$; $s = -\frac{16.90}{5^7}$ &c. atque aequationis radix erit:

$$f = 1 - \frac{1}{5} - \frac{3}{5^3} - \frac{14}{5^5} - \frac{60}{5^7} - \&c. = 0,770751.$$

Ponatur nunc $x = 0,77$, & quia eſt $y = x^3 + 2x - 2$; $p = \frac{1}{3xx+2}$; $q = -6p^3x$; $r = 9xxp^5 - 12p^5$; atque $s = -2160p^7x^3 + 720p^7x$; habebitur logarithmis adhibendis:

$lx = 9,8864907$	$x = 0,77$
$lx^2 = 9,7729814$	$x^2 = 0,5929$
$lx^3 = 9,6594721$	$x^3 = 0,456533$
	$2x = 1,54$
	$x^3 + 2x = 1,996533$
Ergo	$y = -0,003467$

$l-y$

$l{-}y = 7,5399538$; $3xx+2 = 3,7787$

$lp = 9,4226575$; $l(3xx+2) = 0,5773424$

$l{-}py = 6,9626113$; $-py = 0,000917511$

$lp^3 = 8,2679725$

$lx = 9,8864907$

$l3 = 0,4771213$

$ly^2 = 5,0799076$

$l{-}\frac{1}{2}qyy = 3,7114922$ $\quad -\frac{1}{2}qyy = 0,000000514$.

Ergo radix $f = 0,770916997$, quae vix in vltima figura a vero aberrabit.

EXEMPLUM II.

Sit propoſita aequatio $x^4 - 2xx + 4x = 8$.

Ponatur $y = x^4 - 2xx + 4x - 8$, erit $dy = 4dx(x^3 - x + 1)$

$$p = \frac{1}{4(x^3-x+1)}; \quad \frac{dp}{dx} = \frac{-3xx+1}{4(x^3-x+1)^2} \text{ Ergo}$$

$$q = \frac{-3xx+1}{16(x^3-x+1)^3}; \quad \frac{dq}{dx} = \frac{21x^4-12xx-3}{16(x^3-x+1)^4} \quad \&$$

$$r = \frac{21x^4-12xx-3}{64(x^3-x+1)^5} \quad \&c.$$

ex quibus erit radix aequationis propoſitae:

$$f = x - \frac{y}{4(x^3-x+1)} - \frac{(3xx-1)yy}{32(x^3-x+1)^3} - \frac{(7x^4-4xx+1)y^3}{128(x^3-x+1)^5} - \&c.$$

Oportet ergo ipſi x idoneum valorem tribui, quo ſeries iſta fiat conuergens. Primum autem perſpicuum eſt, ſi

 ipſi

ipſi x tribueretur talis valor, quo fieret $x^3-x+1=0$, tum omnes ſeriei terminos praeter primum euadere infinitos, neque adeo exinde quicquam concludi poſſe. Conuenit ergo ipſi x eiusmodi valorem aſſignare, quo & y fiat exiguum & x^3-x+1 non admodum paruum. Sit $x=1$, erit $y=-5$, &

$$f=1+\frac{5}{4}-\frac{25}{16}+\frac{125}{64}-\&c.$$

vbi cum tres termini $\frac{5}{4}-\frac{25}{16}+\frac{125}{64}$ congruant cum progreſſione geometrica, cuius ſumma eſt $\frac{5}{9}$ erit circiter $f=\frac{14}{9}$. Statuamus ergo $x=\frac{3}{2}$, erit $y=-\frac{23}{16}$; & $x^3-x+1=\frac{23}{8}$, vnde fit:

$$f=\frac{3}{2}+\frac{1}{8}-\frac{1}{64}+\frac{407}{256.529}-\&c.=1,61.$$

Ponatur nunc $x=1,61$; erit:

$lx=0,2068259$	$x=1,61$	ſit $x^3-x+1=z$
$lx^2=0,4136518$	$x^2=2,5921$	
$lx^3=0,6204777$	$x^3=4,173281$	
$lx^4=0,8273036$	$x^4=6,718983$	
	hinc	
$l-y=8,4016934$	$y=-0,025217$	
$lz=0,5518502$	$z=3,563281$	
$l\frac{-y}{z}=7,8498432$		

$l4=$

$l4 = 0,6020600$	
$l\frac{-y}{4z} = 7,2477832$	$\frac{-y}{4z} = 0,0017692$
$l(3xx-1) = 0,8309926$	$3xx-1 = 6,7763$
$ly^2 = 6,8033868$	
$7,6343794$	
$lz^3 = 1,6555506$	
$5,9788288$	
$l32 = 1,2041200$	$\frac{(3xx-1)y^2}{32z^3} = 0,000005952$
$4,7747088$	

Ergo $f = 1,6117632$.

242. Methodus haec inueniendi radices aequationum proxime aeque patet ad quantitates tranſcendentes. Quaeramus numerum x, cuius logatithmus ex quocunque canone deſumtus ad ipſum numerum datam habeat rationem vt 1 ad n, atque habebitur iſta aequatio $x - nlx = 0$: ſit autem k modulus horum logarithmorum, ita vt iſti logarithmi obtineantur, ſi logarithmi hyperbolici multiplicentur per k, erit $d.lx = \frac{kdx}{x}$. Ponatur ergo $x - nlx = y$, ſitque f valor ipſius x quaeſitus, qui reddat $x = nlx$. Cum igitur ſit $y = x - nlx$, erit $dy = dx - \frac{kndx}{x} = \frac{dx(x-kn)}{x}$ & $\frac{dx}{dy} = p = \frac{x}{x-kn}$; vnde $dp = -\frac{kndx}{(x-kn)^2}$ ergo

$$\frac{dp}{dy} = q = \frac{-knx}{(x-kn)^3};\quad dq = \frac{2knxdx + k^2n^2dx}{(x-kn)^4}$$

$$\frac{dq}{dy} = r = \frac{knx(2x+kn)}{(x-kn)^5}\quad \&c.$$

Qua-

Quare fiet:

$$f = x - \frac{xy}{x-kn} - \frac{knxyy}{2(x-kn)^3} - \frac{knxy^3(2x+kn)}{6(x-kn)^5} - \&c.$$

Infra autem oſtendemus hoc problema ſolutionem non admittere, niſi ſit $kn > e$, exiſtente e numero cuius logarithmus hyperbolicus eſt $= 1$, ſeu debet eſſe $kn > 2,7182818$.

EXEMPLUM.

Quaeratur numerus praeter 10, *cuius logarithmus tabularis aequetur decimae parti ipſius numeri.*

Quia de logarithmis tabularibus quaeſtio inſtituitur, erit $k = 0,434294481903\,25$, atque ob $n = 10$ habebitur $kn = 4,342944819032\,5$. Facto iam $x = 1$, erit $y = 1$, fietque $f = 1 + \frac{1}{3,3429} + \frac{2,1714724}{(3,3429)^3} - \&c.$ ſicque proxime erit $f = 1,37$. Statuatur ergo $x = 1,37$, erit $lx = 0,136720567156406$, & ob $y = x - 10lx$, erit $y = 0,002794328435\,03$, & $-x + kn = 2,9729448190325$. Fiat ergo

$$\begin{array}{r} lx = 0,1367205 \\ ly = 7,4462773 \\ \hline 7,5829978 \\ l(kn-x) = 0,4731866 \\ \hline 1,1098112 \end{array} \quad \frac{-xy}{x-kn} = 0,00128769.$$

De-

Deinde cum ſit tertius terminus $\frac{knxyy}{2(x-kn)^3} = \frac{kny}{2(x-kn)^2} \cdot \frac{xy}{x-kn}$ erit:

$$l\frac{-xy}{x-kn} = 7,1098112$$
$$ly = 7,4462773$$
$$lkn = 0,6377842$$
$$\overline{5,1938527}$$
$$l(kn-x)^2 = 0,9463732$$
$$\overline{4,2474995}$$
$$l2 = 0,3010300$$

l tert. term.	$= 3,9464695$
I. term. x	$= 1,37$
II. term.	$= 0,00128769$
III. term.	$= 0,00000088$
f	$= 1,37128857$
lf	$= 0,137128857.$

243. Si aequatio fuerit exponentialis, ea ad logarithmicam reduci poterit; ita ſi quaeratur valor ipſius x, vt ſit $x^x = a$, erit $xlx = la$. Quare poſito $y = xlx - la$, fiet $dy = dxlx + dx$, & $\frac{dx}{dy} = p = \frac{1}{1+lx}$. Tumque

$dp =$

$$dp = \frac{-dx}{x(1+lx)^2}\,; \quad \& \quad \frac{dp}{dy} = q = \frac{-1}{x(1+lx)^3}$$

$$dy = \frac{dx}{xx(1+lx)^3} + \frac{3dx}{xx(1+lx)^4}\,, \quad \text{ideoque}$$

$$\frac{dq}{dy} = r = \frac{1}{xx(1+lx)^4} + \frac{3}{xx(1+lx)^5}\,; \quad \text{porro erit}$$

$$dr = \frac{-2dx}{x^3(1+lx)^4} - \frac{10dx}{x^3(1+lx)^5} - \frac{15dx}{x^3(1+lx)^6}\,; \quad \text{ergo}$$

$$s = \frac{-2}{x^3(1+lx)^5} - \frac{10}{x^3(1+lx)^6} - \frac{15}{x^3(1+lx)^7}\,, \quad \&$$

$$t = \frac{6}{x^4(1+lx)^6} + \frac{40}{x^4(1+lx)^7} + \frac{105}{x^4(1+lx)^8} + \frac{105}{x^4(1+lx)^9}$$

$$u = \frac{-24}{x^5(1+lx)^7} - \frac{196}{x^5(1+lx)^8} - \frac{700}{x^5(1+lx)^9} - \frac{1260}{x^5(1+lx)^{10}} - \frac{945}{x^5(1+lx)^{11}}.$$

Hinc ergo ſi verus valor ipſius x ſit $= f$, ita vt ſit $ff = a$; erit:

$$\begin{aligned} f = x - \frac{y}{(1+lx)} - \frac{yy}{2x(1+lx)^3} - \frac{y^3}{2xx(1+lx)^5} - \frac{5y^4}{8x^3(1+lx)^7} - \frac{7y^5}{8x^4(1+lx)^9} \\ - \frac{y^3}{6x^2(1+lx)^4} - \frac{5y^4}{12x^3(1+lx)^6} - \frac{7y^5}{8x^4(1+lx)^8} \\ - \frac{y^4}{12x^3(1+lx)^5} - \frac{y^5}{3x^4(1+lx)^7} \\ - \frac{y^5}{20x^4(1+lx)^6} \end{aligned}$$

&c.

Haec

Haec ergo expreſſio in infinitum continuata, quicunque valor pro x ſtatuatur, ſumto $y = x\,lx - la$ verum ipſius f dabit valorem. Sic ſi ponatur $x = 1$, erit $y = -la$, &

$$f = 1 + la - \frac{(la)^2}{2} + \frac{2(la)^3}{3} - \frac{9(la)^4}{8} + \frac{32(la)^5}{15} - \frac{625(la)^6}{144} - \&c.$$

vbi notandum eſt eſſe la logarithmum hyperbolicum ipſius a.

EXEMPLUM.

Quaeratur numerus f, *vt ſit* $f^f = 100$.

Cum ſit $a = 100$, & $y = x\,lx - la = x\,lx - l100$, quia patet eſſe $f > 3$ & < 4, ſtatuatur

$x = \frac{7}{2}$; eritque $lx = 1,25276296849$

$$x\,lx = 4,38467034972$$
$$l100 = 4,60517018599$$
$$y = -0,22049983627$$
$$1 + lx = 2,25276296849$$

Hinc erit logarithmis vulgaribus adhibendis:

$$l-y = 9,3434083$$
$$l(1+lx) = 0,3527156$$
$$8,9906927;\quad \frac{-y}{1+lx} = 0,0978797$$
$$ly^2 = 8,6868166$$
$$3l(1+lx) = 1,0581468$$
$$= 7,6286698$$
$$l2x = l7 = 0,8450980$$
$$6,7835718 \quad \frac{y^2}{2x(1+lx)^3} = 0,0006075$$

Ergo proxime erit $f = 3,5972722$

ſequentibus vero inſuper terminis

ſumtis erit $f = 3,5972852$

244. Praeterea autem calculus differentialis inſignem habet vſum in reſolutione aequationum, ſi quaepiam relatio, quae inter radices intercedit, fuerit cognita. Sit propoſita aequatio $y = 0$, in qua ſit y functio quaecunque ipſius x. Si iam verbi gratia conſtet, duas huius aequationis radices inter ſe differre quantitate data a, hae duae radices facile inuenientur ſequenti modo. Denotet x harum duarum radicum minorem, erit maior $= x + a$, quare cum functio y euaneſcat, ſi x ſignificet vnam ex radicibus aequationis $y = 0$, euaneſcet quoque y, ſi loco x ponatur $x + a$. Quocirca erit:

$$0 = y + \frac{a\,dy}{dx} + \frac{a^2\,ddy}{2dx^2} + \frac{a^3\,d^3y}{6dx^3} + \&c.$$

Vnde cum ſit $y = 0$ erit quoque

$$0 = \frac{dy}{dx} + \frac{a\,ddy}{2dx^2} + \frac{a^2\,d^3y}{6dx^3} + \frac{a^3\,d^4y}{24dx^4} + \&c.$$

quae duae aequationes ſimul ſumtae per methodum eliminationis dabunt valorem illius radicis x, quam alia radix ſuperat quantitate a.

EXEMPLUM.

Sit propoſita haec aequatio

$$x^5 - 24x^3 + 49xx - 36 = 0,$$

quam vndecunque conſtet, habere duas radices vnitate differentes.

Poſito

Poſito $y = x^5 - 24x^3 + 49xx - 36$ erit:

$$\frac{dy}{dx} = 5x^4 - 72x^2 + 98x$$

$$\frac{ddy}{2dx^2} = 10x^3 - 72x + 49$$

$$\frac{d^3y}{6dx^3} = 10x^2 - 24$$

$$\frac{d^4y}{24dx^4} = 5x$$

$$\frac{d^5y}{120dx^5} = 1$$

Jam ob $a = 1$ erit:

A . . . $5x^4 + 10x^3 - 62x^2 + 31x + 26 = 0$. At eſt

B . . . $x^5 - 24x^3 + 49xx - 36 = 0$.

Multiplicetur ſuperior per x & inferior per 5, alteraque ab altera ſubtracta relinquet:

$10x^4 + 58x^3 - 214x^2 + 26x + 180 = 0$ ſeu

C . . . $5x^4 + 29x^3 - 107x^2 + 13x + 90 = 0$

a qua prima A ſubtracta relinquet:

D . . . $19x^3 - 45x^2 - 18x + 64 = 0$

D.$5x$ . . . $95x^4 - 225x^3 - 90x^2 + 320x = 0$

A.19 . . . $95x^4 + 190x^3 - 1178x^2 + 589x + 494 = 0$

E . . . . $415x^3 - 1088x^2 + 269x + 494 = 0$

D.415 . . . . $7885x^3 - 18675x^2 - 7470x + 26560 = 0$

E. 19 . . . . $7885x^3 - 20672x^2 + 5111x + 9386 = 0$

F . . . . $1997x^2 - 12581x + 17174 = 0$

$$\begin{array}{ll}
D.247 \ldots & 4693x^3 - 11115x^2 - 4446x + 15808 = 0 \\
E.\ 32 \ldots & 13280x^3 - 34816x^2 + 8608x + 15808 = 0 \\
\hline
& 8587x^3 - 23701x^2 + 13054x = 0
\end{array}$$

$$\begin{array}{ll}
G \ldots\ldots\ldots & 8587x^3 - 23701x + 13054 = 0 \\
F.8587 \ldots & 17148239x^2 - 108033047x + 147473138 = 0 \\
G.1997 \ldots & 17148239x^2 - 47330897x + 26068838 = 0 \\
\hline
& 60702150x - 121404300 = 0
\end{array}$$

Ex qua aequatione ſequitur $x = 2$, ac propterea quoque radix aequationis erit $x = 3$, quorum vterque valor aequationis ſatisfacit.

245. Poteſt autem haec operatio abſolui ſine ſubſidio calculi differentialis, propterea quod eadem aequatio, quam calculus differentialis ſuppeditauit, prodit ſi in ipſa aequatione propoſita ponatur $x + a$ loco x. Ceterum vero haec methodus eliminandi nimium eſt operoſa, &, ſi aequationes eſſent altioris gradus, labor penitus foret inſuperabilis; ex quo multo minus in aequationibus transcendentibus locum habere poteſt. Quod ſi autem ponamus duas aequationis propoſitae $y = 0$ radices inter ſe eſſe aequales, tum ob $a = 0$, aequatio differentialis abit in hanc $\frac{dy}{dx} = 0$. Quoties ergo quaepiam aequatio $y = 0$ habuerit duas radices aequales, toties erit $\frac{dy}{dx} = 0$; atque hae duae aequationes coniunctae praebebunt eum ipſius x valorem, cui binae radices ſunt aequa-

aequales. Vnde viciſſim ſi ambae aequationes $y = 0$ & $\frac{dy}{dx} = 0$ communem habeant radicem, ea erit radix duplex aequationis $y = 0$. Euenit autem hoc, ſi poſtquam quantitas x ope duarum iſtarum aequationum $y = 0$ & $\frac{dy}{dx} = 0$ penitus fuerit eliminata, perueniatur ad aequationem identicam. Sic ſi proponatur aequatio:

$$x^3 - 2xx - 4x + 8 = 0$$

erit quoque $3xx - 4x - 4 = 0$, cuius duplum ad eam additum dat $x^3 + 4xx - 12x = 0$ ſeu $xx + 4x - 12 = 0$ cuius triplum eſt $3xx + 12x - 36 = 0$

ſubtrahatur $3xx - 4x - 4 = 0$

$$16x - 32 = 0$$

$$x - 2 = 0$$

Cum ergo prodierit $x = 2$, ſubſtituatur hic valor in vna praecedentium $3xx - 4x - 4 = 0$, & prodibit aequatio identica $12 - 8 - 4 = 0$, vnde colligitur aequationem propoſitam $x^3 - 2xx - 4x + 8 = 0$ duas habere radices aequales, nempe 2.

246. Si igitur habeatur aequatio algebraica quotcunque dimenſionum:

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + Dx^{n-4} - \&c. = 0$$

quae duas habeat radices inter ſe aequales, erit quoque

$$nx^{n-1} + (n-1)Ax^{n-2} + (n-2)Bx^{n-3} + (n-3)Cx^{n-4} + (n-4)Dx^{n-5} + \&c. = 0.$$

 Scili-

Scilicet illius aequationis radix duplex ſimul erit radix iſtius aequationis. Multiplicetur illa per n, ab eaque haec per x multiplicata ſubtrahatur, prodibitque haec noua aequatio:

$$Ax^{n-1} + 2Bx^{n-2} + 3Cx^{n-3} + 4Dx^{n-4} + \&c. = 0.$$

Nunc addantur prima per a & haec per b multiplicata erit:

$$ax^n + (a+b)Ax^{n-1} + (a+2b)Bx^{n-2} + (a+3b)Cx^{n-3} + \&c. = 0$$

quae aequatio cum ipſa propoſita coniuncta monſtrabit radices aequales, ſi quas habet propoſita. Cum igitur quantitates a & b pro lubitu aſſumi queant, coefficientes a, $a+b$, $a+2b$, &c. progreſſionem quamcunque arithmeticam repraeſentant. Quamobrem ſi aequatio quaecunque habeat duas radices aequales, eae inuenientur, ſi ſinguli aequationis propoſitae termini multiplicentur per terminos cuiusuis progreſſionis arithmeticae reſpectiue; noua enim aequatio hoc modo reſultans eam radicem, quae in propoſita bis ineſt, quoque continebit. Sic aequatio:

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + \&c. = 0$$

ſi eius termini multiplicentur per progreſſionem arithmeticam hanc:

$$a;\ a+b;\ a+2b;\ a+3b;\ a+4b;\ \&c.$$

prodibit noua aequatio haec:

$$ax^n + (a+b)Ax^{n-1} + (a+2b)Bx^{n-2} + (a+3b)Cx^{n-3} + \&c. = 0$$

quae cum illa coniuncta radices aequales oſtendet. Haecque eſt regula ſatis cognita inueniendi radices aequales cuiuscunque aequationis.

247.

247. Si aequatio $y = 0$ tres habeat radices aequales non ſolum erit $\frac{dy}{dx} = 0$, ſed etiam erit $\frac{ddy}{dx^2} = 0$; ſi quidem pro x ſtatuatur eius radicis valor, quae in aequatione $y = 0$ ter ineſt. Ad hoc oſtendendum ponamus aequationem $y = 0$ tres habere radices huiusmodi x, $x + a$, & $x + b$, quae primum interuallis finitis a & b a ſe inuicem diſcrepent; & quia y euaneſcit, ſi loco x tam $x + a$, quam $x + b$ ſcribatur, erit:

$$y = 0$$

$$y + \frac{a\,dy}{dx} + \frac{a^2\,ddy}{2\,dx^2} + \frac{a^3\,d^3y}{6\,dx^3} + \frac{a^4\,d^4y}{24\,dx^4} + \&c. = 0$$

$$y + \frac{b\,dy}{dx} + \frac{b^2\,ddy}{2\,dx^2} + \frac{b^3\,d^3y}{6\,dx^3} + \frac{b^4\,d^4y}{24\,dx^4} + \&c. = 0$$

a quibus binis poſterioribus ſi prima ſubtrahatur erit:

$$\frac{dy}{dx} + \frac{a\,ddy}{2\,dx^2} + \frac{a^2\,d^3y}{6\,dx^3} + \frac{a^3\,d^4y}{24\,dx^4} + \&c. = 0$$

$$\frac{dy}{dx} + \frac{b\,ddy}{2\,dx^2} + \frac{b^2\,d^3y}{6\,dx^3} + \frac{b^3\,d^4y}{24\,dx^4} + \&c. = 0$$

Subtrahantur quoque hae a ſe inuicem, diuiſioneque per $a - b$ facta erit:

$$\frac{ddy}{2\,dx^2} + \frac{(a+b)\,d^3y}{6\,dx^3} + \frac{(aa+ab+bb)\,d^4y}{24\,dx^4} + \&c. = 0.$$

Ponatur iam $a = 0$ & $b = 0$, ita vt tres illae radices inter ſe ſint aequales, eritque ob terminos euaneſcentes:

$$y = 0; \quad \frac{dy}{dx} = 0; \quad \& \quad \frac{ddy}{dx^2} = 0.$$

248. Quoties ergo aequatio $y = 0$, tres habent radices aequales puta f, f, f, tum iſta quantitas f erit quoque radix non ſolum huius aequationis $\frac{dy}{dx} = 0$, ſed etiam huius $\frac{ddy}{dx^2} = 0$. Hinc manifeſtum eſt, cum f ſit radix communis aequationis $\frac{dy}{dx} = 0$, & eius differentialis $\frac{ddy}{dx^2} = 0$, eam in aequatione $\frac{dy}{dx} = 0$, bis ineſſe debere, per ea quae ante de binis radicibus aequalibus oſtendimus. Quare ſi aequatio:

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + \&c. = 0$$

tres contineat radices aequales f, f, f, ſi eius termini per terminos progreſſionis arithmeticae cuiusuis multiplicentur, tum aequatio reſultans binas habebit radices aequales f & f: quamobrem ea denuo per progreſſionem arithmeticam quamcunque multiplicari poterit, vt prodeat aequatio eandem radicem f ſemel complectens. Obtinebuntur ergo tres aequationes communem radicem f habentes, ex quarum combinatione haec ipſa radix facile elicietur. Si enim eiusmodi progreſſiones arithmeticae eligantur, quarum vel primus vel vltimus terminus ſit $= 0$, tum aequatio prodibit vno gradu inferior, ſicque eliminatio eo facilior euadet.

249. Simili modo oſtendetur, ſi aequatio $y = 0$ quatuor habeat radices aequales f, f, f, f, tum poſito $x = f$ non

non ſolum fieri $y = 0$, $\frac{dy}{dx} = 0$, & $\frac{ddy}{dx^2} = 0$, ſed etiam fore $\frac{d^3y}{dx^3} = 0$. Scilicet vti aequatio $y = 0$ quater continet radicem $x = f$; ita aequatio $\frac{dy}{dx}$ eandem radicem ter; aequatio vero $\frac{ddy}{dx^2} = 0$ bis, & aequatio $\frac{d^3y}{dx^3} = 0$ ſemel complectetur. Hoc quoque facilius perſpicietur, ſi perpendamus functionem y hoc caſu huiusmodi formam $(x-f)^4 X$ habere debere, denotante x functionem quamcunque ipſius x. Hac forma aſſumta erit: $\frac{dy}{dx} = (x-f)^3 \left(4X + \frac{(x-f)dX}{dx}\right)$, ideoque per $(x-f)^3$ diuiſibilis. Similiter porro habebit $\frac{ddy}{dx^2}$ factorem $(x-f)^2$, & $\frac{d^3y}{dx^3}$ factorem $x-f$; ex quo perſpicuum eſt, ſi radix $x = f$ in aequatione $y = 0$ quater inſit, eam in aequatione $\frac{dy}{dx} = 0$ ter, in aequatione $\frac{ddy}{dx^2} = 0$ bis, atque in $\frac{d^3y}{dx^3} = 0$ ſemel adhuc ineſſe debere.

CAPUT X.

DE MAXIMIS ET MINIMIS.

250.

Si functio ipſius x ita fuerit comparata, vt creſcentibus valoribus x ipſa continuo creſcat vel decreſcat, tum iſta functio nullum habebit valorem maximum minimumve. Quicunque enim huius functionis valor conſideretur, ſequentes erunt maiores, praecedentes vero minores. Huiusmodi functio eſt $x^3 + x$, cuius valor creſcentibus x continuo creſcit, decreſcentibus vero x, continuo decreſcit; maximum ergo haec functio valorem alium induere nequit, niſi ipſi x valor maximus, hoc eſt infinitus tribuatur; ſimilique modo minimum obtinebit valorem, ſi ponatur $x = -\infty$. Niſi autem functio ita fuerit comparata, vt creſcente x continuo creſcat decreſcatue, maximum vel minimum alicubi habebit valorem; hoc eſt eiusmodi valorem, qui ſit vel maior vel minor, quam antecedentes & ſequentes. Sic iſta functio $xx - 2x + 3$ minimum valorem induit, ſi ponatur $x = 1$, quicunque enim alius valor ipſi x tribuatur, perpetuo functio maiorem adipiſcetur valorem.

251. Quo autem natura maximorum ac minimorum clarius perſpiciatur, ponamus y eiusmodi eſſe functionem ipſius x, quae maximum obtineat valorem, ſi ponatur $x = f$; atque intelligitur, ſi x ponatur ſiue maius

ius ſiue minus quam f, tum valorem ipſius y inde oriundum minorem fore illo, quem induit ſi ponatur $x = f$. Simili modo, ſi poſito $x = f$, functio y minimum obtineat valorem, neceſſe eſt vt, ſiue x ponatur maius quam f ſiue minus, ſemper maior ipſius y valor reſultet: haecque eſt definitio maximorum & minimorum abſolutorum. Praeterea autem quoque functio y maximum valorem recipere dicitur poſito verbi gratia $x = f$, dummodo iſte valor maior fuerit quam proximi ſiue ſequentes ſiue antecedentes, qui oriuntur, ſi x aliquantillum ſiue maius ſiue minus quam f ſtatuatur; etiamſi aliis valoribus loco x ſubſtituendis functio y maiores forte valores recipiat. Similiter functio y minimum valorem recipere dicitur poſito $x = f$, dummodo ille valor minor fuerit iis, quos induit, ſi loco x valores proxime ſiue maiores ſiue minores quam f ſubſtituantur. Atque in hac poſteriori ſignificatione iſtis maximorum & minimorum vocabulis vtemur.

252. Antequam autem modum oſtendamus haec maxima & minima inueniendi, notari conuenit hanc inveſtigationem proprie in iis tantum ipſius x functionibus locum habere, quas ſupra vniformes vocauimus, & quae ſunt ita comparatae, vt pro ſingulis ipſius x valoribus ſingulos pariter valores recipiat. Biformes autem & multiformes functiones vocauimus, quae pro ſingulis valoribus ipſius x binos pluresue valores inducunt, cuiusmodi functiones ſunt radices aequationum quadraticarum & plurium dimenſionum. Si igitur y huiusmodi fuerit functio ipſius x vel biformis vel multiformis, tum proprie

 dici

dici nequit, eam poſito $x = f$ valorem ſiue maximum ſiue minimum induere: quoniam enim poſito $x = f$ vel duos pluresue valores ſimul obtinet, atque praecedentes aeque ac ſequentes ſint numero plures, diiudicatio maximi minimiue non tam facile inſtituitur: niſi forte omnes functionis y valores, qui ſingulis ipſius x valoribus reſpondent, ſint imaginarii praeter vnum; quo caſu huiusmodi functiones ſpeciem functionum vniformium mentiuntur. Primum ergo functiones vniformes harumque ſpeciem mentientes contemplabimur, tum vero quomodo iudicium ad multiformes accommodari debeat, indicabimus.

253. Sit igitur y functio ipſius x vniformis, quae propterea, quicunque valor pro x ſubſtituatur, ſemper vnum recipiat valorem realem, denotetque x eum valorem, qui functioni y maximum minimumue valorem inducat. Priori ergo caſu, ſiue loco x ſubſtituatur $x + \alpha$ ſiue $x - \alpha$, valor ipſius y minor erit, quam ſi $\alpha = 0$, poſteriori vero caſu maior. Cum igitur poſito $x + \alpha$ loco x functio y abeat in

$$y + \frac{\alpha dy}{dx} + \frac{\alpha^2 ddy}{2dx^2} + \frac{\alpha^3 d^3y}{6dx^3} + \&c.$$

at poſito $x - \alpha$ loco x in

$$y - \frac{\alpha dy}{dx} + \frac{\alpha^2 ddy}{2dx^2} - \frac{\alpha^3 d^3y}{6dx^3} + \&c.$$

ne-

neceſſe eſt vt caſu maximi ſit

$$y > y + \frac{\alpha dy}{dx} + \frac{\alpha^2 ddy}{2dx^2} + \frac{\alpha^3 d^3y}{6dx^3} + \&c. \quad \&$$

$$y > y - \frac{\alpha dy}{dx} + \frac{\alpha^2 ddy}{2dx^2} - \frac{\alpha^3 d^3y}{6dx^3} + \&c.$$

In caſu autem, quo valor ipſius y ſit minimus erit:

$$y < y + \frac{\alpha dy}{dx} + \frac{\alpha^2 ddy}{2dx^2} + \frac{\alpha^3 d^3y}{6dx^3} + \&c.$$

$$y < y - \frac{\alpha dy}{dx} + \frac{\alpha^2 ddy}{2dx^2} - \frac{\alpha^3 d^3y}{6dx^3} + \&c.$$

254. Quoniam haec euenire debent, ſi α denotet quantitatem minimam, ſtatuamus α tam paruum, vt eius poteſtates altiores reiici queant; debebitque tam pro caſu maximi quam minimi eſſe $\frac{\alpha dy}{dx} = 0$. Niſi enim $\frac{\alpha dy}{dx}$ eſſet $= 0$, neque valor ipſius y maximus neque minimus eſſe poſſet. Hinc tam pro maximis quam pro minimis inueſtigandis haec habetur regula communis, vt differentiale propoſitae y nihilo aequale ponatur, eritque ille ipſius x valor, qui functionem reddit vel maximam vel minimam, radix iſtius aequationis. Vtrum vero valor hoc modo inuentus ipſius y futurus ſit maximus an minimus, incertum relinquitur; quin etiam fieri poteſt, vt y neque maximum neque minimum ſit futurum: tantum enim inuenimus vtroque caſu fore $\frac{dy}{dx} = 0$, ne-

 que

que viciſſim affirmauimus, quoties ſit $\frac{dy}{dx} = 0$, toties quoque valorem pro y prodire vel maximum vel minimum.

255. Interim tamen ad caſus, quibus valor ipſius y vel maximus vel minimus euadat, inueſtigandos haec prima operatio inſtituenda eſt, vt differentiale functionis propoſitae nihilo aequetur, atque ex aequatione $\frac{dy}{dx} = 0$ omnes ipſius x valores eliciantur. Quibus inuentis deinceps diſpiciendum erit, vtrum iis functio y maximum induat valorem, an minimum, an neutrum? oſtendemus enim fieri poſſe, vt neque maximum neque minimum locum habeat, etiamſi ſit $\frac{dy}{dx} = 0$. Sit f valor ſeu vnus ex valoribus ipſius x, quem obtinet ex aequatione $\frac{dy}{dx} = 0$, hicque valor ſubſtituatur in expreſſionibus $\frac{ddy}{dx^2}$; $\frac{d^3y}{dx^3}$; &c. fiatque hac ſubſtitutione $\frac{ddy}{dx^2} = p$; $\frac{d^3y}{dx^3} = q$; $\frac{d^4y}{dx^4} = r$ &c. Abeat autem functio ipſa y poſito f loco x in F, atque ſi loco x ponatur $f + \alpha$, iſta functio abibit in

$$F + \frac{1}{2}\alpha^2 p + \frac{1}{6}\alpha^3 q + \frac{1}{24}\alpha^4 r + \&c.$$

ſin autem loco x ponatur $f - \alpha$ prodibit

$$F + \frac{1}{2}\alpha^2 p - \frac{1}{6}\alpha^3 q + \frac{1}{24}\alpha^4 r - \&c.$$

vnde

vnde patet, ſi p fuerit quantitas affirmatiua, vtrumque valorem maiorem fore quam F, ſaltem ſi α quantitatem valde paruam denotet, ac propterea valorem F, quem functio y induit poſito $x = f$, fore minimum. Sin autem p ſit quantitas negatiua, tum valor $x = f$ functioni y inducet valorem maximum.

256. Quodſi autem fuerit $p = 0$, tum ſpectari debet valor ipſius q, qui ſi non fuerit $= 0$, valor ipſius y neque maximus erit neque minimus; nam poſito $x = f + \alpha$ erit $F + \frac{1}{6}\alpha^3 q > F$ & poſito $x = f - \alpha$ erit $F - \frac{1}{6}\alpha^3 q < F$. Sin autem quoque fuerit $q = 0$, ad quantitatem r erit reſpiciendum, quae ſi habuerit valorem affirmatiuum valor functionis F, quem recipit poſito $x = f$, erit minimum; ſin autem r habeat valorem negatiuum, erit F maximum. At ſi quoque r euaneſcat iudicium ex ſequentis litterae s valore erit petendum, quod ſimile erit illi, quod ex littera q formauimus. Scilicet ſi s non fuerit $= 0$, functio F neque maximum erit neque minimum; ſin autem ſit quoque $s = 0$, tum ſequens littera t, ſi habeat valorem affirmatiuum indicabit minimum, ſin autem habeat valorem negatiuum, indicabit maximum. Verum ſi & haec littera t euaneſcat, tum in iudicando vlterius eſt procedendum eodem prorſus modo, quo in caſibus praecedentibus ſumus vſi. Sicque de qualibet radice aequationis $\frac{dy}{dx} = 0$ indagabitur, vtrum functioni y inducat valorem maximum an mini-

minimum, an neutrum; atque hoc modo omnia maxima & minima, quae quidem functio y recipere poteſt, inuenientur.

257. Si ergo aequatio $\frac{dy}{dx} = 0$ duas radices habeat aequales, ita vt factorem habeat quadratum $(x - f)^2$, tum poſito $x = f$ ſimul $\frac{ddy}{dx^2}$ euaneſcet, eritque $p = 0$, non autem q. Hoc ergo caſu functio y neque maximum neque minimum valorem induet. Sin autem aequatio $\frac{dy}{dx} = 0$ tres radices habeat aequales, ſeu $\frac{dy}{dx}$ factorem cubicum $(x - f)^3$, tum poſito $x = f$, fiet $\frac{ddy}{dx^2} = 0$ & $\frac{d^3y}{dx^3} = 0$; non autem $\frac{d^4y}{dx^4}$. Huius ergo termini valor ſi fuerit affirmatiuus indicabit minimum, ſin negatiuus, maximum. Judicium ergo ante explicatum huc redit, vt ſi expreſſio $\frac{dy}{dx}$ factorem habuerit $(x-f)^n$, exiſtente n numero impari, functio y, ſi in ea ponatur $x = f$, valorem ſit acceptura vel maximum vel minimum; ſin autem exponens n fuerit numerus par, tum ſubſtitutio $x = f$ neque maximum neque minimum valorem producat.

258. Deinde inuentio maximi ac minimi ſaepenumero non mediocriter adiuuabitur ſequentibus conſiderationibus. Quibus ſcilicet caſibus functio y fit maximum vel minimum, iisdem caſibus fiet quoduis eius multiplum ay, ſi qui-

quidem a fuerit quantitas affirmatiua, itemque y^3, y^5, y^7, &c. atque generaliter ay^n, ſi quidem n fuerit numerus affirmatiuus impar, pariter maximum vel minimum; quoniam huiusmodi formulae ita ſunt comparatae, vt creſcente y creſcant, & decreſcente y decreſcant. Quibus autem caſibus fit y maximum vel minimum, iisdem caſibus $-y$, $-ay$, $b-ay$, & generaliter $b-ay^n$ exiſtente n numero affirmatiuo impari, fiet ordine inverſo vel minimum vel maximum. Similiter quibus caſibus y fit maximum vel minimum, iisdem caſibus formulae hae $\frac{a}{y}$, $\frac{a}{y^3}$, $\frac{a}{y^5}$, & generaliter $\frac{a}{y^n}\pm b$, denotante a quantitatem affirmatiuam & n numerum affirmatiuum imparem, fient inuerſo ordine vel minimum vel maximum; ſin autem a fuerit quantitas negatiua, tum iſtae formulae maximum impetrabunt valorem, ſi y fuerit maximum, & minimum, ſi y ſit minimum.

259. Ad poteſtates autem pares haec non item traduci poſſunt: quoniam enim, ſi y valores recipit negatiuos, eius poteſtates pares y^2, y^4, &c. valores affirmativos inducunt, fieri poteſt, vt dum y minimum valorem negatiuum ſcilicet recipit, eius poteſtates pares fiant maxima. Huius igitur conditionis ratione habita affirmare poterimus, ſi y fuerit maximum vel minimum, exiſtente eius valore affirmatiuo, tum eius poteſtates pares y^2, y^4, &c. quoque fore maxima vel minima. Sin autem valor ipſius y negatiuus fuerit maximum, tum eius quadratum yy accepturum eſſe valorem minimum, & contra ſi valor

 ipſius

ipſius y negatiuus ſit minimum, tum y^2, y^4, &c. fore maximum. Quodſi vero exponentes ipſius y pares fuerint negatiui, tum contrarium eueniet. Ceterum quae hic de exponentibus paribus & imparibus annotauimus, ea non ſolum pro numeris integris valent, ſed etiam pro fractis, quorum denominatores ſunt numeri impares; in hoc enim negotio fractiones $\frac{1}{3}$, $\frac{5}{3}$, $\frac{7}{3}$, $\frac{1}{5}$, $\frac{3}{5}$, &c. numeris imparibus, at $\frac{2}{3}$, $\frac{4}{3}$, $\frac{2}{5}$, $\frac{4}{5}$, $\frac{6}{7}$, &c. numeris paribus aequiualent.

260. Sin autem denominatores fuerint numeri pares, tum, quoniam ſi y negatiuum habet valorem, eius poteſtates $y^{\frac{1}{2}}$; $y^{\frac{3}{4}}$; &c. fiunt imaginariae; hoc tantum de iis affirmari poterit, ſi valor ipſius y affirmatiuus fuerit maximum vel minimum, tum quoque $y^{\frac{1}{2}}$; $y^{\frac{3}{2}}$; $y^{\frac{1}{4}}$; &c. fore pariter vel maxima vel minima; contra autem $y^{-\frac{1}{2}}$, $y^{-\frac{3}{2}}$, $y^{-\frac{1}{4}}$, &c. minima vel maxima. Quia autem haec irrationalia ſimul geminos valores habent, alterum affirmatiuum, alterum negatiuum, de negatiuis contrarium erit tenendum, quod hic de affirmatiuis diximus. Sin autem valor ipſius y negatiuus euadat maximum vel minimum, tum quia huiusmodi poteſtates omnes fiunt imaginariae, neque maximis neque minimis annumerari poterunt. His igitur ſubſidiis inueſtigatio maximi & minimi ſaepe admodum reddetur facilis, quae alias futura eſſet vehementer difficilis.

261. Quoniam haec proprie ad functiones rationales, quippe quae ſunt ſolae uniformes, pertinent, primum functiones integras euoluamus, atque maxima minimaque quae in ipſis occurrunt indagemus. Cum igitur huiusmodi functiones ad hanc formam referantur:

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \&c.$$

primum patet earum valorem maiorem fieri non poſſe, quam ſi ponatur $x = \infty$; tum vero ſi $x = -\infty$ valor huius formulae prodit $= \infty^n$, ſi n ſit numerus par, at $-\infty^n$ ſi n ſit numerus impar, qui propterea valor erit omnium minimus. Dantur autem praeterea ſaepe alia maxima & minima eo ſenſu, quem his vocibus attribuimus, quae ſequentibus exemplis illuſtrabimus.

EXEMPLUM I.

Inuenire valores ipſius x, *quibus haec functio* (x — a)ⁿ *ſit maximum vel minimum.*

Poſito $(x-a)^n = y$, erit $\frac{dy}{dx} = n(x-a)^{n-1}$, quo poſito $= 0$, fiet $x = a$. Cum igitur $\frac{dy}{dx}$ factorem habeat $(x-a)^{n-1}$, ex §. 257. intelligitur y maximum minimumue eſſe non poſſe, niſi ſit $n-1$ numerus impar, ſeu n numerus par. Quia autem dum fit

$$\frac{d^n y}{dx^n} = n(n-1)(n-2) \ldots . 1$$

hoc eſt numerus affirmatiuus, ſequitur, valorem ipſius y poſito $x = a$ proditurum eſſe minimum. Quod quidem

 faci-

facile patet; nam poſito $x = a$ fit $y = 0$; & ſi x ponatur vel maius vel minus quam a, ob n numerum parem, accipiet y valorem poſitiuum hoc eſt nihilo maiorem; ſin autem n fuerit numerus impar, tum functio $y = (x-a)^n$ neque maximum neque minimum admittit. Perſpicuum autem porro eſt hoc idem valere, ſi n fuerit numerus fractus ſiue impar ſiue par. Scilicet $(x-a)^{\frac{\mu}{\nu}}$ fiet poſito $x = a$ minimum, ſi μ fuerit numerus par & ν impar; ſin autem vterque fuerit impar, neque maximum dabitur neque minimum.

EXEMPLUM II.

Inuenire caſus quibus valor huius formulae $xx + 3x + 2$ *ſit maximum vel minimum.*

Ponatur $xx + 3x + 2 = y$, erit $\frac{dy}{dx} = 2x + 3$, $\frac{ddy}{2dx^2} = 1$. Statuatur ergo $2x + 3 = 0$, fiet $x = -\frac{3}{2}$; qui caſus vtrum maximum an minimum producat, cognoſcetur ex valore $\frac{ddy}{2dx^2} = 1$, qui cum ſit affirmatiuus, quicquid ſit x, indicat minimum. Poſito autem $x = -\frac{3}{2}$ fit $y = -\frac{1}{4}$, & ſi alii quicunque valores ipſi x tribuantur, valor ipſius y inde oriundus perpetuo maior erit quam $-\frac{1}{4}$. Ex natura quoque ipſius formulae $xx + 3x + 2$ perſpicitur, eam minimum valorem habere debere; nam cum in infinitum excreſcat ſiue ponatur $x = +\infty$ ſiue $x = -\infty$, neceſſe eſt, vt quispiam valor ipſius x ipſi y omnium minimam quantitatem inducat.

EXEM-

EXEMPLUM III.

Inuenire caſus, quibus expreſſio haec $x^3 - axx + bx - c$ *maximum minimumue valorem accipit.*

Poſito $y = x^3 - axx + bx - c$; erit $\frac{dy}{dx} = 3xx - 2ax + b$ & $\frac{ddy}{2dx^2} = 3x - a$; $\frac{d^3y}{6dx^3} = 1$. Statuatur ergo $\frac{dy}{dx} = 3xx - 2ax + b = 0$; erit $x = \frac{a \pm \sqrt{(aa - 3b)}}{3}$, ex quo intelligitur, niſi ſit $aa > 3b$, formulam propoſitam neque maximum neque minimum eſſe habituram. Sin autem ſit $aa > 3b$, duobus caſibus ſit maximum vel minimum. Hinc vero oritur $\frac{ddy}{2dx^2} = \pm \sqrt{(aa - 3b)}$, vnde intelligitur niſi ſit $aa = 3b$, valorem $x = \frac{a + \sqrt{(aa - 3b)}}{3}$ reddere formulam $y = x^3 - axx + bx - c$ minimam, alterum vero $x = \frac{a - \sqrt{(aa - 3b)}}{3}$ maximam. Quanti autem futuri iſti ſint ipſius y valores, cum ſit $3xx - 2ax + b = 0$ ſeu $x^3 - \frac{2}{3}axx + \frac{1}{3}bx = 0$; erit $y = -\frac{1}{3}axx + \frac{2}{3}bx - c$ & ob $\frac{1}{3}axx - \frac{2aa}{9}x + \frac{ab}{9} = 0$, fit $y = \frac{2}{9}(3b - aa)x + \frac{ab}{9} - c = -\frac{2a(aa-3b)}{27} \mp \frac{2(aa-3b)\sqrt{(aa-3b)}}{27} + \frac{ab}{9} - c$, ſiue $y = -\frac{2a^3}{27} + \frac{ab}{3} - c \mp \frac{2}{27}(aa - 3b)^{\frac{3}{2}}$, vbi ſignum ſuperius valet pro minimo, inferius autem

 pro

pro maximo. Reſtat ergo caſus quo $aa = 3b$, in quo cum fiat $\frac{ddy}{dx^3} = 0$, ſequens vero terminus $\frac{d^3y}{6dx^3} = 1$, non ſit $= 0$, ſequitur hoc caſu formulam propoſitam neque maximum neque minimum eſſe recepturam.

EXEMPLUM. IV.

Inuenire caſus quibus haec functio ipſius X;

$$X^4 - 8X^3 + 22X^2 - 24X + 12$$

ſit maximum vel minimum.

Poſito $y = x^4 - 8x^3 + 22x^2 - 24x + 12$; erit

$$\frac{dy}{dx} = 4x^3 - 24x^2 + 44x - 24$$

$$\frac{ddy}{2dx^2} = 6x^2 - 24x + 22$$

Statuatur nunc $\frac{dy}{dx} = 4x^3 - 24x^2 + 44x - 24 = 0$ ſeu $x^3 - 6x^2 + 11x - 6 = 0$, orientur tres valores reales pro x; I. $x = 1$; II. $x = 2$; III. $x = 3$. Ex primo valore fit $\frac{ddy}{2dx^2} = 4$, ideoque poſito $x = 1$ functio propoſita fit minimum. Ex ſecundo valore $x = 2$ fit $\frac{ddy}{2dx^2} = -2$, ideoque functio propoſita maximum. Ex tertio valore $x = 3$ fit $\frac{ddy}{2dx^2} = +4$, ideoque functio propoſita iterum minimum.

EXEM-

EXEMPLUM V.

Propoſita ſit haec functio $y = x^5 - 5x^4 + 5x^3 + 1$, *quae, quibus caſibus fiat maximum minimumue quaeritur.*

Cum ſit $\frac{dy}{dx} = 5x^4 - 20x^3 + 15xx$, formetur aequatio $x^4 - 4x^3 + 3xx = 0$, cuius radices ſunt I. & II. $x = 0$; III. $x = 1$; IV. $x = 3$. Quoniam prima & ſecunda radices ſunt aequales, ex iis neque maximum neque minimum ſequitur, ſit enim $\frac{ddy}{dx^2} = 0$, at $\frac{d^3y}{2dx^3}$ non euaneſcit. Tertia radix autem $x = 1$, ob $\frac{ddy}{2dx^2} = 10x^3 - 30x^2 + 15x$ praebet $\frac{ddy}{2dx^2} = -5$, hocque ergo caſu functio fit maximum. Ex quarta radice $x = 3$ fit $\frac{ddy}{2dx^3} = 45$, ideoque functio propoſita minimum.

EXEMPLUM VI.

Inuenire caſus quibus haec formula

$$y = 10x^6 - 12x^5 + 15x^4 - 20x^3 + 20$$

ſit maximum vel minimum.

Erit ergo $\frac{dy}{dx} = 60x^5 - 60x^4 + 60x^3 - 60x^2$, & $\frac{ddy}{60dx^2} = 5x^4 - 4x^3 + 3x^2 - 2x$. Formetur aequatio $x^5 - x^4 + x^3 - xx = 0$, quae cum in factores res reſoluta ſit $x^2(x-1)(xx+1) = 0$, duas habet radices

dices aequales $x = 0$, & praeterea radicem $x = 1$, duasque inſuper ex $xx + 1 = 0$ imaginarias. Cum igitur binae radices aequales $x = 0$ neque maximum neque minimum exhibeant, tantum conſideranda ſupereſt radix $x = 1$, ex qua fit $\frac{ddy}{60\,dx^2} = 2$, cuius valor affirmatiuus indicat minimum.

262. Determinatio ergo maximorum & minimorum pendet a radicibus aequationis differentialis $\frac{dy}{dx} = 0$, cuius poteſtas ſumma, cum ſit vno gradu inferior quam in ipſa functione propoſita y, ſi quidem haec fuerit functio rationalis integra: manifeſtum eſt ſi in genere proponatur haec functio:

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + \&c. = y$$

eius maxima & minima determinari per radices huius aequationis:

$$nx^{n-1} + (n-1)Ax^{n-2} + (n-2)Bx^{n-3} + (n-3)Cx^{n-4} + \&c. = 0.$$

Ponamus huius aequationis radices reales ſecundum ordinem quantitatis diſpoſitas eſſe α, β, γ, δ, &c. ita vt α ſit maxima $\beta < \alpha$, $\gamma < \beta$, &c. Ac primo quidem ſi hae radices omnes fuerint inaequales, vnaquaeque formulae propoſitae y inducet valorem maximum vel minimum; totque idcirco functio y habebit maxima vel minima, quo aequatio $\frac{dy}{dx} = 0$ habuerit radices reales inaequales. Sin autem duae pluresue radices inter ſe fuerint aequales, res ita ſe habebit, vt duae radices aequales

les neque maximum neque minimum exhibeant; ternae vero radices aequales vni aequiualeant: atque in genere ſi numerus radicum aequalium fuerit par, nullum inde reſultat maximum minimumue; ſin autem ſit impar, vnum inde oritur ſiue maximum ſiue minimum.

263. Quaenam autem radices maxima, & quae minima producant, ſine ſubſidio regulae ante traditae ita definiri poterit. Cum functio y poſito $x = \infty$ fiat pariter infinita, neque valores ipſius x intra limites ∞ & α, vllum producant ſiue maximum ſiue minimum; perſpicuum eſt valores functionis y, dum loco x ſucceſſiue valores ab ∞ vsque ad α ſubſtituantur, continuo decreſcere oportere; ideoque valor $x = \alpha + \omega$ functioni y maiorem valorem inducet, quam valor $x = \alpha$: vnde cum $x = \alpha$ maximum minimumue producat, neceſſe eſt, vt hoc caſu functio y fiat minimum. Vlterius ergo x diminuendo ſeu ponendo $x = \alpha - \omega$, valor ipſius y iterum creſcet, donec fiat $x = \beta$, quae eſt ſecunda aequationis $\frac{dy}{dx} = 0$ radix maximum minimumue producens: quare haec ſecunda radix $x = \beta$ maximum praebebit, & valor $x = \beta - \omega$ minorem efficiet functionem y, quam $x = \beta$, donec perueniatur ad $x = \gamma$, quae conſequenter iterum minimum generabit. Ex quo ratiocinio perſpicitur, radices aequationis $\frac{dy}{dx} = 0$ primam, tertiam, quintam, &c. minima, ſecundam autem, quartam, ſextam &c. maxima exhibere. Simul autem hinc intelligitur in caſu

 dua-

duarum radicum aequalium maximum & minimum coaleſcere, ſicque neutrum locum habere.

264. Si ergo in functione propoſita

$$y = x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \&c.$$

maximus exponens n fuerit numerus par, aequatio

$$\frac{dy}{dx} = x^{n-1} + (n-1)Ax^{n-2} + \&c. = 0;$$

erit gradus imparis, ideoque vel vnam habebit radicem realem, vel tres, vel quinque, vel numero impares. Si vnica radix fuerit realis, ea dabit minimum; ſin tres fuerint reales, maxima praebebit minimum, media maximum, & minima iterum minimum; & ſi quinque radices fuerint reales, functio y tria habebit minima & duo maxima; ſicque porro. At ſi exponens n fuerit numerus impar, aequatio $\frac{dy}{dx} = 0$ ad gradum parem pertinebit, ideoque vel nullam habebit radicem realem, vel duas, vel quatuor, vel ſex, &c. Primo caſu functio y neque maximum habebit neque minimum: altero caſu, quo duae dantur radices, earum maior minimum, minor autem maximum indicabit: quatuor autem radicum prima (quae eſt maxima) & tertia minimum, ſecunda vero & quarta maximum producunt. Perpetuo autem quotcunque radices fuerint reales, maxima & minima ſe mutuo alternatim inſequuntur.

265. Progrediamur ad functiones rationales fractas, quibus altera ſpecies functionum vniformium conſtituitur.

Sit

Sit igitur $y = \frac{P}{Q}$, exiſtentibus P & Q functionibus quibuscunque ipſius x; ac primo quidem apparet, ſi ipſi x eiusmodi valor tribuatur, vt fiat $Q = 0$, niſi ſimul P euaneſcat, functionem y euadere infinitam, quod vtique maximum videatur. Nihilo vero minus iſte caſus pro maximo haberi nequit; cum enim fractio inuerſa $\frac{Q}{P}$ iisdem caſibus fiat minimum, quibus propoſita $\frac{P}{Q}$ fit maximum, deberet fractio $\frac{Q}{P}$ fieri minimum, ſi Q euaneſcit; hoc autem non ſemper euenit, propterea quod adhuc minores valores, negatiuos ſcilicet, induere poſſet. Hoc igitur dubio exemto, ſimul regula ante data confirmatur, quod maxima & minima ex aequatione $\frac{dy}{dx} = 0$ elici debeant. Fiet ergo caſu propoſito $\frac{dy}{dx} = \frac{QdP - PdQ}{QQdx}$; ideoque $QdP - PdQ = 0$; huiusque aequationis radices efficient functionem y vel maximum vel minimum. Atque ſi dubium ſit, vtrum maximum an minimum locum habeat, confugiendum eſt ad valorem $\frac{ddy}{dx^2}$, qui ſi fuerit affirmatiuus minimum indicabit, ſin autem ſit negatiuus maximum. Quod ſi vero & hic valor $\frac{ddy}{dx^2}$ euaneſcat, quod euenit, ſi aequatio $\frac{dy}{dx} = 0$ habeat duas pluresue ra-

 dices

dices aequales; perpetuo tenendum eſt radices aequales numero pares neque maximum neque minimum producere.

EXEMPLUM I.

Inuenire caſus, quibus functio $\frac{x}{1+xx}$ *fit maximum vel minimum.*

Primum quidem apparet, hanc functionem in nihilum abire caſibus tribus $x=\infty$, $x=0$ & $x=-\infty$, vnde ad minimum duo recipiet ſiue maxima ſiue minima. Ad quae inuenienda ponatur $y=\frac{x}{1+xx}$; eritque $\frac{dy}{dx}=\frac{1-xx}{(1+xx)^2}$ & $\frac{ddy}{dx^2}=-\frac{6x+2x^3}{(1+xx)^3}$. Jam ſtatuatur $\frac{dy}{dx}=0$, erit $1-xx=0$, & vel $x=+1$ vel $x=-1$. Priori caſu $x=+1$ fit $\frac{ddy}{dx^2}=-\frac{4}{2^3}$; ideoque y maxim. $=\frac{1}{2}$, poſteriori $x=-1$ fit $\frac{ddy}{dx^2}=+\frac{4}{2^3}$; ideoque y minim. $=-\frac{1}{2}$. Haec quoque facilius inueniuntur, ſi fractio propoſita $\frac{x}{1+xx}$ inuertatur, ponendo $y=\frac{1+xx}{x}=x+\frac{1}{x}$, dummodo recordemur tum quae maxima inueniuntur. in minima & viciſſim transmutari debere. Erit autem $\frac{dy}{dx}=1-\frac{1}{xx}$, & $\frac{ddy}{dx^2}=\frac{2}{x^3}$. Statuto ergo $\frac{dy}{dx}=0$, fit $xx-1=0$, indeque vel $x=+1$ vel $x=-1$ vt ante.

Atque

Atque caſu $x = +1$ fit $\frac{ddy}{dx^2} = 2$, ideoque y minimum, & formula propoſita $\frac{1}{y}$ maximum. Caſu autem $x = -1$, fit $\frac{ddy}{dx^2} = -2$, vnde y maximum & $\frac{1}{y}$ minimum.

EXEMPLUM II.

Inuenire caſus quibus formula $\frac{2 - 3x + xx}{2 + 3x + xx}$ *ſit maximum vel minimum.*

Poſito $y = \frac{xx - 3x + 2}{xx + 3x + 2}$, erit $\frac{dy}{dx} = \frac{6x^2 - 12}{(xx + 3x + 2)^2}$ & $\frac{ddy}{dx^2} = -\frac{12x^3 + 72x + 72}{(xx + 3x + 2)^3}$. Statuatur $\frac{dy}{dx} = 0$, fiet vel $x = +\sqrt{2}$ vel $x = -\sqrt{2}$. Priori caſu $x = +\sqrt{2}$, erit $\frac{ddy}{dx^2} = \frac{48\sqrt{2} + 72}{(4 + 3\sqrt{2})^3}$, ideoque affirmatiuum, ob denominatorem affirmatiuum: hinc erit y minimum $= \frac{4 - 3\sqrt{2}}{4 + 3\sqrt{2}} = 12\sqrt{2} - 17 = -0,02943725$. Poſteriori caſu $x = -\sqrt{2}$ fit $\frac{ddy}{dx^2} = -\frac{48\sqrt{2} + 72}{(4 - 3\sqrt{2})^3} = \frac{24(3 - 2\sqrt{2})}{(4 - 3\sqrt{2})^3}$, cuius valor ob numeratorem affirmatiuum & denominatorem negatiuum erit negatiuus, ideoque y fiet maximum $= \frac{4 + 3\sqrt{2}}{4 - 3\sqrt{2}} = -12\sqrt{2} - 17 = -33,97056274$. Qui valor etſi minor eſt quam prior minimus, tamen ideo eſt maximus, quod maior ſit contiguis proximis, qui ori-

 un-

untur, ſi loco x vel aliquantillum maiores vel minores valores quam $-\sqrt{2}$ ſubſtituantur. Cum igitur $\sqrt{2}$ inter limites $\frac{4}{3}$ & $\frac{3}{2}$ contineatur, probatio facile inſtituetur hoc modo:

ſi $x = \frac{4}{3}$ fit $y = -\frac{2}{70} = -0,0285$

ſi $x = \sqrt{2}$ fit $y = 12\sqrt{2} - 17 = -0,0294$ minimum

ſi $x = \frac{3}{2}$ fit $y = -\frac{1}{35} = -0,0285$

ſi $x = -\frac{4}{3}$ fit $y = -35$

ſi $x = -\sqrt{2}$ fit $y = -33,970$ maximum

ſi $x = -\frac{3}{2}$ fit $y = -35$.

EXEMPLUM III.

Inuenire caſus, quibus formula $\frac{xx - x + 1}{xx + x - 1}$ *fit maximum vel minimum.*

Ponatur $y = \frac{xx - x + 1}{xx + x - 1}$, eritque $\frac{dy}{dx} = \frac{2xx - 4x}{(xx + x - 1)^2}$ & $\frac{ddy}{dx^2} = -\frac{4x^3 + 12xx + 4}{(xx + x - 1)^3}$. Statuatur $\frac{dy}{dx} = 0$, erit vel $x = 0$ vel $x = 2$: priori caſu fit $\frac{ddy}{dx^2} = \frac{4}{-1}$, ideoque erit y maximum $= -1$. Poſteriori caſu $x = 2$ fit $\frac{ddy}{dx^2} = \frac{20}{5^3}$, ideoque y minimum $= \frac{3}{5}$, etiamſi illud maximum

ximum minus ſit quam hoc minimum. Probatio patebit ex his poſitionibus:

ſi $x = -\frac{1}{3}$; erit $y = -\frac{13}{11}$

ſi $x = 0$; . . $y = -1$ maximum

ſi $x = +\frac{1}{3}$; . . $y = -\frac{7}{5}$

ſi $x = 2-\frac{1}{3}$; erit $y = \frac{19}{31}$

ſi $x = 2$; . . $y = \frac{3}{5}$ minimum.

ſi $x = 2+\frac{1}{3}$; . . $y = \frac{37}{61}$

Quod autem, ſi ponatur $x = 1$, fiat $y = 1$, ideoque > -1, cauſa eſt, quod inter valores ipſius x, 0 & 1 contineatur vnus, quo fit $y = \infty$.

EXEMPLUM VI.

Quaerantur caſus, quibus haec fractio $\frac{x^3+x}{x^4-xx+1}$ *fiat maxima vel minima.*

Poſito $y = \frac{x^3+x}{x^4-xx+1}$, erit $\frac{dy}{dx} = -\frac{x^6-4x^4+4xx+1}{(x^4-xx+1)^2}$ & $\frac{ddy}{dx^2} = \frac{2x^9+18x^7-24x^5-16x^3+12x}{(x^4-xx+1)^3}$. Habebimus ergo hanc aequationem: $x^6+4x^4-4xx-1=0$, quae reſoluitur in has duas: $xx-1=0$ & $x^4+5x^2+1=0$, quarum illius radices ſunt $x=+1$ & $x=-1$; haec vero

vero reſoluta dat $xx = -\frac{5 \pm \sqrt{21}}{2}$, ex qua nulla radix realis emergit. Duarum igitur radicum inuentarum prior $x = +1$ facit $\frac{ddy}{dx^2} = -8$, ac propterea y maximum $= 2$; altera radix $x = -1$ facit $\frac{ddy}{dx^2} = +8$ ac propterea y minimum $= -2$.

EXEMPLUM V.

Inuenire caſus, quibus haec fractio $\frac{x^3 - x}{x^4 - xx + 1}$ *ſit maximum vel minimum.*

Poſito $y = \frac{x^3 - x}{x^4 - xx + 1}$, erit $\frac{dy}{dx} = -\frac{x^6 + 2x^4 + 2x^2 - 1}{(x^4 - x^2 + 1)^2}$ & $\frac{ddy}{dx^2} = \frac{2x^9 - 6x^7 - 18x^5 + 20x^3}{(x^4 - x^2 + 1)^3}$. Facto autem $\frac{dy}{dx} = 0$, erit $x^6 - 2x^4 - 2x^2 + 1 = 0$, quae diuiſa per $xx + 1$ dat $x^4 - 3x^2 + 1 = 0$, haecque vlterius reſoluitur in $xx - x - 1 = 0$ & $xx + x - 1 = 0$, vnde ſequentes quatuor oriuntur radices reales:

$$\text{I. } x = \frac{1 + \sqrt{5}}{2}; \quad \text{II. } x = \frac{1 - \sqrt{5}}{2}$$

$$\text{III. } x = -\frac{1 + \sqrt{5}}{2}; \quad \text{IV. } x = -\frac{1 - \sqrt{5}}{2}.$$

Quae cum omnes in aequatione $x^4 - 3xx + 1 = 0$ contineantur, poſito $x^4 = 3xx - 1$ fiet pro omnibus

ddy

$$\frac{ddy}{dx^2} = \frac{2x(10-20xx)}{8x^6} = \frac{5(1-2xx)}{2x^5} = \frac{5(1-2xx)}{2x(3xx-1)}$$

$$\& \quad y = \frac{x^3 - x}{2xx} = \frac{xx - 1}{2x}.$$

Pro duabus autem prioribus ex aequatione $xx = x + 1$ ortis erit $\frac{ddy}{dx^2} = -\frac{5(2x+1)}{2x(3x+2)} = -\frac{5(2x+1)}{2(5x+3)}$, & $y = \frac{1}{2}$. Prima igitur radix $x = \frac{1+\sqrt{5}}{2}$ dat $\frac{ddy}{dx^2} = -\frac{5(2+\sqrt{5})}{11+5\sqrt{5}}$, ideoque eſt y maximum. Secunda radix $x = \frac{1-\sqrt{5}}{2}$; dat $\frac{ddy}{dx^2} = -\frac{5(2-\sqrt{5})}{11-5\sqrt{5}} = -\frac{5(\sqrt{5}-2)}{5\sqrt{5}-11}$; ideoque $y = \frac{1}{2}$ erit quoque maximum. Duae reliquae radices dant $y = -\frac{1}{2}$ minimum.

266. In his igitur exemplis exploratio, vtrum valor quiſpiam inuentus maximum an minimum producat, facilius inſtitui poterit: cum enim ſit $\frac{dy}{dx} = 0$, valor termini $\frac{ddy}{dx^2}$, eius aequationis ratione habita, ſimplicius exprimi poterit. Sit enim propoſita fractio $y = \frac{P}{Q}$; cum ſit $dy = \frac{QdP - PdQ}{QQ}$, & $QdP - PdQ = 0$; erit $ddy = \frac{d(QdP - PdQ)}{Q^2} - \frac{2dQ(QdP - PdQ)}{Q^3}$. At vero ob $QdP - PdQ = 0$ hic poſterior terminus euanes-

cit, eritque $ddy = \frac{d.(QdP - PdQ)}{QQ} = \frac{QddP - PddQ}{Q^2}$; Quoniam vero iudicium ex huius termini valore ſiue affirmatiuo ſiue negatiuo petitur, denominator autem Q^2 perpetuo ſit affirmatiuus; ex ſolo numeratore negotium ita confici poterit, vt quoties $QddP - PddQ$ ſeu $\frac{d(QdP - PdQ)}{dx^2}$ fuerit affirmatiuum, minimum pronuncietur, ſin ſit negatiuum maximum. Siue poſtquam inuentum fuerit $\frac{dy}{dx}$, cuius forma erit huiusmodi $\frac{R}{QQ}$, tantum quaeratur $\frac{dR}{dx}$; & quae radix huic expreſſioni valorem affirmatiuum inducit, ex ea proueniet minimum, & contra maximum.

267. Si denominator fractionis propoſitae fuerit quadratum ſeu altior poteſtas quaecunque, ita vt ſit $y = \frac{P}{Q^n}$, fiet $dy = \frac{QdP - nPdQ}{Q^{n+1}}$, & poſito $\frac{QdP - nPdQ}{dx} = R$, erit $\frac{dy}{dx} = \frac{R}{Q^{n+1}}$; & maxima minimaque determinabuntur ex radicibus aequationis $R = 0$. Cum deinde ſit $\frac{ddy}{dx} = \frac{QdR - (n+1)RdQ}{Q^{n+2}}$, ob $R = 0$, fiet $\frac{ddy}{dx} = \frac{dR}{Q^{n+1}}$; cuius valor affirmatiuus indicabit minimum, negatiuus autem maximum. Perſpicuum autem eſt, ſi n fuerit numerus impar, ob Q^{n+1} ſemper affirmati-

matiuum, iudicium ex ſolo $\frac{dR}{dx}$ perfici poſſe; ſin autem n ſit numerus par, adhibeatur formula $\frac{Q\,dR}{dx}$. Ponamus autem porro proponi huiusmodi fractionem $\frac{P^m}{Q^n} = y$, erit $dy = \frac{(mQdP - nPdQ)P^{m-1}}{Q^{n+1}}$; ſi itaque ponatur $\frac{mQdP - nPdQ}{dx} = R$, aequationis $R = 0$ radices indicabunt caſus, quibus functio y fit vel maximum vel minimum. Cum igitur ſit $\frac{dy}{dx} = \frac{P^{m-1}R}{Q^{n+1}}$, erit:

$$\frac{ddy}{dx} = \frac{P^{m-2}R[(m-1)QdP - (n+1)PdQ]}{Q^{n+2}} + \frac{P^{m-1}dR}{Q^{n+1}},$$

& ob $R = 0$, fiet $\frac{ddy}{dx^2} = \frac{P^{m-1}dR}{Q^{n+1}dx}$; quae inſuper per quodcunque quadratum $\frac{P^{2\mu}}{Q^{2\nu}}$ diuidi poteſt, ad iudicium abſoluendum. Praeterea vero quoque aequatio $P = 0$ dabit maximum vel minimum, ſi m fuerit numerus par; atque ſimili modo formulam inuerſam $\frac{Q^n}{P^m}$ ſpectando, prodibit maximum vel minimum ponendo $Q = 0$, ſi n fuerit numerus par, vti ſupra §. 257. oſtendimus: hic autem ad maxima vel minima hinc oriunda non reſpicimus, ſed tantum ad vſum methodi explicandum ea indagamus, quae oriuntur ex aequatione $R = 0$.

EXEMPLUM I.

Proponatur fractio $\frac{(\alpha+\beta x)^m}{(\gamma+\delta x)^n}$, *quae, quo caſu fiat maximum vel minimum, quaeritur.*

Poſito $y=\frac{(\alpha+\beta x)^m}{(\gamma+\delta x)^n}$, primo quidem patet fore $y=0$ ſi $x=-\frac{\alpha}{\beta}$, & $y=\infty$ ſi $x=-\frac{\gamma}{\delta}$: quorum caſuum ille dabit minimum, hic vero maximum, ſi m & n fuerint numeri pares. Praeterea vero erit: $\frac{dy}{dx}=\frac{(\alpha+\beta x)^{m-1}}{(\gamma+\delta x)^{n+1}}[(m-n)\beta\delta x+m\beta\gamma-n\alpha\delta]$, ideoque $R=(m-n)\beta\delta x+m\beta\gamma-n\alpha\delta$. Quare poſito $R=0$, erit $x=\frac{n\alpha\delta-m\beta\gamma}{(m-n)\beta\delta}$. Deinde ob $\frac{dR}{dx}=(m-n)\beta\delta$, diſpiciendum eſt, vtrum

$$\frac{P^{m-1}dR}{Q^{n+1}dx}=\frac{m^{m-1}\beta^{n+1}}{n^{n+1}\delta^{m-1}}\left(\frac{\alpha\delta-\beta\gamma}{m-n}\right)^{m-n-2}\frac{dR}{dx}$$

ſit quantitas affirmatiua an negatiua? priori caſu formula propoſita erit minimum, poſteriori maximum. Sic ſi fuerit $y=\frac{(x+3)^3}{(x+2)^2}$, fiet $\frac{P^{m-1}dR}{Q^{n+1}dx}=\frac{9}{8}$, ideoque formula $\frac{(x+3)^3}{(x+2)^2}$ fiet minimum, ſi ponatur $x=0$. Sin autem ſit $y=\frac{(x-1)^m}{(x+1)^n}$, erit $\frac{P^{m-1}dR}{Q^{n+1}dx}=\frac{m^{m-1}}{n^{n+1}}\left(\frac{n-m}{2}\right)^{n-m+2}(m-n)$,

&

& $x = \frac{n+m}{n-m}$. Cum autem m & n ponantur numeri affirmatiui, iudicium petendum erit ex formula $(n-m)^{n-m+2}(m-n)$ ſeu $(n-m)^{n-m}(m-n)$. Si igitur fuerit $n > m$, valor erutus $x = \frac{n+m}{n-m}$ ſemper dabit maximum; ſin autem ſit $n < m$, numerus $m-n$ par dabit minimum, at impar, maximum: ſic $\frac{(x-1)^3}{(x+1)^2}$ fiet maximum poſito $x = -5$; fit enim $y = -\frac{6^3}{4^2} = -\frac{27}{2}$.

EXEMPLUM II.

Proponatur formula $y = \frac{(1+x)^3}{(1+xx)^2}$.

Erit $$\frac{dy}{dx} = \frac{(1+x)^2}{(1+xx)^3}(3-4x-xx),$$

& $$\frac{P^{m-1}}{Q^{n+1}} \cdot \frac{dR}{dx} = -\frac{(1+x)^2}{(1+xx)^3}(2x+4),$$

vbi cum $(1+x)^2$ & $(1+xx)^3$ ſemper habeant valorem affirmatiuum, iudicium relinquetur formulae $-x-2$, quae ſi fuerit affirmatiua minimum, ſin negatiua maximum indicat. At vero ex aequatione $3-4x-xx=0$ ſequitur vel $x=-2+\sqrt{7}$ vel $x=-2-\sqrt{7}$. Priori caſu fit $-x-2=-\sqrt{7}$, ideoque fractio propoſita erit maximum, poſteriori vero caſu minimum ob $-x-2=+\sqrt{7}$. Poſito autem $x=-2+\sqrt{7}$, erit $1+x=-1+\sqrt{7}$, & $1+xx=12-4\sqrt{7}$, vnde

 $y=$

$$y = \left(\frac{-1+\sqrt{7}}{12-4\sqrt{7}}\right)^2 (\sqrt{7}-1) = \frac{(2+\sqrt{7})^2(\sqrt{7}-1)}{16} = \frac{17+7\sqrt{7}}{16}$$
$$= 2,220.$$

Poſito autem $x = -2-\sqrt{7}$ fiet $y = \frac{17-7\sqrt{7}}{16} = -0,0950.$

268. Dantur etiam functiones irrationales & transcendentes, quae proprietatem functionum vniformium habent, & hancobrem maxima & minima eodem modo inueniri poſſunt. Radices enim cubicae & omnium imparium poteſtatum reuera ſunt vniformes, cum nonniſi vnicum valorem realem exhibeant: radices autem quadratae atque omnium poteſtatum parium, etſi reuera, quoties ſunt reales, geminum valorem indicant, alterum affirmatiuum alterum negatiuum, tamen vnusquisque ſeorſim ſpectari poteſt, hocque ſenſu etiam maxima & minima inueſtigari poſſunt. Sic ſi y fuerit functio quaecunque ipſius x, etſi $\sqrt{y}$ geminum habet valorem tamen vterque ſeorſim tractari poterit. Scilicet $+\sqrt{y}$ maximum vel minimum habebit valorem, ſi y talem habuerit, dummodo fuerit affirmatiuus; quia alioquin $\sqrt{y}$ euaderet imaginarium. Vice verſa autem $-\sqrt{y}$ fiet minimum vel maximum, iisdem caſibus, quibus $+\sqrt{y}$ fit maximum vel minimum. Poteſtas autem quaecunque $y^{\frac{m}{n}}$, iisdem caſibus fiet maximum vel minimum, ſiquidem n fuerit numerus impar; at ſi n fuerit numerus par, ii tantum caſus valent, quibus y induit valorem affirmatiuum: hisque caſibus ob ancipitem valorem gemina prodibunt maxima vel minima.

269.

269. Quoniam aequatio differentialis, quae ex poteſtate functionis y^m naſcitur, eſt $\frac{y^{m-1}dy}{dx} = 0$, cuius radices ſimul caſus, quibus poteſtas ſurda $y^{\frac{m}{n}}$ fit maximum vel minimum, indicant, ad hoc indagandum duplex habetur aequatio, altera $y^{m-1} = 0$, altera $\frac{dy}{dx} = 0$, quarum illa abit in $y = 0$, atque tum ſolum maxima & minima exhibet, ſi $m - 1$ fuerit numerus impar, ſeu ſi m fuerit numerus par, ob rationes §. 257. allegatas. Quare cum n ſit numerus impar, ſi m fuerit numerus par, ſi numeros pares per 2μ & impares per $2\nu - 1$ indicemus, functio $y^{2\mu:(2\nu-1)}$ euadet maxima vel minima tribuendis ipſi x valoribus, quos tam ex hac aequatione $y = 0$ quam ex hac $\frac{dy}{dx} = 0$ adipiſcitur. Sin autem m ſit numerus impar, functio $y^{(2\mu-1):(2\nu-1)}$ vel $y^{(2\mu-1):2\nu}$ tum ſolum fit maxima vel minima, cum loco x ſubſtituitur valor ex hac aequatione $\frac{dy}{dx} = 0$. Ac poſteriori quidem caſu $y^{(2\mu-1):2\nu}$ maxima & minima tantum proueniunt, ſi y ab inuentis ex aequatione $\frac{dy}{dx} = 0$ valoribus, affirmatiuos recipiat valores.

270. Sic iſta formula $x^{\frac{2}{3}}$ fit minimum, ponendo $x = 0$, propterea quod hoc caſu x^2 fit minimum. Niſi au-

autem formulam $x^{\frac{2}{3}}$ ad formam x^2 reducamus, methodus ante tradita hoc minimum indicaret; propterea quod caſu $x = 0$, termini ſeriei $y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \&c.$ vnde iudicium peti debet, praeter primum omnes fiunt infiniti. Facto enim $y = x^{\frac{2}{3}}$ erit:

$$\frac{dy}{dx} = \frac{2}{3x^{\frac{1}{3}}}; \quad \frac{ddy}{dx^2} = \frac{-2}{9x^{\frac{4}{3}}}; \quad \frac{d^3y}{dx^3} = \frac{2.4}{27x^{\frac{7}{3}}}; \quad \&c.$$

Hinc neque aequatio $\frac{dy}{dx} = \frac{2}{3x^{\frac{1}{3}}} = 0$ oſtendit valorem $x = 0$, neque termini ſequentes rationem maximi minimiue indicant. Cum igitur aſſumſimus ſeriem $y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \&c.$ fieri conuergentem, ſi ω ſtatuatur quantitas valde parua; ii caſus vtique methodum generalem effugiunt, quibus haec ſeries fit divergens, quod euenit exemplo hic allato $y = x^{\frac{2}{3}}$, ſi ponatur $x = 0$. Quamobrem his caſibus ea reductione, qua ante vſi ſumus, erit opus, quo expreſſio propoſita ad aliam formam reuocetur, quae huic incommodo non ſit ſubiecta. Hoc autem tantum pauciſſimis caſibus vſu venit, qui in formula $y^{\frac{2\mu}{2\nu - 1}}$ continentur, vel ad eam facile reducuntur. Sic ſi requirantur maxima minimaue formulae $y^{\frac{2\mu}{2\nu - 1}} z$, exiſtente z functione quacun-

cunque ipſius x, inueſtigetur forma haec $y^{2\mu} z^{2\nu-1}$, quippe quae iisdem caſibus fit maxima vel minima, quibus ipſa propoſita.

271. Hoc caſu excepto, qui iam facile expeditur, functiones, quae continent quantitates irrationales, eodem modo quo rationales tractari, earumque maxima & minima determinari poſſunt, id quod ſequentibus exemplis illuſtrabimus.

EXEMPLUM I.

Propoſita ſit formula $\sqrt{(aa+xx)}-x$, *quae quibus caſibus fiat maxima vel minima, quaeritur.*

Poſito $y=\sqrt{(aa+xx)}-x$, erit $\frac{dy}{dx}=\frac{x}{\sqrt{(aa+xx)}}-1$, & $\frac{ddy}{dx^2}=\frac{aa}{(aa+xx)^{3:2}}$. Facto ergo $\frac{dy}{dx}=0$, erit: $x=\sqrt{(aa+xx)}$, ideoque $x=\infty$, ac fit $\frac{ddy}{dx^2}=0$. Simili vero modo fiunt ſequentes termini $\frac{d^3y}{dx^3}$; $\frac{d^4y}{dx^4}$; &c. omnes $=0$; ex quo iudicium incertum relinquitur, vtrum ſit maximum an minimum? ratio eſt quod reuera tam fiat $x=-\infty$, quam $x=+\infty$. Interim ponendo $x=+\infty$, ob $\sqrt{(aa+xx)}=x+\frac{aa}{2x}$, fit $y=0$, qui valor omnium eſt minimus.

 EXEM-

EXEMPLUM II.

Quaerantur caſus, quibus haec forma $V(aa+2bx+mxx)-nx$ *ſit maximum vel minimum.*

Poſito $y=V(aa+2bx+mxx)-nx$, erit: $\frac{dy}{dx}=\frac{b+mx}{V(aa+2bx+mxx)}-n$, quo facto $=0$, erit: $bb+2mbx+mmxx=nnaa+2nnbx+mnnxx$, ſeu $xx=\frac{2bx(nn-m)+nnaa-bb}{mm-mnn}$, ideoque $x=\frac{(nn-m)b\pm V[mnn(m-nn)aa-nn(m-nn)bb]}{m(m-nn)}$, ſiue $x=-\frac{b}{m}\pm\frac{n}{m}V\frac{maa-bb}{m-nn}$: vnde fit $V(aa+2bx+mxx)=\frac{b+mx}{n}=\pm V\frac{maa-bb}{m-nn}$. Cum igitur ſit $\frac{ddy}{dx^2}=\frac{maa-bb}{(aa+2bx+mxx)^{\frac{3}{2}}}$, erit: $\frac{ddy}{dx^2}=\frac{maa-bb}{\pm\left(\frac{maa-bb}{m-nn}\right)^{\frac{3}{2}}}=\frac{\pm(m-nn)V(m-nn)}{V(maa-bb)}$. Niſi ergo fuerit $\frac{m-nn}{maa-bb}$ quantitas affirmatiua, maximum minimumue plane non datur. Sin autem ſit quantitas affirmatiua, ſignum ſuperius dabit minimum ſi $m>nn$, maximum vero ſi $m<nn$: contrarium euenit, ſi ſignum inferius valeat. Si ergo ſit $m=2$, $n=1$, & $b=0$, formula

$V(aa+2xx)-x$ fit minimum ponendo $x=+\frac{1}{2}V2aa=\frac{a}{Va}$, at maximum ponendo $x=-\frac{a}{V2}$. Erit ergo minimum $=aV2-\frac{a}{V2}=\frac{a}{V2}$, & maximum $=aV2+\frac{a}{V2}=\frac{3a}{V2}$.

EXEMPLUM III.

Quaerantur caſus, quibus haec expreſſio

$\sqrt[4]{(1+mx^4)}+\sqrt[4]{(1-nx^4)}$

fiat maximum vel minimum.

Cum ſit $\frac{dy}{dx}=\frac{mx^3}{(1+mx^4)^{\frac{3}{4}}}-\frac{nx^3}{(1-nx^4)^{\frac{3}{4}}}$, fiet $mx^3(1-nx^4)^{\frac{3}{4}}=nx^3(1+mx^4)^{\frac{3}{4}}$, ideoque $m^4(1-nx^4)^3=n^4(1+mx^4)^3$, ſeu $n^4-m^4+3mn(n^3+m^3)x^4+3m^2n^2(n^2-m^2)x^8+m^3n^3(n+m)x^{12}=0$. Niſi ergo haec aequatio radicem poſitiuam habeat pro x^4, maximum minimumue prorſus non datur. Quia haec aequatio generaliter commode reſolui nequit, fiet enim $x^4=\frac{m^{\frac{4}{3}}-n^{\frac{4}{3}}}{mn(\sqrt[3]{m}+\sqrt[3]{n})}$ ſeu $x^4=\frac{m-\sqrt[3]{m^2n}+\sqrt[3]{mn^2}-n}{mn}$

ponamus pro caſu ſpeciali $m=8n$, eritque

$-4095+24.513nx^4-3.63.64n^2x^8+9.512n^3x^{12}=0$

ſeu $512n^3x^{12}-1344n^2x^8+1368nx^4-455=0$

ponatur $8nx^4=z$, erit $z^3-21z^2+171z-455=0$

quae diuiſorem habet $z-5$, alterque factor erit:

 zz—

$zz - 16z + 91 = 0$ radices continens imaginarias. Erit ergo tantum $z = 8nx^4 = 5$, ideoque $x = \sqrt[4]{\frac{5}{8n}}$ qui valor reddet expreſſionem $\sqrt[4]{(1+8nx^4)} + \sqrt[4]{(1-nx^4)}$ maximum vel minimum. Quorum vtrum eueniat? quaeratur $\frac{ddy}{dx^2} = \frac{3mxx}{(1+mx^4)^{\frac{7}{4}}} - \frac{3nxx}{(1-nx^4)^{\frac{7}{4}}}$. At ob $m = 8n$, poſito $x^4 = \frac{5}{8n}$, erit $\frac{ddy}{dx^2} = \left(\frac{24n}{6^{\frac{7}{4}}} - \frac{3n}{(3:8)^{\frac{7}{4}}}\right) xx = - \frac{360nxx}{6^{\frac{7}{4}}}$ ideoque negatiuum, ergo fiet $\sqrt[4]{(1+8nx^4)} + \sqrt[4]{(1-nx^4)}$ maximum poſito $x = \sqrt[4]{\frac{5}{8n}}$. Erit vero hoc maximum $= \sqrt[4]{6} + \sqrt[4]{\frac{3}{8}} = \frac{3\sqrt[4]{6}}{2}$. Si loco nx^4 ponamus u, patet hanc expreſſionem $\sqrt[4]{(1+8u)} + \sqrt[4]{(1-u)}$ fieri maximam, poſito $u = \frac{5}{8}$, huncque valorem maximum fore $= \frac{3\sqrt[4]{6}}{2} = 2,347627$. Quicunque ergo valor praeter $\frac{5}{8}$ pro u ſcribatur, expreſſio minorem accipiet valorem.

272. Simili modo maxima ac minima determinabuntur, ſi quantitates quoque tranſcendentes in expreſſione propoſita inſint. Niſi enim functio propoſita fuerit

rit multiformis, atque aliquot eius ſignificatus ſimul conſiderari debeant, radices aequationis differentialis oſtendent maxima vel minima, niſi affuerint radices aequales, quarum numerus ſit par. Hanc ergo inueſtigationem in aliquot exemplis declarabimus.

EXEMPLUM I.

Inuenire numerum, qui ad ſuum logarithmum minimam teneat rationem.

Dari huiusmodi rationem minimam $\frac{x}{lx}$ inde patet, quod haec ratio tam poſito $x = 1$, quam $x = \infty$ fiat infinita. Viciſſim ergo habebit fractio $\frac{lx}{x}$ alicubi maximum valorem, eodem ſcilicet caſu, quo $\frac{x}{lx}$ fit minimum. Ad hunc caſum indagandum ponatur $y = \frac{lx}{x}$, fietque $\frac{dy}{dx} = \frac{1}{xx} - \frac{lx}{xx}$. Quo nihilo aequali poſito erit $lx = 1$, & quia hic logarithmum hyperbolicum aſſumſimus, ſi e ponatur numerus cuius logarithmus hyperbolicus ſit $= 1$, erit $x = e$. Cum igitur omnes logarithmi ad hyperbolicos in data ſint ratione, erit in quocunque logarithmorum canone $\frac{e}{le}$ minimum, ſeu $\frac{le}{e}$ maximum. Quoniam in logarithmis tabularibus eſt $le = 0,4342944819$, fractio $\frac{lx}{x}$ perpetuo erit minor

 quam

quam $\frac{4342944819}{27182818284}$, ſeu proxime quam $\frac{47}{305}$: neque vllus datur numerus, qui ad ſuum logarithmum minorem teneat rationem quam 305 ad 47. Eſſe autem hoc caſu $\frac{lx}{x}$ maximum inde patet, quod ob $\frac{dy}{dx} = \frac{1 - lx}{xx}$, fiat $\frac{ddy}{dx^2} = -\frac{1}{x^3} - \frac{2(1-lx)}{x^3} = -\frac{1}{x^3}$, propter $1-lx = 0$, ideoque negatiuum.

EXEMPLUM II.

Inuenire numerum x, *vt haec poteſtas* $x^{1:x}$ *fiat maximum.*

Dari huius formulae valorem maximum inde patet, quod numeris loco x ſubſtituendis ſit

$$1^{1:1} = 1,000000$$
$$2^{1:2} = 1,414213$$
$$3^{1:3} = 1,442250$$
$$4^{1:4} = 1,414213$$

Ponatur ergo $x^{1:x} = y$, eritque $\frac{dy}{dx} = x^{1:x}\left(\frac{1}{xx} - \frac{lx}{xx}\right)$. Quo valore nihilo aequali poſito, erit $lx = 1$ & $x = e$, exiſtente $e = 2,718281828$. Et cum ſit $\frac{dy}{dx} = (1-lx)\frac{x^{1:x}}{xx}$, erit $\frac{ddy}{dx^2} = -\frac{x^{1:x}}{x^3} + (1-lx)\,d.\frac{x^{1:x}}{xx} = -\frac{x^{1:x}}{x^3}$ ob $1-lx = 0$. Quare cum ſit $\frac{ddy}{dx^2}$ quantitas negatiua, fiet $x^{1:x}$ maximum

caſu

caſu $x = e$. Cum autem ſit $e = 2{,}718281828$, reperitur fore $e^{\frac{1}{e}} = 1{,}444667861009764$, qui valor obtinetur facile ex ſerie $e^{\frac{1}{e}} = 1 + \frac{1}{e} + \frac{1}{2e^2} + \frac{1}{6e^3} + \frac{1}{24e^4} + \&c.$

Hoc exemplum quoque ex praecedenti reſoluitur: ſi enim ſit $x^{1:x}$ maximum, quoque eius logarithmus, qui eſt $\frac{lx}{x}$, debebit eſſe maximum; quod quo fiat, debet eſſe $x = e$, vti inuenimus.

EXEMPLUM III.

Inuenire arcum x, *vt ſit eius ſinus maximus vel minimus.*

Poſito ſin $x = y$ erit $\frac{dy}{dx} = \cos x$, ideoque $\cos x = 0$, vnde prodeunt ſequentes valores pro x: $\pm\frac{\pi}{2}$; $\pm\frac{3\pi}{2}$; $\pm\frac{5\pi}{2}$ &c. Fit autem $\frac{ddy}{dx^2} = -\sin x$. Cum igitur hi valores pro x ſubſtituti dent pro ſin x vel $+1$ vel -1, illi erunt maximi, hi vero minimi, vti conſtat.

EXEMPLUM. IV.

Inuenire arcum x, *vt rectangulum* x. *ſin* x *fiat maximum.*

Dari maximum inde patet, quod poſito vel $x = 0$ vel $x = 180°$ vtroque caſu rectangulum propoſitum euaneſcat. Sit igitur $y = x \sin x$; erit $\frac{dy}{dx} = \sin x + x\cos x$,

ideo-

ideoque tg $x = -x$. Sit $x = 90° + u$, erit tg $x = -\cot u$, ergo $\cot u = 90 + u$. Ad quam aequationem modo ſupra tradito reſoluendam ponatur $z = 90 + u - \cot u$, ſitque f valor arcus u quaeſitus. Cum ſit $dz = du + \frac{du}{\text{ſin}\, u^2}$, erit $p = \frac{du}{dz} = \frac{\text{ſin}\, u^2}{1 + \text{ſin}\, u^2}$; $dp = \frac{2\, du\, \text{ſin}\, u \cos u}{(1 + \text{ſin}\, u)^2}$; ideoque $\frac{dp}{dz} = q = \frac{2\, \text{ſin}\, u^3 \cos u}{(1 + \text{ſin}\, u^2)^3}$; $dq = \frac{6\, du\, \text{ſin}\, u^2 \cos u^2 - 2\, du\, \text{ſin}\, u^4}{(1 + \text{ſin}\, u^2)^3}$ $- \frac{12\, du\, \text{ſin}\, u^4\, \text{cſ}\, u^2}{(1 + \text{ſin}\, u^2)^4}$. Ergo $\frac{dq}{dz} = r = \frac{6\, \text{ſin}\, u^4\, \text{cſ}\, u^2 - 2\, \text{ſin}\, u^6}{(1 + \text{ſin}\, u^2)^4}$ $- \frac{12\, \text{ſin}\, u^6 \cos u^2}{(1 + \text{ſin}\, u^2)^5} = \frac{6\, \text{ſin}\, u^4 - 14\, \text{ſin}\, u^6 + 4\, \text{ſin}\, u^8}{(1 + \text{ſin}\, u^2)^5}$. Ex quibus erit $f = u - pz + \frac{1}{2} qzz - \frac{1}{6} rz^3 + \&c.$ Ponatur, poſtquam aliquot tentaminibus proximus ipſius f valor eſt detectus, $u = 26°, 15'$, erit $90 + u = 116°, 15'$, & arcus cotangenti u aequalis ita definiatur:

A	$l \cot u =$	10,3070250
ſubtrahatur		4,6855749
		5,6214501
Ergo	$\cot u =$	418263, 7''
ſeu	$\cot u =$	116°, 11', $3\frac{7}{10}$''
vnde	$z =$	3', $56\frac{3}{10}$'' = 236, 3''.

Iam

Iam ad valorem termini pz inueniendum, iſte inſtituatur calculus:

$l\,\text{ſin}\,u =$	9,6457058
$l\,\text{ſin}\,u^2 =$	9,2914116
$1+\text{ſin}\,u^2 =$	1,19561
$l(1+\text{ſin}\,u^2) =$	0,0775895
$lp =$	9,2138221
$lz =$	2,3734637
$lpz =$	1,5872858
Ergo $pz =$	38,6621 ſecundis
ſeu $pz =$	$38''$, $39'''$, $43''''$
ab $u =$	$26°$, $15'$
fiet $f =$	$26°$, $14'$, $21''$, $20'''$, $17''''$

& arcus quaeſitus $x = 116$, 14 , 21 , 20 , 17

Tertius vero terminus $\frac{1}{2}qzz = \frac{\text{ſin}\,u^3}{(1+\text{ſin}\,u^2)^3}\frac{\text{coſ}\,u}{}zz$ inſuper addi debet.

Cuius valor vt inueniatur, vnum z in partibus radii exprimi debet, hoc modo:

$$lz'' = 2,3734637$$

add. $4,6855749$

$$7,0590386$$

add. $l\frac{\text{ſin}\, u^2}{1+\text{ſin}\, u^2}z = 1,5872858$

$$8,6463244$$

add. $l\,\text{ſin}\, u = 9,6457058$

$$l\,\text{coſ}\, u = 9,9527308$$

$$8,2447600$$

ſubtr. $l(1+\text{ſin}\, u^2)^2 = 0,1551790$

$$l\tfrac{1}{2}qzz = 8,0895810$$

Ergo $\tfrac{1}{2}qzz = 0,012291$

ſeu $\tfrac{1}{2}qzz = 44'''', 15'''''$.

Vnde & hoc termino adhibito fiet arcus quaeſitus

$$x = 116°, 14', 21'', 21''', 0''''$$

maioribus autem logarithmis adhibitis reperitur

$$x = 116°, 14', 21'', 20''', 35'''', 47'''''.$$

CAPUT

CAPUT XI.

DE MAXIMIS ET MINIMIS FUNCTIONUM MULTIFORMIUM PLURESQUE VARIABILES COMPLECTENTIUM.

273.

Si y fuerit functio multiformis ipſius x, ita vt pro vnoquoque valore ipſius x ea plures obtineat valores reales; tum variato x plures illi ipſius y valores ita inter ſe connectentur, vt plures ſeries valorum ſucceſſiuorum repraeſentent. Si enim y tanquam applicatam lineae curuae conſideremus, x exiſtente abſciſſa, quot y habuerit valores reales diuerſos, totidem diuerſi eiusdem curuae rami eidem abſciſſae x reſpondebunt: atque hinc illi ipſius y valores ſucceſſiui, qui eundem ramum conſtituunt, cohaerere cenſendi ſunt; valores autem ad diverſos ramos relati erunt inter ſe disiuncti. Tot igitur ſeries valorum cohaerentium ipſius y habebimus, quot diuerſos valores reales pro quouis ipſius x valore receperit; atque in qualibet ſerie valores ipſius y, dum x creſcens aſſumitur, vel creſcent vel decreſcent, vel poſtquam creuerint iterum decreſcent, vel vice verſa. Ex quo perſpicuum eſt, in unaquaque valorum cohaerentium ſerie aeque dari maxima minimaue, atque in functionibus uniformibus.

274. Ad haec maxima minimaue determinanda eadem quoque methodus valebit, quam capite praecedente pro functionibus uniformibus tradidimus. Cum enim, ſi variabilis x incremento ω augeatur, functio y perpetuo recipiat hanc formam $y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2\,dx^2} + \frac{\omega^3 d^3 y}{6\,dx^3} + \&c.$ neceſſe eſt vt caſu maximi minimiue terminus $\frac{\omega\,dy}{dx}$ euaneſcat, fiatque $\frac{dy}{dx} = 0$. Radices ergo huius aequationis $\frac{dy}{dx} = 0$ eos ipſius x valores indicabunt, quibus in ſingulis valorum ipſius y cohaerentium ſeriebus, maxima minimaue reſpondeant. Neque vero ambiguum erit, in quanam valorum cohaerentium ſerie detur maximum minimumue. Cum enim in aequatione $\frac{dy}{dx} = 0$ ambae inſint variabiles x & y, valores ipſius x definiri nequeunt, niſi ope aequationis, qua relatio functionis y ab x continetur, variabilis y eliminetur; antequam autem hoc fit, peruenitur ad aequationem, qua valor ipſius y per functionem rationalem ſeu vniformem ipſius x exprimitur. Hinc inuentis valoribus ipſius x, cuique reſpondens valor ipſius y reperietur, qui erit maximus vel minimus in ſerie valorum ſucceſſiuorum cohaerentium, ad quam pertinet.

275. Iudicium autem, vtrum iſti valores ipſius y ſint maximi an minimi? inſtituetur eodem modo, quem ante

ante indicauimus. Scilicet quaeratur valor ipſius $\frac{ddy}{dx^2}$ finitis terminis expreſſus, in eoque loco x ſubſtituatur vnusquisque ipſius x valor inuentus ſucceſſive, ſimul autem pro y ponatur valor, qui ipſi pro quolibet ipſius x valore conuenit; quo facto diſpiciatur, vtrum expreſſio $\frac{ddy}{dx^2}$ adeptura ſit valorem affirmatiuum an negatiuum? priorique caſu minimum, poſteriori vero maximum indicabitur. Quodſi vero & $\frac{ddy}{dx^2}$ euaneſcat, tum procedendum erit ad formulam $\frac{d^3y}{dx^3}$, quae ſi eodem caſu non euaneſcat, neque maximum habebitur neque minimum: ſin autem quoque $\frac{d^3y}{dx^3}$ euaneſcat, iudicium formari oportebit ex formula $\frac{d^4y}{dx^4}$ eodem modo, quo ratione formulae $\frac{ddy}{dx^2}$ praecipimus. Atque ſi quoque $\frac{d^4y}{dx^4}$ quopiam caſu euaneſcat, ad differentiale quintum ipſius y erit progrediendum: perpetuo autem quousque progredi neceſſe fuerit, iudicia ex differentialibus ordinum imparium ſimilia ſunt illi, quod de formula $\frac{d^3y}{dx^3}$ dedimus. His ſcilicet caſibus in formulis $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$ &c. eousque erit pergendum, quoad

perueniatur ad talem, quae propoſito caſu non euaneſcat; quae ſi fuerit differentialis ordinis imparis, neque maximum neque minimum indicabitur, ſin autem fuerit ordinis paris, eius valor affirmatiuus minimum, negatiuus vero maximum innuet.

276. Ponamus functionem y determinari ex x per aequationem quamcunque: quae aequatio ſi differentietur, induet huiusmodi formam $Pdx + Qdy = 0$. Facto ergo $\frac{dy}{dx} = 0$, erit $\frac{P}{Q} = 0$ ideoque vel $P = 0$ vel $Q = \infty$. Poſterior quidem aequatio, ſi relatio inter x & y exprimatur per aequationem rationalem integram, locum habere nequit; quia vel x vel y vel vtramque fieri oporteret infinitam. Quare iudicium relinquetur aequationi $P = 0$, cuius radices, ſeu valores ipſius x, quos adipiſcitur, poſtquam ope aequationis propoſitae variabilis y penitus fuerit eliminata, indicabunt caſus, quibus valores ipſius y fiunt maximi vel minimi. Ad iudicium vero, vtrum prodeat maximum an minimum? abſoluendum, examinetur formula $\frac{ddy}{dx^2}$. Aequatio vero differentialis $Pdx + Qdy = 0$ denuo differentiata, ſi ponamus $dP = Rdx + Sdy$ & $dQ = Tdx + Vdy$, dabit (poſito dx conſtante):

$$Rdx^2 + Sdxdy + Tdxdy + Vdy^2 + Qddy = 0.$$

Cum autem iam ſit $\frac{dy}{dx} = 0$, aequatione per dx^2 diviſa fiet $R + \frac{Qddy}{dx^2} = 0$, ideoque $\frac{ddy}{dx^2} = -\frac{R}{Q}$. Hinc in

in aequatione differentiali $Pdx + Qdy = 0$, differentietur tantum quantitas P, ponendo y conſtans, prodibitque Rdx, tum indagetur, valor fractionis $\frac{R}{Q}$, qui ſi fuerit affirmatiuus, maximum, ſin negatiuus minimum indicabit.

277. Sit y functio biformis ipſius x, quae determinetur per hanc aequationem $yy + py + q = 0$, denotantibus p & q functiones quascunque ipſius x vniformes. Erit ergo differentiando $2ydy + pdy + ydp + dq = 0$, ideoque $Pdx = ydp + dq$. Poſito igitur $P = 0$ erit $ydp + dq = 0$ prodibitque $y = -\frac{dq}{dp}$, ſicque y per functionem ipſius x vniformem exprimitur, ita vt, quicunque valor pro x fuerit inuentus, ex eo & y valorem determinatum vnicum acquirat. Eliminatio vero nunc ipſius y erit facilis; nam ſi in aequatione propoſita $yy + py + q = 0$ loco y valor $-\frac{dq}{dp}$ ſubſtituatur, habebitur $dq^2 - pdpdq + qdp^2 = 0$, quae aequatio diuiſa per dx^2 & reſoluta praebebit valores ipſius x omnes, quibus maxima vel minima reſpondent: quod clarius fiet ſequentibus exemplis.

EXEMPLUM I.

Propoſita aequatione $yy + mxy + aa + bx + nxx = 0$ *definire maxima vel minima functionis* y.

Differentiata aequatione habebimus:

$$2ydy + mxdy + mydx + bdx + 2nxdx = 0$$

vnde

vnde fit $P = my + b + 2nx$ & $Q = 2y + mx$. Poſito ergo $P = 0$ fiet $y = -\frac{b-2nx}{m}$; qui valor in ipſa aequatione ſubſtitutus dat:

$$\begin{array}{l} \frac{4nn}{mm}xx + \frac{4nb}{mm}x + \frac{bb}{mm} \\ -2nxx - bx + aa \\ +nxx + bx \end{array} = 0$$

ſeu $$xx = \frac{4nbx + bb + mmaa}{mmn - 4nn};$$ vnde fit

$$x = \frac{2nb \pm \sqrt{[mmnbb + mmn(mm-4n)aa]}}{mmn - 4nn}$$

ſeu $$x = \frac{2nb \pm m\sqrt{[nbb + n(mm-4n)aa]}}{n(mm - 4n)}$$

& $$y = \frac{-mb \mp 2\sqrt{[nbb + n(mm-4n)aa]}}{mm - 4n}.$$

Tum poſito ſolo x variabili fit $dP = 2ndx$, ideoque $R = 2n$. At eſt $Q = 2y + mx = \pm\frac{\sqrt{[nbb+n(mm-4n)aa]}}{n}$, vnde $\frac{R}{Q} = \frac{\pm 2nn}{\sqrt{[nbb+n(mm-4n)aa]}}$, cuius numerator $2nn$ cum ſit perpetuo affirmatiuus, ſi ſignum ſuperius valeat, prodibit pro y valor maximus, ſin inferius prodibit minimus. Vbi ſequentia annotari debent.

I. Si fuerit $m = 0$, ex aequatione $P = 0$ ſtatim ſequitur $x = -\frac{b}{2n}$, vt nulla eliminatione opus ſit. Huicque valori

lori geminus ipſius y reſpondet ob $y = \pm \frac{1}{2n} \sqrt{(nbb - 4mmaa)}$ quorum alter affirmatiuus eſt maximus, alter negatiuus minimus.

II. Si ſit $n = 0$, fit $y = -\frac{b}{m}$, & x in infinitum excreſcit, atque y per ſpatium infinitum eundem valorem retinet, ita vt neque maximus ſit neque minimus.

III. Si ſit $mm = 4n$, erit $4nbx + bb + mmaa = 0$ ſeu $x = \frac{bb + mmaa}{-mmb}$; fietque $y = -\frac{b - 2nx}{m} = -\frac{2b - mmx}{m} = -\frac{2b}{m} + \frac{bb + mmaa}{mb} = \frac{mmaa - bb}{mb}$. Huic ergo valori ipſius $x = -\frac{mmaa - bb}{mmb}$ alter ipſius y valor, qui reſpondet $\frac{mmaa - bb}{mb}$, erit maximus vel minimus. Quia autem, vt iſte ipſius y valor prodeat, in expreſſione $y = -\frac{mb \mp 2\sqrt{[nbb + n(mm - 4n)aa]}}{mm - 4n}$ ſignum inferius valere debet, erit valor ipſius y minimus.

EXEMPLUM II.

Propoſita aequatione $yy - xxy + x - x^3 = 0$ *definire valores ipſius* y *maximos vel minimos.*

Differentiata aequatione prodit:

$$2ydy - xxdy - 2xydx + dx - 3xxdx = 0.$$

 Fit-

Fitque $P = 1 - 3xx - 2xy$ & $Q = 2y - xx$. Quare poſito $P = 0$, erit $y = \frac{1-3xx}{2x}$; ideoque hoc valore ſubſtituto:

$$\frac{1}{4xx} - \frac{3}{2} + \frac{9xx}{4} - \frac{x}{2} + \frac{3}{2}x^3 + x - x^3 = 0$$

ſeu $1 - 6xx + 2x^3 + 9x^4 + 2x^5 = 0$. Cuius vna radix eſt $x = -1$, cui reſpondet $y = 1$. At poſito y conſtante fit $R = -6x - 2y$, ergo $\frac{ddy}{dx^2} = \frac{2y+6x}{2y-xx}$; quod caſu $x = -1$ & $y = 1$ abit in -4, ita vt valor ipſius $y = 1$ ſit maximus. Ipſi $x = -1$ autem geminus valor ipſius y reſpondet ex aequatione $yy - y = 0$: alter ergo eſt $y = 0$, qui neque maximus eſt neque minimus. Quodſi aequatio illa quinti gradus per $x+1$ diuidatur, prodit aequatio, cuius radices ſimpliciter exhiberi nequeunt.

EXEMPLUM III.

Sit propoſita haec aequatio: $yy + 2xxy + 4x - 3 = 0$ *ex qua maximi minimiue valores ipſius* y *requiruntur.*

Per differentiationem ergo prodibit haec aequatio:

$$2ydy + 2xxdy + 4xydx + 4dx = 0$$

Factoque $\frac{dy}{dx} = 0$ erit $xy + 1 = 0$, ideoque $y = -\frac{1}{x}$, qui valor ſubſtitutus in ipſa aequatione propoſita oritur,

$$\frac{1}{xx} - 2x + 4x - 3 = 0 = 2x^3 - 3xx + 1$$

cuius

cuius radices ſunt $x = 1$; $x = 1$; & $x = -\frac{1}{2}$. Quia nunc eſt $\frac{dy}{dx} = -\frac{4xy-4}{2y+2xx} = -\frac{2xy-2}{y+xx}$, erit differentiando $\frac{ddy}{dx^2} = -\frac{2y}{y+xx}$, poſito y conſtanti ob $dy = 0$ & facto $xy + 1 = 0$. Quare iſti valores ita ſe habebunt

x	y	$\frac{ddy}{dx^2}$
1	-1	∞
1	-1	∞
$-\frac{1}{2}$	2	$-\frac{16}{9}$ pro maximo.

Quoniam pro radicibus aequalibus fit $\frac{ddy}{dx^2} = \infty$, vtrum hoc caſu maximum an minimum prodeat? non determinatur. Quia autem ſimul fit $y + xx = 0$; nequidem hoc caſu erit $\frac{dy}{dx} = 0$; ob $P = 0$ & $Q = 0$ in fractione $\frac{dy}{dx} = -\frac{P}{Q}$; quare cum primaria proprietas deſit, neque maximum nec minimum habet locum. Indicatur autem hoc caſu $x = 1$, ambos ipſius y valores inter ſe fieri aequales. Quam indolem infra fuſius ſumus expoſituri, cum ad vſum calculi differentialis in doctrina de lineis curuis perueniemus. Etiamſi enim haec materia & huc pertineat; tamen ne eam bis attingere opus ſit, eam totam ſequenti tractationi reſeruamus.

278. Datur vero inſuper in functionibus multiformibus alia ſpecies maximorum ac minimorum, quae methodo hactenus tradita non inuenitur, cuius natura ex functionibus biformibus facillime explicari poteſt. Sit enim y functio quaecunque biformis ipſius x, ita vt, quicunque valor ipſi x tribuatur, pro y oriantur bini valores vel ambo reales vel ambo imaginarii. Ponamus hos ipſius y valores fieri imaginarios, ſi ponatur $x > f$, reales autem eſſe, ſi ſtatuatur $x < f$; atque poſito $x = f$ ambo ipſius y valores in vnum coaleſcent, qui ſit $y = g$. Cum igitur ſi ſumatur $x > f$; functio y nullum habeat valorem realem: ſi eueniat, vt poſito $x < f$ ambo ipſius y valores fiant vel maiores quam g, vel minores quam g: priori caſu valor $y = g$ erit minimus, poſteriori maximus; quoniam illo caſu minor eſt, quam ambo praecedentes, hoc vero maior. Neque hoc maximum minimumue methodo hactenus tradita reperietur, propterea quod hic non ſit $\frac{dy}{dx} = 0$. Sunt autem quoque haec maxima vel minima generis diuerſi, cum talia non ſint ratione valorum antecedentium & conſequentium in ſerie cohaerentium; ſed ratione binorum valorum disiunctorum vel antecedentium vel ſequentium tantum.

279. Euenit hoc ſi aequatio propoſita fuerit huiusmodi $y = p \pm (f - x) V(f - x)q$, exiſtentibus p & q functionibus ipſius x per $f - x$ non diuiſibilibus; obtineatque q valorem affirmatiuum, ſi ponatur vel $x = f$ vel

ali-

aliquanto maius minuſue. Fiat $p = g$ poſito $x = f$: & manifeſtum eſt, caſu $x = f$ ambos ipſius y valores in vnum $y = g$ coaleſcere; poſito autem $x > f$ ambo valores ipſius y fient imaginarii. Si igitur ponamus x aliquanto minus quam f, puta $x = f - \omega$, functio p abibit in $g - \frac{\omega dp}{dx} + \frac{\omega^2 ddp}{2dx^2} - \&c.$ & q in $q - \frac{\omega dq}{dx} + \frac{\omega^2 ddq}{2dx^2} - \&c.$ vnde hoc caſu erit $y = g - \frac{\omega dp}{dx} + \frac{\omega^2 ddp}{2dx^2} + \&c.$ $\pm \omega \surd \omega \left(q - \frac{\omega dq}{dx} + \frac{\omega^2 ddq}{2dx^2} - \&c.\right)$. Ponamus ω minimum, vt prae ω altiores eius poteſtates euaneſcant, eritque $y = g - \frac{\omega dp}{dx} \pm \omega \surd \omega q$; qui valores ambo ipſius y minores erunt quam g, ſi $\frac{dp}{dx}$ fuerit affirmatiuum, maiores autem, ſi negatiuum. Vnde valor duplex ipſius $y = g$ illo caſu erit maximus, hoc vero minimus.

280. Haec igitur maxima atque minima inde ortum ſuum habent, quod primo poſito $x = f$ ambo ipſius y valores fiant aequales: poſito autem $x > f$ imaginarii, at poſito $x < f$ reales. Deinde quod poſito $x = f - \omega$ alterum membrum irrationale praebeat altiores poteſtates ipſius ω, quam membrum rationale. Hoc ergo euenit quoque ſi fuerit $y = p \pm (f-x)^n \surd (f-x) q$, dummodo ſit n numerus integer > 0. Cum autem non ſolum radix quadrata ſed etiam quaecunque alia radix poteſtatis paris eandem ambiguitatem ſignorum introducat; idem eueniet, ſi fuerit

$y = p \pm (f-x)^{\frac{2n+1}{2m}} q$, dummodo ſit $2n+1 > 2m$, erit ergo $(y-p)^{2m} = (f-x)^{2n+1} q^{2m}$ ſeu $(y-p)^{2m} = (f-x)^{2n+1} Q$. Quoties ergo functio y per huiusmodi aequationem exprimitur, ita vt ſit $2n+1 > 2m$, toties poſito $x = f$, valor ipſius y fiet maximus vel minimus: prius quidem ſi fuerit $\frac{dp}{dx}$ quantitas affirmatiua, poſterius vero ſi ſit $\frac{dp}{dx}$ quantitas negatiua poſito $x = f$. Sin autem fiat hoc caſu $\frac{dp}{dx} = 0$, tum erit $y = g + \frac{\omega^2 ddp}{2 dx^2} \pm \omega^{\frac{2n+1}{2m}} q$. Niſi ergo ſit $\frac{2n+1}{2m} > 2$, neque maximum neque minimum locum habebit; at ſi $\frac{2n+1}{2m} > 2$, tum $y = g$ erit maximum, ſi $\frac{ddp}{dx^2}$ habuerit valorem negatiuum, minimum vero, ſi affirmatiuum: ſicque vlterius ſi quoque $\frac{ddp}{dx^2}$ euaneſcat, iudicium erit inſtituendum.

281. Si igitur y fuerit huiusmodi functio ipſius x, fieri poteſt, vt praeter maxima & minima, quae prior methodus exhibet, etiam maxima minimaue huius alterius ſpeciei adſint, quae modo hic expoſito explorari poterunt. Id quod ſequentibus exemplis declarabimus.

EXEM-

EXEMPLUM I.

Determinare maxima ac minima functionis y, *quae definitur hac aequatione :*

$$yy - 2xy - 2xx - 1 + 3x + x^3 = 0.$$

Ad maxima minimaue primae ſpeciei inueſtiganda differentietur aequatio, eritque

$2ydy - 2xdy - 2ydx - 4xdx + 3dx + 3xxdx = 0$,

poſitoque $\frac{dy}{dx} = 0$, erit $y = \frac{3}{2} - 2x + \frac{3}{2}xx$,

qui valor in prima aequatione ſubſtitutus dat:

$9x^4 - 32x^3 + 42xx - 24x + 5 = 0$, quae reſolvitur in $9xx - 14x + 5 = 0$ & $xx - 2x + 1 = 0$. Poſterior bis dat $x = 1$, fitque $y = 1$, vnde hoc caſu in fractione $\frac{dy}{dx} = \frac{2y + 3 - 4x + 3xx}{2y - 2x}$ denominator quoque euaneſcit, ſicque maximum minimumue primi generis non datur: prior vero aequatio $9xx - 14x + 5 = 0$ dabit $x = 1$ & $x = \frac{5}{9}$, quorum valorum ille eodem incommodo laborat, quo praecedentes. Poſito autem $x = \frac{5}{9}$, fit $y = \frac{3}{2} - \frac{10}{9} + \frac{25}{54} = \frac{23}{27}$. Et cum ſit $\frac{dy}{dx} = \frac{2y - 3 + 4x - 3xx}{2y - 2x}$, fiet $\frac{ddy}{dx^2} = \frac{+4 - 6x}{2y - 2x} = \frac{-3x + 2}{y - x}$ ob $dy = 0$ & numeratorem $= 0$. Erit ergo $\frac{ddy}{dx^2} = \frac{9}{8}$, vnde hic valor

$x =$

$x = \frac{5}{9}$ dat minimum primi generis. Deinde cum ſit $(y-x)^2 = (1-x)^3$, erit $y = x \pm (1-x)\sqrt{(1-x)}$; ideoque poſito $x = 1$ prodit maximum ſecundae ſpeciei: facto enim $x = 1-\omega$, erit $y = 1-\omega \pm \omega\sqrt{\omega}$, quorum vterque minor eſt quam vnitas, ſiquidem ω ſumatur minimum.

EXEMPLUM II.

Inuenire maxima ac minima functionis:

$$y = 2x - xx \pm (1-x)^2\sqrt{(1-x)}.$$

Pro primi generis maximis & minimis differentietur aequatio; eritque

$$\frac{dy}{dx} = 2 - 2x \mp \frac{5}{2}(1-x)\sqrt{(1-x)}$$

qui valor poſitus $= 0$ prodit primo $x = 1$, & cum ſit $\frac{ddy}{dx^2} = -2 \pm \frac{15}{4}\sqrt{(1-x)}$, erit y hoc caſu maximum primi generis, ſitque $y = 1$. Aequatione vero $\frac{dy}{dx} = 0$ per $1-x$ diuiſa erit $4 \mp 5\sqrt{(1-x)} = 0$ ſeu $16 = 25 - 25x$, vnde ſit $x = \frac{9}{25}$, & $\frac{ddy}{dx^2} = -2 \pm 3$. Quare ſi ſignum ſuperius valet, erit $y = \frac{2869}{3125}$ minimum; ſin autem ſignum inferius valeat, erit $y = \frac{821}{3125}$, quod maximum videatur: at vero tantum ſignum ſuperius locum habere poteſt, quoniam $4 \mp 5\sqrt{(1-x)}$ nequit eſſe $= 0$, niſi ſit $\sqrt{(1}$

$V(1-x) = +\frac{4}{5}$. Primi ergo generis inuenimus maximum caſu $x = 1$ & $y = 1$, atque minimum caſu $x = \frac{9}{25}$ & $y = \frac{2869}{3125}$. Ex genere vero altero maximum quoque prodit, ſi $x = 1$, quo caſu fit $y = 1$. Nam poſito $x = 1 - \omega$, erit $y = 1 - \omega\omega \pm \omega^2 V\omega$ vtroque caſu < 1. Hic itaque, ſi $x = 1$, maxima duo primae & alterius ſpeciei coaleſcunt, maximumque quaſi mixtum conſtituunt.

282. Ex his exemplis non ſolum natura huius alterius ſpeciei maximorum & minimorum elucet; ſed etiam pro lubitu iſtiusmodi functiones formari poſſunt, quae maxima vel minima ſecundae ſpeciei admittant. Quemadmodum autem, ſi propoſita fuerit functio quaecunque, explorari poſſit, vtrum eiusmodi maximis minimisue ſit praedita nec ne? id in ſequenti ſectione oſtendemus: propterea quod natura linearum curuarum hac inueſtigatione maxime illuſtratur. Ceterum vero facile intelligitur, ſi fuerit y eiusmodi functio ipſius x, quae maximum minimumue ſecundae ſpeciei recipiat, tum quoque viciſſim x eiusmodi fore functionem ipſius y. Nam quia ex hac aequatione $(y-x)^2 = (1-x)^3$, facto $x = 1$, obtinet y valorem maximum ſecundae ſpeciei; ſi variabiles y & x permutentur, haec aequatio $(x-y)^2 = (1-y)^3$ exhibet pro y quoque eiusmodi functionem ipſius x, quae habeat maximum ſecundae ſpeciei. Facto enim $x = 1$, fiet $(1-y)^2 = (1-y)^3$,

 hinc-

hincque erit bis $y = 1$ & ſemel $y = 0$. Sin autem ponatur $x = 1 + \omega$, erit $(1 + \omega - y)^2 = (1 - y)^3$; vnde ſi ſtatuamus $y = 1 + \phi$ erit $(\omega - \phi)^2 = (-\phi)^3 = -\phi^3$ ideoque ϕ debet eſſe negatiuum. Sit ergo $y = 1 - \phi$ erit $(\omega + \phi)^2 = \phi^3$, atque cum ſumto ϕ minimo, ϕ^3 prae ϕ^2 euaneſcat, debebit neceſſario ω eſſe negatiuum: hinc valori $x = 1 + \omega$ nulli valores reales ipſius y reſpondent. At poſito $x = 1 - \omega$, & $y = 1 - \phi$. ob $(\phi - \omega)^2 = \phi^3$, erit $\phi = \omega \pm \omega \sqrt{\omega}$, ideoque $y = 1 - \omega \mp \omega \sqrt{\omega}$, vnde vterque valor ipſius y reſpondens ipſi $x = 1 - \omega$ minor eſt valore $y = 1$, qui reſpondet valori $x = 1$; eritque conſequenter iſte ipſius y valor maximus.

283. Hactenus tantum functiones biformes ſumus contemplati, quarum maxima vel minima, quia ambo valores facile per reſolutionem aequationis quadraticae exprimi poſſunt, ad examen reuocari poſſunt. Sin autem functio y per aequationem altiorem exprimatur, methodus ante tradita, qua maxima minimaque primae ſpeciei indagauimus, eodem ſucceſſu adhiberi poterit. Inuentionem vero maximorum ac minimorum ſecundae ſpeciei ſequenti ſectioni reſeruamus. Functiones ergo triformes ac multiformes, quemadmodum tractari oporteat, aliquot exemplis oſtendamus.

EXEM-

EXEMPLUM I.

Definiatur functio y, *cuius maxima vel minima quaeruntur, per hanc aequationem:*

$$y^3 + x^3 = 3axy.$$

Differentiata hac aequatione fit $3y^2dy + 3xxdx = 3axdy + 3aydx$, ideoque $\frac{dy}{dx} = \frac{ay - xx}{yy - ax}$. Maximum ergo vel minimum dabitur, ſi fuerit $ay = xx$, ſeu $y = \frac{xx}{a}$, qui valor in aequatione propoſita ſubſtitutus dat:

$$\frac{x^6}{a^3} + x^3 = 3x^3 \quad \text{ſeu} \quad x^6 = 2a^3x^3;$$

Erit ergo ter $x = 0$, quo caſu quoque fit denominator $yy - ax = 0$, ob $y = \frac{xx}{a} = 0$. Vtrum ergo hoc caſu maximum minimumue prodeat? patebit ſi ipſi x valorem tribuamus minime ab o diſcrepantem. Sit ergo $x = \omega$, & $y = \phi$, ob $\phi^3 + \omega^3 = 3a\omega\phi$, fiet vel $\phi = \alpha V\omega$ vel $\phi = \beta\omega^2$. Priori caſu erit $\alpha^3\omega V\omega = 3\alpha a\omega V\omega$, ideoque $\alpha = V3a$. Hinc poſito $x = \omega$ erit $y = \pm V3a\omega$. Vnde etiamſi ω negatiue accipi nequeat; tamen binorum ipſius y valorum alter maior erit quam o, alter minor; hincque $y = 0$ neque maximum erit neque minimum. Sin autem ſtatuatur $\phi = \beta\omega^2$ erit $\omega^3 = 3a\beta\omega^3$, ideoque $\beta = \frac{1}{3a}$ & $\phi = \frac{\omega^2}{3a}$. Ergo hoc caſu ſiue x capiatur $= +\omega$ ſiue $= -\omega$, valor

 ipſius

ipſius $y = \phi$ nihilo erit maior, ideoque hoc caſu $y = 0$ erit minimum. Reſtat ergo tertius caſus ex aequatione $x^3 = 2a^3$ examinandus, qui dat $x = a\sqrt[3]{2}$, & $y = a\sqrt[3]{4}$. Qui vtrum ſit maximus an minimus? ex aequatione $\frac{dy}{dx} = \frac{ay - xx}{yy - ax}$ quaeratur differentiale ſecundum, quod ob $dy = 0$ & $ay - xx = 0$ erit $\frac{ddy}{dx^2} = \frac{-2x}{yy - ax}$, cuius valor praeſenti caſu eſt $-\frac{2a\sqrt[3]{2}}{2a^2\sqrt[3]{2} - aa\sqrt[3]{2}} = -\frac{2}{a}$, qui indicat valorem ipſius y eſſe maximum.

EXEMPLUM II.

Si functio y definiatur per hanc aequationem:

$$y^4 + x^4 + ay^3 + ax^3 = b^3x + b^3y,$$

inuenire eius maximos minimosue valores.

Cum per differentiationem oriatur

$$4y^3dy + 3ayydy - b^3dy = b^3dx - 3axxdx - 4x^3dx;$$

erit $\frac{dy}{dx} = \frac{b^3 - 3axx - 4x^3}{4y^3 + 3ayy - b^3}$, ponique oportet: $b^3 = 3axx + 4x^3$. Quaeſtio ergo huc reducitur, vt functionis vniformis $b^3 - ax^3 - x^4$ maxima ac minima indagentur, quae ſimul erunt maxima ſeu minima functionis y. Sit $a = 2$ & $b = 3$ ſeu proponatur haec aequatio $y^4 + x^4 + 2y^3 + 2x^3 = 27x + 27y$; erit $\frac{dy}{dx} = \frac{27 - 6xx - 4x^3}{4y^3 + 6yy - 27}$ & $4x^3 + 6xx - 27 = 0$,

quae

quae diuiſa per $2x - 3 = 0$ dat $2xx + 6x + 9 = 0$, cuius poſterioris radices cum ſint imaginariae, erit $x = \frac{3}{2}$ & $y^4 + 2y^3 - 27y = \frac{459}{16}$, cuius ſingulae radices erunt vel maximae vel minimae. Cum autem ſit $\frac{dy}{dx} = \frac{27 - 6xx - 4x^3}{4y^3 + 6yy - 27}$, erit $\frac{ddy}{dx^2} = \frac{-12x - 12xx}{4y^3 + 6yy - 27}$, qui poſito $x = \frac{3}{2}$, ſi fuerit affirmatiuus, indicabit minimum, contra vero maximum.

EXEMPLUM III.

Si fuerit $y^m + ax^n = by^p x^q$; *definire maxima & minima ipſius* y.

Per differentiationem fit $\frac{dy}{dx} = \frac{qby^p x^{q-1} - nax^{n-1}}{my^{m-1} - pby^{p-1}x^q}$, quo poſito $= 0$, erit primo $x = 0$, ſi quidem n & q fuerint vnitate maiores; atque ſimul $y = 0$. Quo caſu an detur maximum vel minimum, valores proximi ſunt inueſtigandi, quoniam quoque denominator fit $= 0$; quae inueſtigatio ab exponentibus potiſſimum pendebit. Praeterea vero aequatio $\frac{dy}{dx} = 0$ dabit $y^p = \frac{na}{qb} x^{n-q}$, qui valor in propoſita ſubſtitutus ponendo $\frac{na}{qb} = g$ dabit $g^{\frac{m}{p}} x^{\frac{mn-mq}{p}} + ax^n = \frac{na}{q} x^n$ ſeu $g^{\frac{m}{p}} x^{\frac{mn-mq-np}{p}} = \frac{(n-q)a}{q}$,

 vnde

vnde fit $x = \left(\frac{(n-q)a}{q}\right)^{p:(mn-mq-np)} : g^{m:(mn-mq-np)}$, ſimulque valor ipſius y innoteſcit. Deinde diſpiciendum eſt, vtrum differentio - differentiale $\frac{ddy}{dx^2} = \frac{q(q-1)byp\,x^{q-2} - n(n-1)ax^{n-2}}{my^{m-1} - pby^{p-1}x^q}$ obtineat valorem affirmatiuum an negatiuum? vt ex priori minimum, ex poſteriori vero maximum pronuncietur.

EXEMPLUM. IV.

Si fuerit $y^4 + x^4 = 4xy - 2$, *maxima & minima functionis* y *aſſignare.*

Differentiatione inſtituta fit $\frac{dy}{dx} = \frac{y-x^3}{y^3-x}$, hincque oritur $y = x^3$, erit ergo $x^{12} = 3x^4 - 2$ ſeu $x^{12} - 3x^4 + 2 = 0$, quae aequatio reſoluitur in has $x^4 - 1 = 0$ & $x^8 + x^4 - 2 = 0$, poſteriorque in $x^4 - 1 = 0$ & $x^4 + 2 = 0$. Hinc erit bis vel $x = +1$ vel $x = -1$; vtroque vero caſu & denominator fractionis $\frac{dy}{dx}$ euaneſcit. Ad inueſtigandum ergo, vtrum his caſibus maximum minimumue locum habeat? ponamus $x = 1 - \omega$ & $y = 1 - \phi$; erit:

$$\begin{aligned} 1 - 4\phi + 1 - 4\omega &= 4 - 4\omega - 4\phi - 2 \\ + 6\phi^2 + 6\omega^2 &\qquad + 4\omega\phi \\ - 4\phi^3 - 4\omega^3 & \\ + \phi^4 + \omega^4 & \end{aligned}$$

ideo-

ideoque $4\omega\phi = 6\phi^2 + 6\omega^2 - 4\phi^3 - 4\omega^3 + \phi^4 + \omega^4$, & ob ω & ϕ minima $4\omega\phi = 6\phi^2 + 6\omega^2$. Valor ergo ipſius ϕ erit imaginarius, ſiue ω capiatur affirmatiue ſiue negatiue. Seu ſi y & x deſignent coordinatas curuae, ea caſu $x = 1$ & $y = 1$ habebit punctum coniugatum. Neque ergo hic valor pro maximo neque pro minimo haberi poteſt, propterea quod antecedentes & conſequentes, cum quibus comparari deberet, fiunt imaginarii.

284. Si aequatio, qua relatio inter x & y exprimitur, ita fuerit comparata, vt functio ipſius y aequetur functioni ipſius x, puta $Y = X$; ad maxima minimaue inuenienda poni debebit $dX = 0$: fiet ergo y maximum vel minimum iisdem caſibus, quibus X fit maximum vel minimum. Simili modo ſi x tanquam functio ipſius y conſideretur, fiet x maximum vel minimum ſi $dY = 0$ hoc eſt ſi Y fuerit maximum vel minimum. Neque tamen hinc ſequitur y & x ſimul fieri maxima vel minima. Nam ſi fuerit $2ay - yy = 2bx - xx$, erit y maximum vel minimum, ſi fuerit $x = b$; eritque $y = a \pm \sqrt{(aa-bb)}$. Contra vero x fit maximum vel minimum, ſi fuerit $y = a$, fitque $x = b \pm \sqrt{(bb-aa)}$, neque ergo fiet y maximum vel minimum, ſi $x = b \pm \sqrt{(bb-aa)}$, quo tamen caſu x eſt maximum minimumue. Ceterum hoc caſu, ſi y habeat valores maximos vel minimos, x hac indole prorſus carebit: namque y maximum minimumue fieri nequit, niſi ſit $a > b$, quo caſu maximum minimumue ipſius x fit imaginarium.

285. Tum vero etiam euenire poteſt, vt non omnes radices aequationis $dX = 0$ praebeant maximos mimimosue valores pro y; ſi enim illa aequatio duas habuerit radices aequales, exinde neque maximum neque minimum conſequitur; hocque idem euenit, ſi quotcunque radices numero pares fuerint inter ſe aequales. Sic ſi proponatur aequatio $b(y-a)^2 = (x-b)^3 + c^3$; quia ſumtis differentialibus fit $2bdy(y-a) = 3dx(x-b)^2$, functio y neque maxima fiet neque minima poſito $x = b$, propterea quod hic occurrunt duae radices aequales. Sin autem x tanquam functio ipſius y ſpectetur, ea fiet maxima vel minima, ſi ſtatuatur $y = a$; eritque $x = b - c$ minimum. Quia denique in huiusmodi aequationibus $Y = X$ variabiles x & y inter ſe non permiſcentur, ſi ipſi x tribuitur valor, qui ſit radix aequationis $dX = 0$, omnes valores ipſius y, quotcunque fuerint reales, erunt maximi vel minimi; quod non euenit, ſi in aequatione ambae variabiles fuerint permixtae.

286. Quae praeterea ſuperſunt de natura maximorum ac minimorum exponenda, ea in ſequentem ſectionem reſeruamus, quoniam commodius ope figurarum menti repraeſentari atque explicari poſſunt. Pergamus ergo ad functiones, quae ex pluribus variabilibus ſunt compoſitae, atque inueſtigemus valores, quos ſingulis variabilibus tribui oportet, vt ipſa functio vel maximum vel minimum valorem obtineat. Ac primo quidem patet, ſi variabiles non fuerint inter ſe permixtae, ita vt functio propoſita ſit huiusmodi $X + Y$, exiſtente

X

X functione ipſius x, & Y ipſius y tantum, tum functionem propoſitam X + Y fore maximum, ſi ſimul X & Y maximum euadat: minimumque, ſi ſimul X & Y fiat minimum. Ad maximum ergo inueniendum inquirantur valores ipſius x, quibus X fiat maximum, ſimilique modo valores ipſius y, quibus Y fit maximum: hique valores pro x & y inuenti efficient functionem X + Y maximam, quod ſimiliter de minimo erit tenendum. Cauendum ergo eſt, ne duo valores ipſarum x & y diuerſae naturae combinentur, quorum ille reddat X maximum, hic vero Y minimum, aut contra. Hoc enim ſi fieret, functio X + Y neque maximum foret neque minimum. At huiusmodi functio X — Y fiet maxima, ſi X fuerit maximum ſimulque Y minimum; contra vero X — Y fiet minimum, ſi X fuerit minimum & Y maximum. Sin autem vtraque functio X & Y ſtatueretur vel maxima vel minima, earum differentia X — Y neque foret maxima, neque minima; quae omnia ſunt ex natura maximorum ac minimorum ante expoſita clara & perſpicua.

287. Si ergo quaerantur maximi minimiue valores functionis duarum variabilium; quaeſtio multo magis cautioni obnoxia eſt, quam ſi vnica fuerit variabilis. Non ſolum enim pro vtraque variabili caſus, quibus maximum minimumue producitur, diligenter ſunt diſtinguendi; ſed etiam ex his bini eiusmodi ſunt coniungendi, vt functio propoſita fiat maximum vel minimum; id quod ex exemplis clarius patebit.

EXEMPLUM I.

Sit propoſita haec duarum variabilium x *&* y *functio*: $y^4 - 8y^3 + 18y^2 - 8y + x^3 - 3xx - 3x$ *& quaerantur valores pro* y *&* x *ſubſtituendi, vt haec functio maximum vel minimum obtineat valorem.*

Quoniam haec expreſſio in duas huiusmodi partes $Y + X$ reſoluitur, quarum illa eſt functio ipſius y, haec vero ipſius x tantum; caſus quibus vtraque ſit maxima vel minima, inueſtigentur. Cum igitur ſit $Y = y^4 - 8y^3 + 18y^2 - 8y$, erit $\frac{dY}{dy} = 4y^3 - 24y^2 + 36y - 8$ qua expreſſione nihilo aequali poſita, fiet per 4 diuiſo $y^3 - 6y^2 + 9y - 2 = 0$, cuius radices ſunt $y = 2$, & $y = 2 \pm \sqrt{3}$. Cum ergo ſit $\frac{ddY}{4dy^2} = 3yy - 12y + 9$; caſu $y = 2$, prodibit maximum. Pro reliquis binis radicibus $y = 2 \pm \sqrt{3}$, quae oriuntur ex aequatione $yy - 4y + 1 = 0$ fiet $\frac{ddY}{12dy^2} = yy - 4y + 3 = 2$, vnde vtraque dat minimum. Erit autem his caſibus vt ſequitur.

$y = 2$	$Y = 8$	maximum
$y = 2 - \sqrt{3}$	$Y = -1$	minimum
$y = 2 + \sqrt{3}$	$Y = -1$	minimum

Simili modo cum ſit $X = x^3 - 3xx - 3x$, erit $\frac{dX}{dx} = 3xx - 6x - 3$, vnde oritur haec aequatio $xx = 2x + 1$

&

& $x = 1 \pm \surd 2$. Eſt vero $\frac{ddX}{6dx^2} = x - 1 = \pm \surd 2$. Ergo radix $x = 1 + \surd 2$ dat minimum, nempe $X = -5 - 4\surd 2$ & $x = 1 - \surd 2$ dat maximum, nempe $X = -5 + 4\surd 2$. Quocirca formula propoſita $X + Y = y^4 - 8y^3 - 18yy - 8y + x^3 - 3xx - 3x$ fiet maxima, ſi ponatur $y = 2$ & $x = 1 - \surd 2$, prodibitque $X + Y = 3 + 4\surd 2$. Eadem autem formula $X + Y$ fiet minima, ſi ſumatur vel $y = 2 - \surd 3$ vel $y = 2 + \surd 3$ & $x = 1 + \surd 2$, vtroque caſu erit $X + Y = -6 - 4\surd 2$.

EXEMPLUM II.

Si proponatur haec functio duarum variabilium:

$$y^4 - 8y^3 + 18y^2 - 8y - x^3 + 3xx + 3x$$

quae quibus caſibus fiat maxima vel minima inueſtigetur.

Poſito vt in praecedente exemplo habuimus, $Y = y^4 - 8y^3 + 18y^2 - 8y$ & $X = x^3 - 3xx - 3x$; formula propoſita erit $Y - X$; ideoque fiet maxima, ſi Y fuerit maximum & X minimum Cum igitur hos caſus iam ante eruerimus, patet $Y - X$ obtinere valorem maximum, ſi ponatur $y = 2$ & $x = 1 + \surd 2$; fietque $Y - X = 13 + 4\surd 2$. Minimus vero valor ipſius $Y - X$ euadet, ſi Y ſit minimum, & X maximum, quod euenit ponendo $y = 2 \pm \surd 3$ & $x = 1 - \surd 2$, fiet autem $Y - X = 4 - 4\surd 2$. Ceterum in vtroque exemplo patet hos valores, quos inuenimus, neque omnium eſſe maximos neque minimos: nam ſi vtrinque poneretur verbi gratia $y = 100$ & $x = 0$, ſine dubio maior prodiret valor eo,

 quem

quem inuenimus: ſimilique modo ponendo $y = 0$ & vel $x = -100$ vel $x = +100$ minor prodiret valor, quam ſunt illi, quos pro caſu minimi inuenimus. Probe ergo tenenda eſt idea ſupra expoſita, quam de natura maximorum ac minimorum dedimus. Scilicet eum valorem vocari maximum, qui maior ſit valoribus tam antecedentibus quam conſequentibus contiguis proximis; minimum autem eſſe eum, qui his valoribus tam antecedentibus quam conſequentibus fuerit minor. Sic in hoc exemplo eſt valor ipſius $Y - X$, qui prodit ponendo $y = 2$ & $x = 1 + \sqrt{2}$ maior eſt iis, qui reſultat ſi ponatur $y = 2 \pm \omega$ & $x = 1 + \sqrt{2} \pm \phi$ ſumtis pro ω & ϕ quantitatibus ſatis exiguis.

288. His exemplis expeditis facilior erit via ad ſolutionem generalem indagandam. Denotet V functionem quamcunque duarum variabilium x & y, ſintque pro x & y valores inueniendi, qui functioni V inducant maximum vel minimum valorem. Cum igitur ad hoc efficiendum vtrique variabili x & y determinatus valor tribui debeat; ponamus alteram y iam habere eum valorem, qui requiritur ad functionem V vel maximam vel minimam reddendam: hocque poſito tantum opus erit, vt pro altera x idoneus quoque valor inueſtigetur, quod fiet, dum functio V differentiatur ponenda ſola x variabili, differentialeque nihilo aequale ſtatuitur. Simili modo ſi fingamus variabilem x iam eum habere valorem, qui aptus ſit ad functionem V vel maximam vel minimam efficiendam, valor ipſius y reperietur differen-

tian-

tiando V poſita ſola y variabili, hocque differentiale nihilo aequali ponendo. Hinc ſi differentiale functionis V fuerit $= P\,dx + Q\,dy$, oportebit eſſe & $P = 0$ & $Q = 0$, ex quibus duabus aequationibus valores vtriusque variabilis x & y erui poterunt.

289. Quoniam vero hoc pacto ſine diſcrimine reperiuntur valores pro x & y, quibus functio V vel maxima vel minima redditur; caſus, quibus vel maximum vel minimum oritur, probe a ſe inuicem ſunt diſtinguendi. Vt enim functio V fiat maxima, neceſſe eſt vt ambae variabiles ad hoc conſpirent; namque ſi altera maximum exhiberet, altera minimum, ipſa functio neque maxima neque minima euaderet. Quocirca inuentis ex aequationibus $P = 0$ & $Q = 0$ valoribus ipſarum x & y inquirendum eſt, vtrum ambo ſimul functioni V vel maximum vel minimum valorem inducant; atque tum demum, cum compertum fuerit vtriusque variabilis valorem hinc erutum pro maximo valere, affirmare poterimus functionem hoc caſu maximum valorem induere. Quod idem de minimo erit tenendum, ita vt functio V minimum valorem adipiſci nequeat, niſi ſimul ambae variabiles x & y minimum producant. Hinc ergo omnes illi caſus reiici debebunt, quibus altera variabilis maximum, altera vero minimum indicare deprehendetur. Interdum vero etiam euenit, vt alterius vel etiam vtriusque variabilis valores ex aequationibus $P = 0$ & $Q = 0$ oriundi neque maximum neque minimum exhibeant, qui caſus proinde pariter tanquam prorſus inepti erunt reiiciendi.

290. Vtrum autem valores pro x & y reperti valeant pro maximo an minimo? de vtroque ſeorſim ſimili modo inueſtigabitur, quo ſupra, cum vnica adeſſet variabilis, ſumus vſi. Ad iudicium ſcilicet de variabili x inſtituendum conſideretur altera y tanquam conſtans, & cum ſit $dV = Pdx$ ſeu $\frac{dV}{dx} = P$, differentietur P denuo poſito y conſtante, vt prodeat $\frac{ddV}{dx^2} = \frac{dP}{dx}$, ac diſpiciatur, vtrum valor ipſius $\frac{dP}{dx}$, poſtquam loco x & y valores ante inuenti fuerint ſubſtituti, fiat affirmatiuus an negatiuus; priori enim caſu indicabitur minimum, poſteriori vero maximum. Simili modo cum poſito x conſtante ſit $dV = Qdy$ ſeu $\frac{dV}{dy} = Q$, differentietur Q denuo poſita ſola y variabili, & examinetur valor $\frac{dQ}{dy}$, ſubſtitutis loco x & y valoribus, qui ex aequationibus $P = 0$ & $Q = 0$ ſunt inuenti; qui ſi fuerit affirmatiuus, declarabit minimum, contra vero maximum. Hinc ergo colligitur, ſi ex valoribus pro x & y inuentis formulae $\frac{dP}{dx}$ & $\frac{dQ}{dy}$ induant valores diuerſis ſignis affectos altera ſcilicet affirmatiuum, altera negatiuum, tum functionem V neque maximam neque minimam effici; ſin autem vtraque formula $\frac{dP}{dx}$ & $\frac{dQ}{dy}$ fiat affirmatiua, minimum reſultabit: contraque, ſi vtraque fiat negatiua, maximum.

291. Quodſi vero altera formula $\frac{dP}{dx}$ & $\frac{dQ}{dy}$, vel etiam vtraque, ſi pro x & y valores inuenti ſubſtituantur, euaneſcat, tum progrediendum erit ad differentialia ſequentia $\frac{ddP}{dx^2}$ & $\frac{ddQ}{dy^2}$, quae niſi pariter euaneſcant, neque maximum neque minimum habebit locum; ſin autem euaneſcant, iudicium ex formulis differentialibus ſequentibus $\frac{d^3P}{dx^3}$ & $\frac{d^3Q}{dy^3}$ erit petendum, ſimilique modo inſtituendum, quo pro formulis $\frac{dP}{dx}$ & $\frac{dQ}{dy}$ eſt factum. Quo autem, quibus caſibus hoc vſu veniat, clarius exponamus, prodierit valor $x = \alpha$, qui ſi formulam $\frac{dP}{dx}$ reddat euaneſcentem, neceſſe eſt vt $\frac{dP}{dx}$ factorem habeat $x - \alpha$; qui factor ſi fuerit ſolitarius, neque ſimul alium ſibi habeat aequalem ſocium, neque maximum neque minimum indicabitur, quod idem euenit ſi $\frac{dP}{dx}$ factorem habuerit $(x-\alpha)^3$, vel $(x-\alpha)^5$, &c. Sin autem factor fuerit $(x-\alpha)^2$, vel $(x-\alpha)^4$, &c. tum quidem maximum vel minimum indicabitur; at inſuper videndum erit, vtrum cum caſu, per y indicato conſentiat.

292. Labor autem his caſibus ad differentialia vlteriora progrediendi mirifice ſubleuari poterit: ſi enim ponamus, vt rem generalius complectamur, inuentum

eſſe

eſſe $\alpha x + \beta = 0$, atque formulam $\frac{dP}{dx}$ factorem habere $(\alpha x + \beta)^2$, ita vt ſit $\frac{dP}{dx} = (\alpha x + \beta)^2 T$, quia eſt $\alpha x + \beta = 0$, fiet $\frac{d^3P}{dx^3} = 2\alpha^2 T$, hincque ob $2\alpha^2$ affirmatiuum, ex ipſa quantitate T iudicium abſolui poterit; quae ſi induet valorem affirmatiuum, pro minimo, contra vero pro maximo pronunciabit. Hocque idem ſubſidium in maximorum minimorumque inueſtigatione, ſi vnica inſit variabilis adhiberi poterit, ita vt nunquam opus ſit ad altiora differentialia aſcendere. Quin etiam nequidem ad differentialia ſecunda procedere opus erit: ſi enim ex aequatione $P = 0$, fiat $\alpha x + \beta = 0$, neceſſe eſt vt P factorem habeat $\alpha x + \beta$; ſit $P = (\alpha x + \beta)T$, & cum ſit $\frac{dP}{dx} = \alpha T + (\alpha x + \beta)\frac{dT}{dx}$, ob $\alpha x + \beta = 0$, erit $\frac{dP}{dx} = \alpha T$, hincque iam ipſe alter factor T, prout valor ipſius αT fuerit vel affirmatiuus vel negatiuus, ſtatim vel minimum vel maximum indicabit.

293. His igitur traditis praeceptis haud difficile erit, ſi functio quaecunque duas variabiles inuoluens fuerit propoſita, caſus inueſtigare, quibus haec functio fiat vel maxima vel minima. Si quae inſuper notanda fuerint, ea ipſa exemplorum euolutio ſuggeret, quamobrem aliquot exemplis regulas datas illuſtrari expediet.

EXEM-

EXEMPLUM I.

Sit propoſita iſta functio duarum variabilium $V = xx + xy + yy - ax - by$, *quae quibus caſibus fiat vel maxima vel minima inquiratur.*

Cum ſit $dV = 2xdx + ydx + xdy + 2ydy - adx - bdy$, ſi comparetur cum forma generali $dV = Pdx + Qdy$ erit $P = 2x + y - a$ & $Q = 2y + x - b$: vnde formabuntur iſtae aequationes $2x + y - a = 0$ & $2y + x - b = 0$, quibus coniunctis eliminando y fiet $x - b = 4x - 2a$, ideoque $x = \frac{2a - b}{3}$, & $y = a - 2x = \frac{2b - a}{3}$. Cum igitur ſit $\frac{dP}{dx} = 2$ & $\frac{dQ}{dy} = 2$, vtraque oſtendit minimum; ex quo concludimus formulam $xx + xy + yy - ax - by$ fieri minimam, ſi ponatur $x = \frac{2a - b}{3}$ & $y = \frac{2b - a}{3}$, prodibitque hoc modo $V = -\frac{3aa + 3ab - 3bb}{9} = -\frac{aa + ab - bb}{3}$, qui cum ſit vnicus, omnium erit minimus. Vnico ergo modo fieri poteſt $xx + xy + yy - ax - by = -\frac{aa + ab - bb}{3}$, & quia minor fieri nequit, erit haec aequatio $xx + xy + yy - ax - by = -\frac{aa + ab - bb}{3} - cc$ impoſſibilis.

EXEMPLUM II.

Si proponatur formula $V = x^3 + y^3 - 3axy$, *quaerantur caſus, quibus* V *adipiſcatur valorem maximum vel minimum.*

Ob $dV = 3xxdx + 3yydy - 3aydx - 3axdy$ erit $P = 3xx - 3ay$ & $Q = 3yy - 3ax$, vnde ſit $ay = xx$ & $ax = yy$. Cum ergo ſit $yy = x^4 : aa = ax$ erit $x^4 - a^3 x = 0$; ideoque vel $x = 0$ vel $x = a$. Priori caſu fit $y = 0$, poſteriori vero $y = a$. Quoniam ergo eſt $\frac{dP}{dx} = 6x$, $\frac{ddP}{dx^2} = 6$ & $\frac{dQ}{dy} = 6y$ atque $\frac{ddQ}{dy^2} = 6$; priori ergo caſu, quo $x = 0$ & $y = 0$, neque maximum neque minimum reſultat. Poſteriori vero caſu quo & $x = a$ & $y = a$ minimum prodit, ſi quidem a fuerit quantitas affirmatiua, fietque $V = -a^3$, qui autem valor tantum minor eſt proximis antecedentibus & conſequentibus: nam ſine dubio V multo minorem induere poteſt valorem, ſi vtrique variabili x & y valores negatiui tribuantur.

EXEMPLUM III.

Propoſita ſit haec functio $V = x^3 + ayy - bxy + cx$, *cuius valores maximi ſeu minimi inquirantur.*

Quia eſt $dV = 3xxdx + 2aydy - bydx - bxdy + cdx$ erit $P = 3xx - by + c$ & $Q = 2ay - bx$, quibus valoribus nihilo aequalibus poſitis erit $y = \frac{bx}{2a}$, ideoque

$3xx$

$3xx - \frac{bbx}{2a} + c = 0$ ſeu $xx = \frac{2bbx - 4ac}{12a}$ vnde fit $x = \frac{bb \pm \surd(b^4 - 48aac)}{12a}$. Niſi ergo ſit $b^4 - 18aac > 0$, neque maximum neque minimum habet locum. Ponamus ergo eſſe $b^4 - 48aac = bbff$, vt ſit $c = \frac{bb(bb-ff)}{48aa}$; erit $x = \frac{bb \pm bf}{12a}$ & $y = \frac{bb(b \pm f)}{24aa}$. Quoniam porro eſt $\frac{dP}{dx} = 6x$ & $\frac{dQ}{dy} = 2a$, fiet $\frac{dP}{dx} = \frac{b(b \pm f)}{2}$. Niſi ergo $2a$ & $\frac{b(b \pm f)}{2a}$ ſint quantitates eiusdem ſigni, neque maximum neque minimum habet locum. At ſi ſint ambae vel affirmatiuae vel ambae negatiuae, quod euenit, ſi earum productum $b(b \pm f)$ fuerit affirmatiuum; tum functio V euadet minimum, ſi a ſit quantitas affirmatiua; contra vero maximum, ſi a ſit quantitas negatiua. Hinc ſi fuerit $f = 0$ ſeu $c = \frac{b^4}{48aa}$, ob bb quantitatem affirmatiuam, functio V euadet minima, ſi a ſit quantitas poſitiua, ponaturque $x = \frac{bb}{12a}$ & $y = \frac{b^3}{24aa}$; contra vero ſi a ſit negatiuum, iſtae ſubſtitutiones producent maximum. Si ſit $f < b$, duobus caſibus oritur vel maximum vel minimum: at ſi $f > b$, tum caſus tantum $x = \frac{b(b+f)}{12a}$ & $y = \frac{bb(b+f)}{24aa}$ praebebit maximum minimumue, prout a fuerit vel negatiuum vel af-

 fir-

firmatiuum. Sit $a = 1$, $b = 3$ & $f = 1$, vt habeatur haec formula $V = x^3 + yy - 3xy + \frac{3}{2}x$, haec fiet minima ob a affirmatiuum, ſi ponatur vel $x = 1$ & $y = \frac{3}{2}$ vel $x = \frac{1}{2}$ & $y = \frac{3}{4}$. Priori caſu oritur $V = \frac{1}{4}$, poſteriori vero $V = \frac{5}{16}$. Interim tamen patet loco x numeris negatiuis ponendis multo minores valores pro V oriri poſſe. Ita ergo intelligi debet valor ipſius $V = \frac{1}{4}$ minor eſſe, quam ſi ponatur $x = 2 + \omega$ & $y = 3 + \phi$; dummodo ſint ω & ϕ numeri parui, ſiue affirmatiui ſiue negatiui; limes autem quem ω transgredi non debet eſt $-\frac{15}{4}$; nam ſi $\omega < -\frac{15}{4}$ fieri poterit vt V fiat minor quam $\frac{1}{4}$.

EXEMPLUM IV.

Inuenire maxima vel minima huius functionis:

$$V = x^4 + y^4 - axxy - axyy + ccxx + ccyy.$$

Sumto differentiali erit $P = 4x^3 - 2axy - ayy + 2ccx$ & $Q = 4y^3 - axx - 2axy + 2ccy$, quibus valoribus nihilo aequalibus poſitis, ſi a ſe inuicem ſubtrahantur erit: $4x^3 - 4y^3 + axx - ayy + 2ccx - 2ccy = 0$, quae cum ſit diuiſibilis per $x - y$, erit primo $y = x$, atque $4x^3 - 3axx + 2ccx = 0$, quae dat $x = 0$ & $4xx = 3ax - 2cc$ ſeu $x = \frac{3a \pm \sqrt{(9aa - 32cc)}}{8}$. Si ſumamus $x = 0$, erit quoque $y = 0$; & ob $\frac{dP}{dx} = 12xx - 2ay + 2cc$, atque $\frac{dQ}{dy} = 12yy - 2ax + 2cc$, fiet functio V minima $= 0$.

Sin

Sin ſtatuamus $x = y = \frac{3a \pm \sqrt{(9aa - 32cc)}}{8}$, ſi quidem fuerit $9aa > 32cc$, ob $4xx = 3ax - 2cc$, erit $\frac{dP}{dx} = \frac{dQ}{dy} =$ $12xx - 2ax + 2cc = 7ax - 4cc = \frac{21aa - 32cc \pm 7a\sqrt{(9aa - 32cc)}}{8}$, qui valor cum ſit ſemper affirmatiuus ob $32cc < 9aa$, valor V hoc quoque caſu fit minimus, eritque $V = \frac{-27}{256}a^4 + \frac{9}{16}aacc - \frac{1}{2}c^4 \mp \frac{a}{256}(9aa - 32cc)^{\frac{3}{2}}$. Diuidamus autem aequationem $4x^3 - 4y^3 + axx - ayy + 2ccx - 2ccy = 0$ per $x - y$ fietque $4xx + 4xy + 4yy + ax + ay + 2cc = 0$. At ex aequatione $P = 0$, erit $yy = -2xy + \frac{4}{a}x^3 + \frac{2ccx}{a}$, quo valore in illa ſubſtituto fit

$$y = \frac{16x^3 + 4axx + aax + 8ccx + 2acc}{4ax - aa}.$$

Verum illa dat $y = -x \pm \sqrt{\frac{4x^3 + axx + 2ccx}{a}}$, vnde efficitur:

$$16x^3 + 8axx + 4ccx + 2acc = (4x - a)\sqrt{(4ax^3 + aaxx + 2accx)},$$

quae ad rationalitatem perducta, dat

$$256x^6 + 192ax^5 + \left.\begin{matrix}80aa \\ +128cc\end{matrix}\right\} x^4 + \left.\begin{matrix}4a^3 \\ +96acc\end{matrix}\right\} x^3 \left.\begin{matrix}-a^4 \\ +48aacc \\ +16c^4\end{matrix}\right\} x^2 \left.\begin{matrix}-2a^3cc \\ +16ac^4\end{matrix}\right\} x + 4a^2c^4 = 0$$

cuius radices, ſi quas habet, reales indicabunt maxima vel minima functionis V, ſi quidem $\frac{dP}{dx}$ & $\frac{dQ}{dy}$ fiant quantitates eodem ſigno affectae.

EXEMPLUM V.

Inuenire maxima & minima huius expreſſionis:

$$x^4 + mxxyy + y^4 + aaxx + naaxy + aayy = V.$$

Facta differentiatione erit:

$$P = 4x^3 + 2mxyy + 2aax + naay = 0$$

$$Q = 4y^3 + 2mxxy + 2aay + naax = 0$$

quae aequationes inuicem vel ſubtractae vel additae dant:

$$(4xx + 4xy + 4yy - 2mxy + 2aa - naa)(x - y) = 0$$

$$(4xx - 4xy + 4yy + 2mxy + 2aa + naa)(x + y) = 0$$

quae diuiſae per $x - y$ & $x + y$, & denuo vel additae vel ſubtractae dant:

$$4xx + 4yy + 2aa = 0 \quad \& \quad 4xy - 2mxy - naa = 0.$$

Ex quarum poſteriori fit $y = \frac{naa}{2(2-m)x}$, prior autem reales valores non admittit. Tres igitur habemus caſus:

I. Si $y = x$, eritque $4x^3 + 2mx^3 + 2aax + naax = 0$, vnde fit vel $x = 0$ vel $2(2+m)xx + (2+n)aa = 0$. Sit $x = 0$, erit quoque $y = 0$, atque ob $\frac{dP}{dx} = 12xx + 2myy + 2aa$ & $\frac{dQ}{dy} = 12yy + 2mxx + 2aa$, hoc caſu fiet $V = 0$ minimum, ſi quidem coefficiens aa fuerit affirmatiuus. Alter caſus dat $xx = -\frac{(n+2)aa}{2(m+2)}$, quae realis eſſe nequit niſi ſit $\frac{n+2}{m+2}$ numerus negatiuus. Sit $\frac{n+2}{m+2} = -2kk$

ſeu

ſeu $n = -2kkm - 4kk - 2$, erit $x = \pm ka$ & $y = \pm ka$. At $\frac{dP}{dx} = 12kkaa + 2mkkaa + 2aa$ & $\frac{dQ}{dy} = 12kkaa + 2mkkaa + 2aa$, quae cum ſint aequales, erit V vel minimum vel maximum, prout iſtae quantitates fuerint vel affirmatiuae vel negatiuae.

II. Sit $y = -x$, eritque $2(m+2)x^3 = (n-2)aax$ ergo vel $x = 0$ vel $xx = \frac{(n-2)aa}{2(m+2)}$. Prior radix $x = 0$ recidit in praecedentem. Poſterior vero erit realis ſi $\frac{(n-2)aa}{2(m+2)}$ fuerit quantitas affirmatiua: & cum fiat $\frac{dP}{dx} = \frac{dQ}{dy}$ prodibit vel maximum vel minimum.

III. Sit $y = \frac{naa}{2(2-m)x}$, erit $4x^3 + \frac{mn^2a^4}{2(2-m)^2x} + 2aax + \frac{nna^4}{2(2-m)x} = 0$ ſeu $4x^4 + 2aaxx + \frac{nna^4}{(2-m)^2} = 0$, cuius aequationis nulla radix eſt realis, niſi ſit aa quantitas negatiua.

EXEMPLUM VI.

Propoſita ſit haec functio determinata:

$$V = x^4 + y^4 - xx + xy - yy,$$

cuius valores maximi vel minimi inueſtigentur.

Cum hinc fiat $P = 4x^3 - 2x + y = 0$ & $Q = 4y^3 - 2y + x = 0$ erit ex priori $y = 2x - 4x^3$, qui in altera ſubſtitutus dat $256x^9 - 384x^7 + 192x^5 - 40x^3 + 3x = 0$.

Cuius

Cuius vna radix eſt $x = 0$, vnde fit quoque $y = 0$. Ergo hoc caſu ob $\frac{dP}{dx} = 12xx - 2$ & $\frac{dQ}{dy} = 12yy - 2$ prodit maximum $V = 0$.

Diuiſa autem aequatione inuenta per x erit:

$$256x^8 - 384x^6 + 192x^4 - 40xx + 3 = 0,$$

quae factorem habet $4xx - 1$, vnde fit $4xx = 1$ & $x = \pm\frac{1}{2}$; atque $y = \pm\frac{1}{2}$, tum vero erit $\frac{dP}{dx} = \frac{dQ}{dy} = 1$, vtroque ergo caſu oritur minimum $V = -\frac{1}{8}$.

Diuidatur illa aequatio per $4xx - 1$, atque obtinebitur:

$$64x^6 - 80x^4 + 28xx - 3 = 0,$$

quae denuo bis continet $4xx - 1 = 0$; ita vt praecedens caſus oriatur. Praeterea vero inde fit $4xx - 3 = 0$, & $x = \frac{\pm\sqrt{3}}{2}$; cui reſpondet $y = \frac{\mp\sqrt{3}}{2}$. Erit igitur quoque $\frac{dP}{dx} = \frac{dQ}{dy} = 7$, ideoque fiet V minimum $= -\frac{9}{8}$; qui eſt valor omnium minimus, quos quidem functio V recipere poteſt: & hanc ob rem iſta aequatio $V = -\frac{9}{8} - cc$ ſemper eſt impoſſibilis. Hinc autem patet via determinandi maxima & minima functionum, quae tres pluresue variabiles inuoluunt.

CAPUT

CAPUT XII.

DE VSU DIFFERENTIALIUM IN INUESTIGANDIS RADICIBUS REALIBUS AEQUATIONUM.

294.

Natura maximorum ac minimorum viam nobis patefacit ad indolem radicum aequationum, vtrum ſint reales an imaginariae, cognoſcendam. Sit enim propoſita aequatio cuiuscunque ordinis:

$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \&c. = 0$

cuius radices ponamus eſſe p, q, r, s, t, &c. ita vt p ſit minima, q ea quae ratione magnitudinis ſequitur, ſicque & reliquae radices ſecundum ordinem quantitatis ſint diſpoſitae: ſcilicet ſit $q > p$; $r > q$; $s > r$; $t > s$ &c. Aſſumamus autem omnes radices aequationis eſſe reales, eritque exponens maximus n ſimul numerus radicum p, q, r, &c. Conſideremus quoque has radices omnes tanquam inter ſe inaequales; hinc tamen aequales radices non excluduntur, propterea quod radices inaequales, ſi earum differentia abeat in infinite paruam, fiant aequales.

295. Quoniam propoſita expreſſio $x^n - Ax^{n-1} + \&c.$ tum ſolum ſit nihilo aequalis, cum loco x aliquis valor ex p, q, r, &c. ſubſtituitur, reliquis vero caſibus omnibus non euaneſcit, ponamus generatim:

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \&c. = z$$

ita vt z ſpectari poſſit tanquam functio ipſius x. Fingamus nunc pro x ſucceſſiue ſubſtitui valores determitos, incipiendo a minimo $x = -\infty$, atque continuo maiores in locum ipſius x collocari; perſpicuumque eſt z nacturum hinc eſſe valores vel nihilo maiores vel nihilo minores, neque prius eſſe euaniturum, quam ponatur $x = p$; quo caſu fiet $z = 0$. Augeantur valores ipſius x vltra p, atque valores ipſius z vel affirmatiui vel negatiui fient, donec perueniatur ad valorem $x = q$; quo caſu iterum erit $z = 0$. Neceſſe ergo eſt, vt cum valores ipſius z ab 0 iterum ad 0 acceſſerint, interea z habuerit valorem vel maximum vel minimum; maximum ſcilicet ſi valores ipſius z, dum x intra limites p & q verſabatur, fuerint affirmatiui, minimum, ſi fuerint negatiui. Simili modo dum x vltra q ad r vsque augetur, functio z maximum vel minimum attinget: maximum nimirum ſi ante fuerit minimum, & contra. Supra enim vidimus maxima & minima ſe mutuo alternatim excipere.

296. Quare cum inter binas quasuis radices ipſius x exiſtat caſus, quo functio z fit maximum vel minimum; erit numerus maximorum & minimorum, quae in functione z implicantur, vnitate minor, quam numerus radicum realium; atque ita quidem alternatim ſe excipient, vt maximi ipſius z valores ſint affirmatiui, minimi negatiui. Quod ſi viciſſim functio z habeat maximum vel ſaltem valorem affirmatiuum caſu $x = f$, atque minimum ſeu ſaltem negatiuum

caſu

caſu $x = g$; quoniam dum valores ipſius x ab f ad g transeunt, functio z ab affirmatiuo abit in negatiuum neceſſe, eſt vt interea per o tranſierit, & hancobrem dabitur radix ipſius x intra limites f & g contenta. Niſi autem haec conditio adſit, vt valores maximi minimique ipſius z fiant alternatim affirmatiui & negatiui, illa concluſio non ſequitur. Si enim dentur functionis z minima, quae quoque ſint affirmatiua, fieri poteſt vt valor ipſius z a maximo ad ſequens minimum tranſeat, cum tamen interea non euaneſcat. Ceterum ex dictis intelligitur, etiamſi aequationis propoſitae non omnes radices fuerint reales, tamen ſemper inter binas quasque dari maximum vel minimum; etiamſi propoſitio conuerſa generatim non valeat, vt inter bina quaeuis maxima ſeu minima radix realis contineatur: valet autem adiecta conditione, ſi alter valor ipſius z fuerit affirmatiuus, alter negatiuus.

297. Quoniam ergo ſupra vidimus, valores ipſius x, quibus functio:

$$z = x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \&c.$$

fit maximum vel minimum, eſſe radices aequationis differentialis huius:

$$\frac{dz}{dx} = nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + \&c. = 0$$

manifeſtum eſt, ſi aequationis $z = 0$ omnes radices, quarum numerus eſt $= n$ fuerint reales, tum quoque omnes radices aequationis $\frac{dz}{dx} = 0$ fore reales. Cum

 enim

enim functio z tot habeat maxima vel minima, quot numerus $n-1$ continet vnitates, neceſſe eſt vt aequatio $\frac{dz}{dx}=0$ totidem habeat radices reales; ideoque omnes eius radices erunt reales. Ex quo ſimul perſpicitur, functionem z plura maxima minimaue habere non poſſe, quam $n-1$. Habemus ergo hanc regulam latiſſime potentem; ſi aequationis $z=0$ omnes radices fuerint reales, tum quoque aequatio $\frac{dz}{dx}=0$ omnes radices habebit reales. Vnde viciſſim ſequitur, ſi aequationis $\frac{dz}{dx}=0$ non omnes radices fuerint reales, tum quoque non omnes aequationis $z=0$ radices reales fore.

298. Quia inter binas quasuis aequationis $z=0$ radices reales datur vnus caſus, quo functio z fit maximum vel minimum; ſequitur ſi aequatio $z=0$ duas habeat radices reales, tum aequationem $\frac{dz}{dx}=0$ neceſſario vnam radicem habituram eſſe realem. Pariter ſi aequatio $z=0$ tres habeat radices reales, tum aequatio $\frac{dz}{dx}=0$ certo duas habebit radices reales. Atque generatim ſi aequatio $z=0$ habeat m radices reales, neceſſe eſt vt aequationis $\frac{dz}{dx}=0$ ad minimum ſint $m-1$ radices reales. Quare ſi aequatio $\frac{dz}{dx}=0$ pauciores habeat radices reales quam $m-1$, tum viciſſim aequatio $z=0$ certo pauciores quam m habebit radices reales. Cauendum autem

tem eſt, ne propoſitio conuerſa pro vera habeatur; etiamſi enim aequatio differentialis $\frac{dz}{dx} = 0$ aliquot vel adeo omnes radices ſuas habeat reales, tamen non ſequitur, aequationem $z = 0$ vllam habituram eſſe radicem realem. Fieri enim poteſt, vt aequationis $\frac{dz}{dx} = 0$ omnes radices ſint reales, cum tamen aequationis $z = 0$ omnes radices ſint imaginariae.

299. Interim tamen, ſi conditio ſupra memorata adiiciatur, propoſitio conuerſa ita proponi poterit, vt ex radicibus realibus aequationis $\frac{dz}{dx} = 0$, numerus radicum realium aequationis $z = 0$ certo cognoſci poſſit. Ponamus enim α, β, γ, δ, &c. eſſe radices reales aequationis $\frac{dz}{dx} = 0$, inter quas α ſit maxima, reliquae vero ordine magnitudinis ſe inuicem ſequantur. His igitur valoribus loco x ſubſtitutis functio z obtinebit vel maximos vel minimos valores alternatim. Cum autem functio z fiat $= \infty$, ſi ponatur $x = \infty$, patet eius valores continuo decreſcere debere, dum valores ipſius x ab ∞ vsque ad α diminuuntur; ex quo, caſu $x = \alpha$, fiet z minimum. Quodſi ergo hoc caſu $x = \alpha$ functio z valorem induat negatiuum, neceſſe eſt vt ante alicubi fuerit $= 0$, ſicque aequationis $z = 0$ radix dabitur realis $x > \alpha$; ſin autem poſito $x = \alpha$ functio z adhuc retineat valorem affirmatiuum, ante nusquam potuit eſſe

minor, alias enim quoque daretur minimum antequam x ad α vſque diminueretur, quod eſſet contra hypotheſin; hinc aequatio $z = o$ nullam habere poterit radicem realem maiorem quam α. Si ergo ponamus poſito $= \alpha$ fieri $z = \mathfrak{A}$, hoc modo iudicari poterit: ſi fuerit $\mathfrak{A}$ quantitas affirmatiua, tum aequatio $z = o$ nullam habebit radicem realem α maiorem; ſin autem $\mathfrak{A}$ fuerit quantitas negatiua, tum aequatio $z = o$ vnam perpetuo habebit radicem realem α maiorem, neque plures.

300. Ad hoc iudicium vlterius perſequendum

ſi ponatur	fiat
$x = \alpha$	$z = \mathfrak{A}$
$x = \beta$	$z = \mathfrak{B}$
$x = \gamma$	$z = \mathfrak{C}$
$x = \delta$	$z = \mathfrak{D}$
$x = \epsilon$	$z = \mathfrak{E}$
&c.	&c.

Quia ergo $\mathfrak{A}$ fuit minimum, erit $\mathfrak{B}$ maximum, & quidem ſi $\mathfrak{A}$ fuerit affirmatiuum, erit quoque $\mathfrak{B}$ affirmatiuum, neque ergo inter limites α & β dabitur radix realis aequationis $z = o$. Quare ſi haec aequatio nullam habeat radicem realem α maiorem, neque vllam habebit, quae eſſet maior quam β. Sin autem $\mathfrak{A}$ fuerit quantitas negatiua, quo caſu vna datur aequationis radix $x > \alpha$; diſpiciatur vtrum valor ipſius $\mathfrak{B}$ ſit affirmatiuus an negatiuus? priori caſu dabitur radix $x > \beta$, poſteriori vero nulla dabitur radix intra limites α & β contenta. Simili modo cum $\mathfrak{B}$ fuerit maximum, erit $\mathfrak{C}$ minimum; quare ſi $\mathfrak{B}$ habuerit valorem negatiuum, multo

to magis $\mathfrak{C}$ erit negatiuum, nullaque hoc caſu dabitur radix intra limites β & γ contenta. Ad ſi $\mathfrak{B}$ fuerit affirmatiuum, radix dabitur realis inter limites β & γ, ſi $\mathfrak{C}$ fiat negatiuum: ſin autem $\mathfrak{C}$ quoque ſit affirmatiuum, tum nulla dabitur radix inter limites β & γ contenta, ſimilique modo iudicium vlterius erit inſtituendum.

301. Quo haec iudicia facilius intelligantur, ea in ſequenti tabella complexus ſum:

Aequatio $z = 0$ vnam habebit radicem realem, quae continetur intra limites	Si fuerit
$x = \infty$ & $x = \alpha$	$\mathfrak{A} = -$
$x = \alpha$ & $x = \beta$	$\mathfrak{A} = -$ & $\mathfrak{B} = +$
$x = \beta$ & $x = \gamma$	$\mathfrak{B} = +$ & $\mathfrak{C} = -$
$x = \gamma$ & $x = \delta$	$\mathfrak{C} = -$ & $\mathfrak{D} = +$
$x = \delta$ & $x = \varepsilon$	$\mathfrak{D} = +$ & $\mathfrak{E} = -$
&c.	&c.

Harumque propoſitionum conuerſae & in negantes transmutatae pariter in omni rigore locum obtinent. Scilicet

Aequatio $z = 0$ nullam habebit radicem realem, quae contineatur inter limites:	ſi non fuerit
$x = \infty$ & $x = \alpha$	$\mathfrak{A} = -$
$x = \alpha$ & $x = \beta$	$\mathfrak{A} = -$ & $\mathfrak{B} = +$
$x = \beta$ & $x = \gamma$	$\mathfrak{B} = +$ & $\mathfrak{C} = -$
$x = \gamma$ & $x = \delta$	$\mathfrak{C} = -$ & $\mathfrak{D} = +$.
$x = \delta$ & $x = \varepsilon$	$\mathfrak{D} = +$ & $\mathfrak{E} = -$
&c.	&c.

Ope

Ope harum ergo regularum ex radicibus aequationis $\frac{dz}{dx} = 0$, ſi eae fuerint cognitae, non ſolum numerus radicum realium aequationis $z = 0$ colligitur, ſed etiam limites innoteſcunt, intra quos ſingulae iſtae radices contineantur.

EXEMPLUM.

Sit propoſita iſta aequatio: $x^4 - 14xx + 24x - 12 = 0$ *quae an habeat radices reales & quot quaeritur.*

Aequatio differentialis erit $4x^3 - 28x + 24 = 0$ ſeu $x^3 - 7x + 6 = 0$, cuius radices ſunt 1, 2, & -3, quae ſecundum ordinem magnitudinis diſpoſitae dabunt

	vnde erit
$\alpha = 2$	$\mathfrak{A} = -4$
$\beta = 1$	$\mathfrak{B} = -1$
$\gamma = -3$	$\mathfrak{C} = -129$

Ob $\mathfrak{A}$ negatiuum ergo aequatio propoſita habebit radicem realem > 2, at ob $\mathfrak{B}$ negatiuum, neque inter limites 2 & 1, neque inter limites 1 & -3 radicem habebit realem. Cum autem poſito $x = -3$ fiat $z = \mathfrak{C} = -129$, ac ſi ſtatuatur $x = -\infty$, fiat $z = +\infty$ neceſſe eſt, vt radix detur realis inter limites -3 & $-\infty$ contenta. Habebit ergo aequatio propoſita duas radices reales, alteram $x > 2$, alteram $x < -3$; ex quo duae radices erunt imaginariae. Simili modo ergo ex vltimo aequationis propoſitae maximo vel minimo iudicari debet,

bet, quo ex primo ſolo. Scilicet ſi aequatio propoſita fuerit ordinis paris, vltimum ſiue maximum ſiue minimum (erit autem hoc caſu minimum), ſi fuerit negatiuum radicem realem, ſin affirmatiuum radicem imaginariam indicat. At pro aequationibus imparium graduum, quia poſito $x = -\infty$ fit $z = -\infty$, ſi vltimum maximum fuerit affirmatiuum, radix realis, ſin negatiuum, imaginaria indicatur.

302. Regula ergo pro cognoſcendis radicibus realibus & imaginariis hoc modo commode exprimi poterit. Propoſita aequatione quacunque $z = 0$, conſideretur eius differentialis $\frac{dz}{dx} = 0$, cuius radices reales ſecundum ordinem quantitatis diſpoſitae ſint $\alpha, \beta, \gamma, \delta$, &c. tum poſito $x = \alpha, \beta, \gamma, \delta, \varepsilon, \zeta$, &c.
fiat $z = \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}, \mathfrak{F}$, &c.
Iam ſi ſigna ſint: $- \quad + \quad - \quad + \quad - \quad +$ &c.
tot aequatio $z = 0$ habebit radices reales, quot habentur litterae α, β, γ, &c. & inſuper vnam. Sin autem vna ex his literis maiuſculis non habeat ſignum infra ſcriptum, tum binae radices imaginariae indicabuntur. Ita ſi $\mathfrak{A}$ haberet ſignum $+$, tum nulla daretur radix intra limites ∞ & β contenta. Si $\mathfrak{B}$ habeat ſignum $-$, nulla dabitur radix inter limites α & γ; &, ſi $\mathfrak{C}$ habeat ſignum $+$, nulla erit radix inter limites β & δ, & ita porro. Generatim autem praeter radices imaginarias hoc modo indicatas, aequatio $z = 0$ inſuper tot habebit imaginarias, quot aequatio $\frac{dz}{dx} = 0$.

303. Si eueniat, vt valorum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. aliquis euaneſcat, tum eo loco aequatio $z = 0$ duas habebit radices aequales. Scilicet ſi fuerit $\mathfrak{A} = 0$, tum habebit duas radices ipſi α aequales; ſin ſit $\mathfrak{B} = 0$, duae erunt radices $= \beta$. Hoc enim caſu aequatio $z = 0$ vnam habebit radicem communem cum aequatione differentiali $\frac{dz}{dx} = 0$; ſupra autem demonſtrauimus, hoc eſſe indicium duarum radicum aequalium. Sin autem aequatio $\frac{dz}{dx} = 0$ duas pluresue radices habeat aequales, tum ſi earum numerus fuerit par, neque maximum neque minimum indicabitur: vnde pro praeſenti inſtituto radices aequales numero pares negligi poterunt. Sin autem numerus radicum aequalium aequationis $\frac{dz}{dx} = 0$ fuerit impar, tum omnes praeter vnam in formatione iudicii reiiciendae ſunt; niſi forte hoc caſu ipſa quoque functio z euaneſcat. Si enim hoc eueniat aequatio $z = 0$ quoque habebit radices aequales & quidem vna plures, quam aequatio $\frac{dz}{dx} = 0$. Sic ſi fuerit $\frac{dz}{dx} = (x - \zeta)^n R$, ita vt haec aequatio habeat n radices aequales ipſi ζ, ſi poſito $x = \zeta$ quoque euaneſcat z, tum aequatio $z = 0$ habebit $n + 1$ radices aequales ipſi ζ.

§ 245 & seqq.

304. Applicemus haec praecepta ad aequationes ſimpliciores, ac primo quidem a quadratica incipiamus. Sit igitur propoſita haec aequatio: $z = x^2 - Ax + B = o$: erit eius differentialis $\frac{dz}{dx} = 2x - A$, qua facta $= o$, erit $x = \frac{1}{2}A$, ſeu $\alpha = \frac{1}{2}A$. Subſtituatur hic valor loco x, fietque $z = -\frac{1}{4}AA + B = \mathfrak{A}$; vnde colligimus, ſi iſte valor ipſius $\mathfrak{A}$ fuerit negatiuus, hoc eſt ſi ſit $AA > 4B$, aequationem $xx - Ax + B = o$ habituram eſſe duas radices reales, alteram maiorem quam $\frac{1}{2}A$ alteram minorem. Sin autem valor ipſius $\mathfrak{A}$ fuerit affirmatiuus ſeu $AA < 4B$, tum ambae aequationis propoſitae radices erunt imaginariae. At ſi fuerit $\mathfrak{A} = o$ ſeu $AA = 4B$, tum aequatio propoſita habebit duas radices aequales, vtramque ſcilicet $= \frac{1}{2}A$. Quae cum ex natura aequationum quadraticarum ſint notiſſima, veritas horum principiorum non mediocriter illuſtratur, ſimulque eorum vtilitas in hoc negotio perſpicitur.

305. Progrediamur ergo ad aequationes cubicas ſimili modo inquirendas. Sit ergo propoſita aequatio $x^3 - Ax^2 + Bx - C = z = o$: cuius differentialis cum ſit $3xx - 2Ax + B = \frac{dz}{dx}$, ſi haec ponatur $= o$, fiet $xx = \frac{2Ax - B}{3}$, cuius aequationis vel ambae radices ſunt imaginariae, vel aequales, vel reales inaequales. Cum igitur hinc ſit $x = \frac{A \pm \sqrt{(A^2 - 3B)}}{3}$, ambae radi-

 ces

ces erunt imaginariae, ſi fuerit $AA < 3B$: hoc ergo caſu aequatio cubica propoſita vnicam habebit radicem realem, cuius alii limites non patent praeter $+\infty$ & $-\infty$. Sint iam ambae radices inter ſe aequales, ſeu $AA = 3B$, erit $x = \frac{A}{3}$. Niſi ergo ſimul fiat $z = 0$, hae duae radices pro nulla reputari debebunt, habebitque aequatio vt ante vnicam radicem realem; ſin autem caſu $x = \frac{A}{3}$ ſimul fiat $z = 0$, quod euenit, ſi fuerit $-\frac{2}{27}A^3 + \frac{1}{3}AB - C = 0$, ſeu $C = \frac{1}{3}AB - \frac{2}{27}A^3$; hoc eſt ſi fuerit $B = \frac{1}{3}A^2$ & $C = \frac{1}{27}A^3$, aequatio habebit tres radices aequales, ſingulas ſcilicet $= \frac{1}{3}A$. Euoluamus nunc tertium caſum, quo ambae radices aequationis differentialis ſunt reales & inter ſe inaequales, quod euenit ſi $AA > 3B$. Sit ergo $AA = 3B + ff$, ſeu $B = \frac{1}{3}AA - \frac{1}{3}ff$, erunt ambae illae radices $x = \frac{A \pm f}{3}$. Fiet ergo $\alpha = \frac{1}{3}A + \frac{1}{3}f$ & $\beta = \frac{1}{3}A - \frac{1}{3}f$. Quaerantur ergo valores ipſius z his reſpondentes $\mathfrak{A}$ & $\mathfrak{B}$; & cum ambae radices contineantur in haec aequatione $xx = \frac{2}{3}Ax - \frac{1}{3}B$, fiet $z = -\frac{1}{3}Axx + \frac{2}{3}Bx - C = -\frac{2}{9}AAx + \frac{1}{9}AB + \frac{2}{3}Bx - C$. Hinc itaque oritur:

$$\mathfrak{A} = -\tfrac{2}{27}A^3 + \tfrac{1}{3}AB - \tfrac{2}{27}A^2f + \tfrac{2}{9}Bf - C = \tfrac{1}{27}A^3 - \tfrac{1}{9}Aff - \tfrac{2}{27}f^3 - C$$
$$\mathfrak{B} = -\tfrac{2}{27}A^3 + \tfrac{1}{3}AB + \tfrac{2}{27}A^2f - \tfrac{2}{9}Bf - C = \tfrac{2}{27}A^3 - \tfrac{1}{9}Aff + \tfrac{2}{27}f^3 - C$$

ob $B = \frac{1}{3}AA - \frac{1}{3}ff$. Si igitur fuerit $\mathfrak{A}$ quantitas negatiua, quod euenit, ſi fuerit $C > \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3$, aequatio $z = 0$ vnam habebit radicem realem $> \alpha$, hoc

eſt

eſt maiorem quam $\frac{1}{3}$ A + $\frac{1}{3}f$. Ponamus ergo eſſe C $>\frac{1}{27}A^3-\frac{1}{9}Aff-\frac{2}{27}f^3$ ſeu eſſe $C=\frac{1}{27}A^3-\frac{1}{9}Aff-\frac{2}{27}f^3+gg$; atque, vt vidimus, aequatio propoſita cubica habebit radicem realem $>\frac{1}{3}A+\frac{1}{3}f$. Quales autem futurae ſint reliquae radices, ex valore $\mathfrak{B}$ intelligetur: erit autem $\mathfrak{B}=\frac{4}{27}f^3-gg$; qui ſi fuerit affirmatiuus, aequatio inſuper duas habebit radices reales, priorem intra limites α & β, hoc eſt intra $\frac{1}{3}A+\frac{1}{3}f$ & $\frac{1}{3}A-\frac{1}{3}f$ contentam, alteram vero minorem quam $\frac{1}{3}A-\frac{1}{3}f$. Sin autem fuerit $gg>\frac{4}{27}f^3$, ſeu $\mathfrak{B}$ negatiuum, tum aequatio habebit duas radices imaginarias. At ſi fuerit $\mathfrak{B}=0$ ſeu $\frac{4}{27}f^3=gg$ tum duae radices euadent aequales, vtraque $=\beta=\frac{1}{3}A-\frac{1}{3}f$. Denique ſi ſit valor ipſius $\mathfrak{A}$ affirmatiuus ſeu $C<\frac{1}{27}A^3-\frac{1}{9}Aff-\frac{2}{27}f^3$, tum aequatio duas habebit radices imaginarias, tertiaque erit realis & $<\frac{1}{3}A-\frac{1}{3}f$. Atque ſi ſit valor ipſius $\mathfrak{A}=0$, duae erunt radices aequales $=\alpha$, manente tertia $<\frac{1}{3}A-\frac{1}{3}f$.

306. Quo igitur aequationis cubicae $x^3-Ax^2+Bx-C=0$ omnes tres radices ſint reales, requiruntur tres conditiones. Primo vt ſit $B<\frac{1}{3}AA$: ſit ergo $B=\frac{1}{3}AA-\frac{1}{3}ff$. Secundo vt ſit $C>\frac{1}{27}A^3-\frac{1}{9}Aff-\frac{2}{27}f^3$. Tertio vt ſit $C<\frac{1}{27}A^3-\frac{1}{9}Aff+\frac{2}{27}f^3$. Quae duae poſteriores conditiones eo redeunt, vt C contineatur intra hos limites $\frac{1}{27}A^3-\frac{1}{9}Aff-\frac{2}{27}f^3$ & $\frac{1}{27}A^3-\frac{1}{9}Aff+\frac{2}{27}f^3$ ſeu intra hos limites $\frac{1}{27}(A+f)^2(A-2f)$ & $\frac{1}{27}(A-f)^2(A+2f)$. Quod ſi ergo harum conditionum vnica deſit, aequatio duas habebit radices imaginarias. Sic ſi fuerit $A=3$, $B=2$,

$B = 2$, erit $\frac{1}{3}ff = \frac{1}{3}AA - B = 1$ & $ff = 3$; vnde iſta aequatio: $x^3 - 3xx + 2x - C = 0$ omnes radices reales habere nequit, niſi C contineatur intra limites $-\frac{2\sqrt{3}}{9}$ & $+\frac{2\sqrt{3}}{9}$. Quare ſi fuerit vel $C < -\frac{2\sqrt{3}}{9}$ ſeu $C < -0,3849$, vel $C > +\frac{2\sqrt{3}}{9}$ ſeu $C > 0,3849$ aut coniunctim $CC > \frac{4}{27}$, aequatio vnicam habebit radicem realem.

307. Quoniam in omni aequatione ſecundus terminus tolli poteſt, ponamus eſſe $A = 0$, ita vt habeamus hanc aequationem cubicam $x^3 + Bx - C = 0$. Vt igitur huius aequationis omnes tres radices ſint reales, neceſſe eſt vt primo ſit $B < 0$, ſeu B debet eſſe quantitas negatiua. Sit ergo $B = -kk$, erit $ff = 3kk$, atque inſuper requiritur, vt quantitas C contineatur intra hos limites $-\frac{2}{27}f^3$ & $+\frac{2}{27}f^3$; hoc eſt inter hos $-\frac{2}{9}kk\sqrt{3kk}$ & $+\frac{2}{9}kk\sqrt{3kk}$. Erit ergo $CC < \frac{4}{27}k^6$ ſeu $CC < -\frac{4}{27}B^3$. Vnica ergo conditione natura aequationum cubicarum, quae omnes tres radices habeant reales comprehendi poterit, dum dicemus eſſe oportere $4B^3 + 27CC$ quantitatem negatiuam. Sic enim iam poſtulatur, vt ſit B quantitas negatiua, quia alioquin $4B^3 + 27CC$ negatiuum fieri non poſſet. Quocirca generatim affirmamus, aequationem $x^3 + Bx \pm C = 0$ omnes tres radices habituram eſſe reales, ſi fuerit $4B^3 + 27CC$ quantitas negatiua. Sin autem haec quantitas fuerit affirmatiua, tum vnicam fore realem, reliquas binas imaginarias; at ſi fiat

$4B^3 + 27CC = 0$, tum omnes quidem radices futuras eſſe reales, at binas inter ſe aequales.

308. Progrediamur ad aequationes biquadratas, in quibus etiam ſecundum terminum deeſſe ponamus. Sit ergo $x^4 + Bx^2 - Cx + D = 0$. Statuamus $x = \frac{1}{u}$, eritque $1 + Bu^2 - Cu^3 + Du^4 = 0$, cuius aequatio differentialis eſt $2Bu - 3Cu^2 + 4Du^3 = 0$, quae vnam habet radicem $u = 0$, tum vero erit $uu = \frac{6Cu - 4B}{8D}$ & $u = \frac{3C \pm \sqrt{(9CC - 32BD)}}{8D}$. Vt igitur omnes quatuor radices ſint reales, primo requiritur, vt ſit $9CC > 32BD$. Ponamus ergo eſſe $9CC = 32BD + 9ff$, erit $u = \frac{3C \pm 3f}{8D}$. Hic C ſemper pro quantitate affirmatiua ſumere poterimus, niſi enim talis fuerit, ponendo $u = -v$ talis euadet. Mox autem demonſtrabimus omnes radices reales eſſe non poſſe, niſi ſit B quantitas negatiua. Sit ergo $B = -gg$, eritque $9CC = 9ff - 32ggD$, & $u = \frac{3C \pm 3f}{8D}$. Atque duo caſus erunt perpendendi, prout D ſit quantitas affirmatiua vel negatiua.

I. Sit D quantitas affirmatiua, eritque $f > C$, ac tres ipſius u radices ſecundum quantitatis ordinem diſpoſitae erunt 1°; $u = \frac{3C + 3f}{8D}$, 2°; $u = 0$, 3°; $u = \frac{3C - 3f}{8D}$.

Aequa-

Aequatio autem $u^4 - \frac{C u^3}{D} + \frac{B u^2}{D} + \frac{1}{D} = 0$, his valoribus loco u ſubſtitutis dabit ſequentes tres valores.

$$\mathfrak{A} = \frac{27(C+f)^3 (C-3f)}{4096 D^4} + \frac{1}{D}$$

$$\mathfrak{B} = \frac{1}{D}$$

$$\mathfrak{C} = \frac{27(C-f)^3 (C+3f)}{4096 D^4} + \frac{1}{D}$$

quorum primus ac tertius debet eſſe negatiuus: vterque quidem ob C affirmatiuum & $C < f$ fit minor quam $\frac{1}{D}$. Oportet itaque eſſe $\frac{1}{D} < \frac{27(C+f)^3 (3f-C)}{4096 D^4}$ & $\frac{1}{D} < \frac{27(f-C)^3(C+3f)}{4096 D^4}$, ſeu $4096 D^3 < 27(f+C)^3 (3f-C)$ & $4096 D^3 < 27(f-C)^3 (C+3f)$. At prior quantitas ſemper longe maior eſt poſteriori; vnde ſufficit, ſi fuerit $D^3 < \frac{27}{4096}(f-C)^3 (C+3f)$, exiſtente $B = \frac{9CC - 9ff}{32 D}$ & $f > C$, atque $D > 0$. Si igitur fuerit D quantitas affirmatiua, C affirmatiua, B negatiua, vt ſit $f > C$, atque $D^3 < \frac{27}{4096}(f-C)^3 (C+3f)$, hoc eſt $D < \frac{3}{16}(f-C)\sqrt[3]{(3f+C)}$, tum aequatio

omnes

omnes radices habebit reales. Sin autem fuerit $D > \frac{3}{16}(f - C)\sqrt[3]{(3f + C)}$, attamen $D < \frac{3}{16}(f + C)\sqrt[3]{(3f - C)}$; tum duae radices erunt reales & duae imaginariae. At ſi adeo fuerit $D > \frac{3}{16}(f + C)\sqrt[3]{(3f - C)}$, tum omnes quatuor radices erunt imaginariae.

II. Sit D quantitas negatiua puta $= -F$, manente C affirmatiua ac B negatiua, ob $B = \frac{9CC - 9ff}{32D} = \frac{9ff - 9CC}{32F}$, erit $C > f$. Cum igitur ſit $u = \frac{3C \pm 3f}{8D} = -\frac{3C \mp 3f}{8F}$, tres valores ipſius u ſecundum ordinem magnitudinis diſpoſiti erunt 1°, $u = 9$; 2°, $u = -\frac{3C + 3f}{8F}$; 3°, $u = -\frac{3C - 3f}{8F}$, qui dabunt ſequentes valores

$$\mathfrak{A} = -\frac{1}{F}$$

$$\mathfrak{B} = \frac{27(C - f)^3(C + 3f)}{4096F^4} - \frac{1}{F}$$

$$\mathfrak{C} = \frac{27(C + f)^3(C - 3f)}{4096F^4} - \frac{1}{F}$$

Cum igitur $\mathfrak{A}$ ſit quantitas negatiua, aequatio iam certo vnam, ac propterea quoque duas habebit radices reales. Vt autem omnes radices ſint reales, oportet vt $\mathfrak{B}$ ſit

 quan-

quantitas affirmatiua, ideoque $27(C-f)^3(C+3f) > 4096F^3$: tum vero neceſſe eſt, vt ſit $\mathfrak{C}$ quantitas negatiua ſeu $27(C+f)^3(C-3f) < 4096F^3$. Quocirca vt omnes radices fiant reales, requiritur vt F^3 contineatur intra hos limites $\frac{27}{4096}(C+f)^3(C-3f)$ & $\frac{27}{4096}(C-f)^3(C+3f)$ ſeu vt F contineatur intra limites $\frac{3}{16}(C+f)\sqrt[3]{(C-3f)}$ & $\frac{1}{16}(C-f)\sqrt[3]{(C+3f)}$; & niſi F contineatur intra hos limites, duae radices erunt imaginariae.

III. Ponamus iam B eſſe quantitatem affirmatiuam, & D pariter affirmatiuam, ob $B = \frac{9CC-9ff}{32D}$, erit $C > f$, & cum ſit $u = \frac{3C \pm 3f}{8D}$, radices ordine magnitudinis diſpoſitae erunt 1°, $u = \frac{3(C+f)}{8D}$; 2°, $u = \frac{3(C-f)}{8D}$ & 3°; $u = 0$, vnde ſequentes oriuntur valores:

$$\mathfrak{A} = \frac{27(C+f)^3(C-3f)}{4096D^4} + \frac{1}{D}$$

$$\mathfrak{B} = \frac{27(C-f)^3(C+3f)}{4096D^4} + \frac{1}{D}$$

$$\mathfrak{C} = \frac{1}{D}$$

vbi cum $\mathfrak{C}$ ſit quantitas affirmatiua, certo duae radices erunt imaginariae. Sin autem fuerit $\mathfrak{A}$ negatiuum, quod

quod euenit, ſi $4096 D^3 < 27(C+f)^3(3f-C)$, duae radices erunt reales: ſin fuerit $4096 D^3 > 27(C+f)^3(3f-C)$, tum omnes quatuor radices erunt imaginariae.

IV. Maneat B affirmatiuum, ſit autem D negatiuum $= -F$, ob $B = \frac{9ff-9CC}{32F}$, erit $f > C$ & ob $u = -\frac{3C \mp 3f}{8F}$, tres ipſius u radices ſecundum ordinem magnitudinis diſpoſitae erunt 1°, $u = \frac{3(f-C)}{8F}$; 2°, $u = 0$; & 3°, $u = -\frac{3(C+f)}{8F}$, vnde iſti valores naſcuntur.

$$\mathfrak{A} = -\frac{27(f-C)^3(C+3f)}{4096 F^4} - \frac{1}{F}$$

$$\mathfrak{B} = - \frac{1}{F}$$

$$\mathfrak{C} = -\frac{27(C+f)^3(3f-C)}{4096 F^4} - \frac{1}{F}$$

vbi ob $\mathfrak{A}$ & $\mathfrak{C}$ negatiua aequatio certo duas habet radices reales, at ob $\mathfrak{B}$ negatiuum, duae radices erunt imaginariae.

309. Si igitur ponamus litteras B, C, D quantitates affirmatiuas denotare, ſequentes oriuntur caſus diverſi diiudicandi, qui ob $f = \sqrt{(CC - \frac{32}{9} BD)}$ huc redeunt.

I. Si aequatio ſit $x^4 - Bx^2 \pm Cx + D = 0$. Omnes radices erunt reales, ſi fuerit

$$D < \tfrac{3}{16}\left[\sqrt{(CC + \tfrac{32}{9} BD)} - C\right] \sqrt[3]{\left[3\sqrt{(CC + \tfrac{32}{9} BD)} + C\right]}$$

 Duae

Duae radices erunt reales, duaeque imaginariae, ſi fuerit

$$D > \tfrac{3}{16}\left[\sqrt{(CC+\tfrac{32}{9}BD)}-C\right]\sqrt[3]{\left[3\sqrt{(CC+\tfrac{32}{9}BD)}+C\right]}$$

at $D < \tfrac{3}{16}\left[\sqrt{(CC+\tfrac{32}{9}BD)}+C\right]\sqrt[3]{\left[3\sqrt{(CC+\tfrac{32}{9}CD)}-C\right]}$

Omnes autem radices erunt imaginariae, ſi fuerit

$$D > \tfrac{3}{16}\left[\sqrt{(CC+\tfrac{32}{9}BD)}+C\right]\sqrt[3]{\left[3\sqrt{(CC+\tfrac{32}{9}BD)}-C\right]}.$$

II. Si aequatio ſit $x^4 - Bx^2 \pm Cx - D = 0$. Duae radices ſemper ſunt reales, reliquae binae quoque erunt reales, ſi quantitas D contineatur intra hos limites.

$$D > \tfrac{3}{16}\left[\sqrt{(CC+\tfrac{32}{9}BD)}+C\right]\sqrt[3]{\left[C-3\sqrt{(CC-\tfrac{32}{9}BD)}\right]}$$

$$D < \tfrac{3}{16}\left[C-\sqrt{(CC-\tfrac{32}{9}BD)}\right]\sqrt[3]{\left[C+3\sqrt{(CC+\tfrac{32}{9}BD)}\right]}$$

niſi autem D contineatur intra hos limites, duae reliqua radices erunt imaginariae.

III. Si aequatio ſit $x^4 + Bx^2 \pm Cx + D = 0$. Duae radices ſemper erunt imaginariae. Reliquae vero duae erunt reales, ſi fuerit

$$D < \tfrac{3}{16}\left[\sqrt{(CC-\tfrac{32}{9}BD)}+C\right]\sqrt[3]{\left[3\sqrt{(CC-\tfrac{32}{9}BD)}-C\right]}$$

Reliquae vero duae quoque erunt imaginariae, ſi fuerit

$$D > \tfrac{3}{16}\left[\sqrt{(CC-\tfrac{32}{9}BD)}+C\right]\sqrt[3]{\left[3\sqrt{(CC-\tfrac{32}{9}BD)}-C\right]}$$

IV. Si aequatio ſit $x^4 + Bx^2 \pm Cx - D = 0$. Huius aequationis duae radices ſemper erunt reales, duae reliquae vero ſemper imaginariae.

EXEM-

EXEMPLUM I.

Si proponatur haec aequatio $x^4 - 2xx + 3x + 4 = 0$ *quaeratur natura radicum, vtrum ſint reales an imaginariae.*

Quia hoc exemplum ad caſum primum pertinet, eſt $B = +2$; $C = 3$ & $D = 4$; vnde $CC + \frac{32}{9}BD = 9 + \frac{32.8}{9} = \frac{337}{9}$ & $\sqrt{(CC + \frac{32}{9}BD)} = \frac{\sqrt{337}}{3}$ vnde conditiones vt omnes radices ſint reales, ſunt

$$4 < \frac{3}{16}\left(3+\frac{\sqrt{337}}{3}\right)\sqrt[3]{(\sqrt{337}-3)} = \frac{1}{16}(9+\sqrt{337})\sqrt[3]{(\sqrt{337}-3)}$$

$$4 < \frac{3}{16}\left(\frac{\sqrt{337}}{3}-3\right)\sqrt[3]{(\sqrt{337}+3)} = \frac{1}{16}(\sqrt{337}-9)\sqrt[3]{(\sqrt{337}+3)}$$

Adhibitis, approximationibus examinari debet ergo, vtrum ſit $4 < \frac{69}{16}$ & $4 < \frac{24}{16}$; quare cum prior tantum conditio locum habeat, aequatio habebit duas radices reales, & duas imaginarias.

EXEMPLUM II.

Propoſita ſit haec aequatio:

$$x^4 - 9xx + 12x - 4 = 0.$$

Quae cum pertineat ad caſum ſecundum, duas habebit radices reales. Ad reliquarum naturam inueſtigan-

dam, ob $B = 9$, $C = 12$ & $D = 4$, erit $\sqrt{(CC - \frac{32}{9}BD)}$ $= \sqrt{(144 - 32.4)} = 4$. Ideoque videndum eſt, vtrum ſit

$$4 > \frac{3}{16}.\ 16 \sqrt[3]{0}, \quad \text{hoc eſt } 4 > 0$$

$$\&\ 4 < \frac{3}{16}.\ 8 \sqrt[3]{24} \quad \text{hoc eſt } 4 < 3\sqrt[3]{3}$$

quorum vtrumque cum eueniat, aequatio propoſita quatuor habebit radices reales.

EXEMPLUM III.

Propoſita ſit haec aequatio:

$$x^4 + xx - 2x + 6 = 0.$$

Quae cum pertineat ad caſum tertium, duae radices certo erunt imaginariae. Tum vero eſt $B = 1$; $C = 2$ & $D = 6$, ideoque $\sqrt{(CC - \frac{32}{9}BD)} = \sqrt{\left(4 - \frac{64}{3}\right)}$, quae quantitas cum ſit imaginaria, & duae reliquae radices certo erunt imaginariae.

EXEMPLUM. IV.

Sit propoſita aequatio haec:

$$x^4 - 4x^3 + 8x^2 - 16x + 20 = 0.$$

Eliminetur primo ſecundus terminus, ſubſtituendo $x = y + 1$ fiet

$$\begin{array}{rcrrrrr}
x^4 & = & y^4 & + 4y^3 & + 6yy & + 4y & + 1 \\
-4x^3 & = & & - 4y^3 & - 12y^2 & - 12y & - 4 \\
+8x^2 & = & & & + 8y^2 & + 16y & + 8 \\
-16x & = & & & & - 16y & - 16 \\
+20 & = & & & & & + 20 \\
\hline
\text{Ergo} & & y^4 & & + 2yy & - 8y & + 9 = 0
\end{array}$$

quae

quae cum pertineat ad caſum tertium, duas radices habebit imaginarias. Tum vero ob $B=2$, $C=8$, $D=9$, erit $\sqrt{(CC - \frac{32}{9}. BD)} = \sqrt{(64 - 64)} = 0$. Comparetur ergo $D=9$ cum $\frac{3}{16}.8 \sqrt[3]{-8} = -3$. Cum ergo ſit $D=9 > -3$, etiam duae reliquae radices erunt imaginariae.

EXEMPLUM V.

Sit propoſita haec aequatio: $x^4 - 4x^3 - 7x^2 + 34x - 24 = 0$
cuius radices conſtat eſſe, 1, 2, 4. & − 3.

Quod ſi autem regulas applicemus, ſublato ſecundo termino ponendo $x = y + 1$ fiet: $y^3 - 13yy + 12y + 0 = 0$ quae cum caſu ſecundo comparata dat $B=13$, $C=12$, $D=0$. Debet ergo eſſe $D > \frac{3}{16}.24. \sqrt[3]{-24}$; ſeu $0 > -9\sqrt[3]{3}$ & $D < 0$; cum igitur D non ſit maius quam 0, aequatio quatuor radices reales habere indicatur. Si enim ſit $D = 0$, altera aequatio abit in $D < \frac{3}{16}\left(\frac{16\,BD}{9\,C}\right)\sqrt[3]{4\,C}$ ideoque $1 < \frac{B}{3C}\sqrt[3]{4\,C}$, ſeu $27\,CC < 4\,B^3$: eſt vero $27.\ 144 < 4.\ 13^3$ ſeu $36.\ 27 < 13^3$.

310. Opus foret maxime difficile, ſi ſimile iudicium ad aequationes altiorum graduum transferre vellemus, propterea quod aequationum differentialium radices plerumque exhiberi non poſſunt; quoties autem has radices aſſignare licet, ex traditis principiis facile colli-

colligitur, quot aequatio propoſita habeat radices reales & imaginarias. Hinc omnis aequationis, quae tantum ex tribus terminis conſtat, radices, vtrum ſint reales an imaginariae? definiri poterunt. Sit enim propoſita haec aequatio generalis:

$$x^{m+n} + Ax^n + B = 0 = z$$

Sumatur eius differentialis $\frac{dz}{dx} = (m+n)x^{m+n-1} + nAx^{n-1}$, qua nihilo aequali poſita, erit primo $x^{n-1} = 0$; vnde ſi n fuerit impar numerus, nulla radix maximum minimumue exhibens oritur: ſin autem ſit n numerus par, vna radix in computum ducenda erit $x = 0$. Tum vero erit $(m+n)x^m + nA = 0$; quae aequatio, ſi m ſit numerus par, & A affirmatiua quantitas, nullam habet radicem realem. Hinc ſequentes caſus erunt expendendi.

I. *Sit* m *numerus par &* n *numerus impar*, & radix $x = 0$ non valebit. Si igitur fuerit A quantitas affirmatiua, nulla prorſus habebitur radix maximum minimumue exhibens; vnde ob $m+n$ numerum imparem aequatio propoſita vnicam habebit radicem realem. Sin autem fuerit A quantitas negatiua; puta $A = -E$, erit $x = \pm \sqrt[m]{\frac{nE}{m+n}}$: vnde $\alpha = +\sqrt[m]{\frac{nE}{m+n}}$ & $\beta = -\sqrt[m]{\frac{nE}{m+n}}$.

Ex quibus valoribus fit:

$$\mathfrak{A} = (x^m - E)x^n + B = -\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m} + B$$

atque $\mathfrak{B} = +\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m} + B$. Si igitur

fuerit

fuerit igitur $\mathfrak{A}$ quantitas negatiua, ſeu $\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{m:m} > B$, aequatio vnam habebit radicem realem $> \alpha$. Si inſuper fuerit $B > -\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m}$, hoc eſt ambas conditiones in vnam complectendo, ſi fuerit $(m+n)^{m+n} B^m < m^m n^n E^{m+n}$, tum aequatio tres habebit radices reales: &, niſi haec conditio locum habeat, aequationis vnica radix erit realis. Valent haec de aequatione $x^{m+n} - Ex^n + B = 0$, ſi fuerit m numerus impar: vbi ſi E fuerit numerus negatiuus, aequatio ſemper vnicam radicem habebit realem.

II. Sint ambo numeri m & n impares, vt ſit $m+n$ numerus par, nullaque radix $x = 0$ in computum veniat. Quia eſt $(m+n)x^m + nA = 0$, erit $x = -\sqrt[m]{\frac{nA}{m+n}}$, quae vnica radix ſi ſit $= \alpha$, fiet $\mathfrak{A} = \frac{mA}{m+n}x^n + B = -\frac{mA}{m+n}\left(\frac{nA}{m+n}\right)^{n:m} + B$. Qui valor ſi fuerit negatiuus, aequatio propoſita duas habebit radices reales, contra nullam. Aequatio ergo propoſita $x^{m+n} + Ax^n + B = 0$ duas habebit radices reales, ſi fuerit $m^m n^n A^{m+n} > (m+n)^{m+n} B^m$; ſin fuerit $m^m n^n A^{m+n} < (m+n)^{m+n} B^m$, nulla prorſus radix erit realis.

III. Sint ambo numeri m & n pares, erit $m+n$ pariter numerus par: vnaque radix $x = 0$ maximum mi-

nimumue praebebit: quae erit vnica, ſi A fuerit quantitas affirmatiua, vnde facto $\alpha = 0$, erit $\mathfrak{A} = B$. Quare ſi fuerit B quoque quantitas affirmatiua, aequatio nullam habebit radicem realem; ſin autem B ſit quantitas negatiua, duae habebuntur radices reales, neque plures, ſi quidem A fuerit quantitas affirmatiua. At ponamus eſſe A quantitatem negatiuam ſeu $A = -E$, erit $x = \pm \sqrt[m]{\frac{nE}{m+n}}$; habebimusque tria maxima vel minima: nempe $\alpha = +\sqrt[m]{\frac{nE}{m+n}}$; $\beta = 0$; $\gamma = -\sqrt[m]{\frac{nE}{m+n}}$. Quibus ipſius $z = x^{m+n} - Ex^n + B = 0$ reſpondent valores $\mathfrak{A} = -\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m} + B$; $\mathfrak{B} = B$; $\mathfrak{C} = -\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m} + B$. Si igitur B ſit quantitas negatiua, ob $\mathfrak{A}$ & $\mathfrak{C}$ negatiuas, aequatio duas tantum habebit radices reales, propterea quod quoque $\mathfrak{B} = B$ ſit negatiuum. At ſi B fuerit quantitas affirmatiua, aequatio quatuor habebit radices reales, ſi ſit $(m+n)^{m+n} B^m < m^m n^n E^{m+n}$. Nullam autem habebit radicem realem, ſi fuerit $(m+n)^{m+n} B^m < m^m n^n E^{m+n}$.

IV. Sit m numerus impar & n numerus par: atque radix $x = 0$ dabit maximum vel minimum. Praeterea vero erit $x = -\sqrt[m]{\frac{nA}{m+n}}$. Si ergo A ſit numerus affir-

affirmatiuus, fiet $\alpha = 0$ & $\beta = -\sqrt[m]{\frac{nA}{m+n}}$; hincque $\mathfrak{A} = B$, & $\mathfrak{B} = \frac{mA}{m+n}\left(\frac{nA}{m+n}\right)^{n:m} + B$. Quare ſi ſit B quantitas negatiua, puta $B = -F$, atque inſuper fuerit $m^m n^n A^{m+n} > (m+n)^{m+n} F^m$, aequatio tres habebit radices reales; contra vnica tantum erit realis. Sin autem ſit A quantitas negatiua puta $A = -E$, fiet $x = +\sqrt[m]{\frac{nE}{m+n}}$ & $\alpha = \sqrt[m]{\frac{nE}{m+n}}$ & $\beta = 0$, quibus reſpondent $\mathfrak{A} = -\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m} + B$ & $\mathfrak{B} = B$. Quare aequatio tres habebit radices reales, ſi fuerit B quantitas affirmatiua, & $m^m n^n E^{m+n} > (m+n)^{m+n} B^m$, quae proprietas niſi locum inueniat, aequatio vnicam habebit radicem realem.

311. Sint omnes coefficientes $= 1$, atque denotantibus μ & ν numeros integros, aequationes ſequentes ita diiudicabuntur:

$x^{2\mu+2\nu-1} + x^{2\nu-1} \pm 1 = 0$ vnicam habebit radicem realem:

$x^{2\mu+2\nu-1} - x^{2\nu-1} \pm 1 = 0$, tres habebit radices reales, ſi fuerit

$(2\mu+2\nu-1)^{2\mu+2\nu-1} < (2\mu)^{2\mu}(2\nu-1)^{2\nu-1}$,

 quod

quod cum nunquam fieri poſſit, aequatio ſemper vnicam radicem realem habebit:

$x^{2\mu+2\nu} \pm x^{2\nu-1} - 1 = 0$ duas habet radices reales.

$x^{2\mu+2\nu} \pm x^{2\nu-1} + 1 = 0$ nullam habet radicem realem.

$x^{2\mu+2\nu} \pm x^{2\nu} + 1 = 0$ nullam habet radicem realem.

$x^{2\mu+2\nu} \pm x^{2\nu} - 1 = 0$ duas habet radices reales.

$x^{2\mu+2\nu+1} + x^{2\nu} \pm 1 = 0$ vnicam habet radicem realem.

$x^{2\mu+2\nu+1} - x^{2\nu} \pm 1 = 0$ vnicam habet radicem realem.

Ceterum quia in caſu tertio ambo exponentes ſunt pares, is ponendo $xx = y$ ad formam ſimpliciorem reduci poteſt, ideoque hic caſus praetermitti poſſet. Quo facto affirmari poterit, nullam aequationem tribus terminis conſtantem plures tribus habere poſſe radices reales.

EXEMPLUM.

Quaerantur caſus, quibus aequatio haec $x^5 \pm Ax^2 \pm B = 0$ *tres habeat radices reales.*

Quia haec aequatio pertinet ad caſum quartum, patet quantitates A & B, eſſe debere ſignis contrariis affectas. Quare niſi huiusmodi habeat formam, vnicam habebit radicem realem: ſin autem aequatio propoſita fuerit huiusmodi

modi $x^5 \pm Ax^2 \mp B = 0$, quo ea habeat tres radices reales, neceſſe eſt vt ſit $3^3 2^2 A^5 > 5^5 B^3$ ſeu $A^5 > \frac{3125}{108} B^3$. Quodſi ergo fuerit $B = 1$, oportet eſſe, $A^5 > \frac{3125}{108}$, ſeu $A > 1,960132$. Si ergo ſit $A = 2$ iſta aequatio $x^5 - 2x^2 + 1 = 0$ tres habet radices reales, quarum cum vna ſit $x = 1$; ſequitur hanc aequationem biquadratam $x^4 + x^3 + x^2 - x - 1 = 0$, duas habere radices reales. Quod quidem tum ex his datis praeceptis intelligi poteſt, tum ex iis, quae in libro ſuperiori ſunt demonſtrata, manifeſtum eſt, vbi oſtendimus, quamuis aequationem paris gradus, cuius terminus abſolutus ſit numerus negatiuus, habere ſemper duas radices reales.

312. Ex his principiis quoque aequationes, quae conſtant quatuor terminis, diiudicari poterunt, dummodo aequationis differentialis radices commode exhiberi queant, quod euenit, ſi exponentes ipſius x vel in tribus anterioribus, vel in tribus poſterioribus terminis ſint in arithmetica progreſſione. Cum autem haec diiudicatio in genere ſuſcepta ad plures perducatur caſus, eam in nonnullis exemplis abſoluamus.

EXEMPLUM I.

Sit propoſita haec aequatio $x^7 - 2x^5 + x^3 - a = 0$.

Facto $z = x^7 - 2x^5 + x^3 - a$, erit $\frac{dz}{dx} = 7x^6 - 10x^4 + 3xx$, quo valore nihilo aequali poſito fiet primo $xx = 0$,

 qui

qui duplex valor pro nullo reputandus. Tum vero erit $7x^4 = 10x^2 - 3$, vnde ſit $x^2 = \frac{5 \pm 2}{7}$; & quatuor valores pro x emergent, qui ſecundum magnitudinem ordinati, ſequentes pro z praebebunt valores:

$\alpha = 1$	$\mathfrak{A} = -a$
$\beta = +\sqrt{\tfrac{3}{7}}$	$\mathfrak{B} = \frac{48}{343}\sqrt{\tfrac{3}{7}} - a$
$\gamma = -\sqrt{\tfrac{3}{7}}$	$\mathfrak{C} = \frac{-48}{343}\sqrt{\tfrac{3}{7}} - a$
$\delta = -1$	$\mathfrak{D} = -a.$

Si ergo ſit a numerus affirmatiuus, erit vel $a > \frac{48}{343}\sqrt{\tfrac{3}{7}}$, vel $a < \frac{48}{343}\sqrt{\tfrac{3}{7}}$, priori caſu ob $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, omnes negatiuas, aequatio propoſita vnicam habebit radicem realem $x > 1$. Poſteriori caſu ſi $a < \frac{48}{343}\sqrt{\tfrac{3}{7}}$ aequatio tres habebit radices reales, primam > 1, ſecundam contentam inter limites 1 & $\sqrt{\tfrac{3}{7}}$, & tertiam intra limites $+\sqrt{\tfrac{3}{7}}$ & $-\sqrt{\tfrac{3}{7}}$.

Sin a ſit quantitas negatiua ponendo $x = -y$, aequatio perducetur ad formam priorem. Quo ergo aequatio propoſita tres habeat radices reales, neceſſe eſt vt ſit $a < 0{,}0916134$ vel $a < \tfrac{1}{11}$.

EXEM-

EXEMPLUM II.

Sit propoſita haec aequatio:

$$ax^8 - 3x^6 + 10x^3 - 12 = 0.$$

Quia hic exponentes trium poſteriorum terminorum ſunt in arithmetica progreſſione, ponatur $x = \frac{1}{y}$ atque aequatio transmutabitur in hanc:

$a - 3y^2 + 10y^5 - 12y^8 = 0$, ponatur ergo $z = 12y^8 - 10y^5 + 3y^2 - a = 0$, eritque differentiando $\frac{dz}{dy} = 96y^7 - 50y^4 + 6y = 0$, ex qua aequatione primo fit $y = 0$; tum vero erit $y^6 = \frac{50y^3 - 6}{96}$ & $y^3 = \frac{25 \pm 7}{96}$, ideoque vel $y = \sqrt[3]{\frac{1}{3}}$ vel $y = \sqrt[3]{\frac{3}{16}}$. His ergo tribus radicibus ſecundum magnitudinem dispoſitis, reſpondentes ipſius z valores ita ſe habebunt:

$\alpha = \sqrt[3]{\frac{1}{3}}$	$\mathfrak{A} = \sqrt[3]{\frac{1}{9}} - a$
$\beta = \sqrt[3]{\frac{3}{16}}$	$\mathfrak{B} = \frac{99}{64}\sqrt[3]{\frac{9}{256}} - a = \frac{99}{256}\sqrt[3]{\frac{9}{4}} - a$
$\gamma = 0$	$\mathfrak{C} = -a.$

Quodſi ergo fuerit $a > \sqrt[3]{\frac{1}{9}}$, aequatio propoſita duas habebit radices reales, alteram $> \sqrt[3]{\frac{1}{9}}$, alteram < 0:

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at praeter has inſuper habebit duas radices reales, ſi ſimul fuerit $\mathfrak{B}$ quantitas affirmatiua, hoc eſt, ſi fuerit $a < \frac{99}{256}\sqrt[3]{\frac{9}{4}}$. Quamobrem aequatio propoſita quatuor habebit radices reales, ſi quantitas a contineatur intra limites $\sqrt[3]{\frac{1}{9}}$ & $\frac{99}{256}\sqrt[3]{\frac{9}{4}}$; qui limites proxime ſunt: 0,48075 & 0,50674. Poſito ergo $a = \frac{1}{2}$, haec aequatio $x^8 - 6x^6 + 20x^3 - 24 = 0$ quatuor habet radices reales intra limites ∞; $\sqrt[3]{\frac{16}{3}}$; $\sqrt[3]{3}$; 0; $-\infty$; ergo tres erunt affirmatiuae & vna negatiua.

CAPUT

CAPUT XIII.

DE CRITERIIS RADICUM IMAGINARIARUM.

313.

In capite praecedenti modum exhibuimus naturam radicum cuiusque aequationis explorandi, ita vt eius beneficio, ſi proponatur aequatio quaecunque, inueniri poſſit, quot ea radices habeat reales, & quot imaginarias. Plerumque quidem haec inueſtigatio difficillime inſtituitur, cum aequatio differentialis ita eſt comparata, vt eius radices exhiberi nequeant. Quanquam autem his caſibus eadem operatio ad aequationem differentialem ipſam accommodari, eiusque radicum natura ex ipſius differentiali indagari, hincque illius radices proxime aſſignari poſſent; tamen labor nimium ſaepiſſime fieret moleſtus. Quamobrem in hoc negotio ſaepenumero ſufficit eiusmodi criteria noſſe, ex quorum praeſentia tuto concludi poſſit, ineſſe in aequatione propoſita radices imaginarias; etiamſi ex eorum abſentia viciſſim inferri nequeat, omnes prorſus radices eſſe reales. Quae cognitio etſi eſt imperfecta, tamen frequenter vſu non deſtituitur: quocirca his criteriis explicandis praeſens caput deſtinauimus.

314. In capite igitur praecedenti vidimus, ſi aequatio quaecunque:

$z = x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \&c. = 0$
omnes radices habeat reales, tum etiam eius differentialem
$\frac{dz}{dx} = nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + \&c. = 0$
omnes ſuas radices habituram eſſe reales. Simul vero oſtendimus, etiamſi aequatio differentialis omnes habeat radices reales; tamen inde non ſequi, ipſius aequationis propoſitae omnes radices futuras eſſe reales. Interim tamen, ſi aequatio differentialis habeat radices imaginarias, tum ſemper recte concludimus, aequationem ipſam propoſitam ad minimum totidem habere debere radices imaginarias. Ad minimum dico: fieri enim poteſt, vt ipſa aequatio plures habeat radices imaginarias. Hoc ergo modo ex aequatione differentiali plus concludi non poteſt, quam, ſi ea habeat radices imaginarias, ipſam propoſitam aequationem eiusmodi radices quoque habere debere, & quidem ad minimum totidem.

315. Si aequatio propoſita multiplicetur per poteſtatem quamcunque x^m, denotante m numerum integrum affirmatiuum; tum quia haec noua aequatio omnes radices habebit reales, ſi quidem propoſitae radices omnes fuerint reales: tum quoque eius differentialis, poſtquam per x^{m-1} fuerit diuiſa, radices erunt reales omnes. Hinc ſi haec aequatio:

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \&c. = 0$$

omnes radices habeat reales, tum quoque iſta aequatio

$$(m+n)x^n - (m+n-1)Ax^{n-1} + (m+n-2)Bx^{n-2} - \&c. = 0$$

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omnes radices habebit reales. Ob eandem rationem, ſi haec multiplicetur per x^k & denuo differentietur, aequatio reſultans:

$$(m+n)(k+n)x^n-(m+n-1)(k+n-1)Ax^{n-1}+(m+n-2)(k+n-2)Bx^{n-2}-\&c.=0$$

omnes adhuc radices habebit reales: ſicque quousque libuerit, vlterius progredi licet. Sin autem huiusmodi aequatio radices imaginarias habere deprehendatur, tum ſimul certum erit, ipſam aequationem propoſitam ſaltem totidem radices imaginarias eſſe habituram.

316. Si aequatio propoſita, antequam differentiatur, per nullam poteſtatem ipſius x multiplicetur, tum iudicium ad aequationem vno gradu inferiorem deducitur. Ita ſi aequatio propoſita

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \&c. = 0$$

omnes radices habeat reales, tum quoque eius differentiales omnium ordinum omnes radices habebunt reales. Quare & ſequentium aequationum omnium radices erunt reales:

$$nx^{n-1}-(n-1)Ax^{n-2}+(n-2)Bx^{n-3}-(n-3)Cx^{n-4}+\&c.=0$$

$$n(n-1)x^{n-2}-(n-1)(n-2)Ax^{n-3}+(n-2)(n-3)Bx^{n-4}-\&c.=0$$

$$n(n-1)(n-2)x^{n-3}-(n-1)(n-2)(n-3)Ax^{n-4}+\&c.=0$$

$$n(n-1)(n-2)(n-3)x^{n-4}-(n-1)(n-2)(n-3)(n-4)Ax^{n-5}+\&c.=0$$

&c.

quae aequationes ad ſequentes formas reuocantur:

$$x^{n-1} - \frac{(n-1)}{n}Ax^{n-2} + \frac{(n-1)(n-2)}{n(n-1)}Bx^{n-3} - \frac{(n-1)(n-2)(n-3)}{n(n-1)(n-2)}Cx^{n-4} + \&c. = 0$$

$$x^{n-2} - \frac{(n-2)}{n}Ax^{n-3} + \frac{(n-2)(n-3)}{n(n-1)}Bx^{n-4} - \frac{(n-2)(n-3)(n-4)}{n(n-1)(n-2)}Cx^{n-5} + \&c. = 0$$

$$x^{n-3} - \frac{(n-3)}{n}Ax^{n-4} + \frac{(n-3)(n-4)}{n(n-1)}Bx^{n-5} - \frac{(n-3)(n-4)(n-5)}{n(n-1)(n-2)}Cx^{n-6} + \&c. = 0$$

$$x^{n-4} - \frac{(n-4)}{n}Ax^{n-5} + \frac{(n-4)(n-5)}{n(n-1)}Bx^{n-6} - \frac{(n-4)(n-5)(n-6)}{n(n-1)(n-2)}Cx^{n-7} + \&c. = 0$$

&c.

317. Hoc igitur modo iudicium ad aequationem dati gradus inferioris, quam eſt ipſa propoſita, reduci poteſt. Sic ſi m fuerit numerus quicunque minor quam n, tum ſi aequatio propoſita omnes radices habeat reales, tum quoque huius aequationis gradus m omnes radices erunt reales:

$$x^{m} - \frac{m}{n}Ax^{m-1} + \frac{m(m-1)}{n(n-1)}Bx^{m-2} - \frac{m(m-1)(m-2)}{n(n-1)(n-2)}Cx^{m-3} + \&c. = 0.$$

Quare ſi ponatur $m = 2$, prodibit iſta aequatio:

$$x^2 - \frac{2}{n}Ax + \frac{2.\,1}{n(n-1)}B = 0,$$

cuius radices debebunt eſſe reales, ſi quidem aequatio propoſita $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \&c. = 0$, omnes habeat radices reales. Cum autem iſta aequatio quadratica radices reales habere nequeat, niſi ſit $\frac{AA}{nn} > \frac{2.\,1}{n(n-1)}B$, ſequitur, aequationis propoſitae radi-

ces

ces omnes reales eſſe non poſſe, niſi ſit $AA > \frac{2n}{n-1}B$. Quamobrem ſi fuerit $AA < \frac{2n}{n-1}B$, hoc certum erit ſignum, aequationis propoſitae ad minimum duas radices fore imaginarias.

318. Hinc ergo aſſecuti ſumus affectionem neceſſariam, qua coefficientes trium primorum terminorum affecti eſſe debent, ſi quidem aequationis propoſitae omnes radices fuerint reales. Hocque eſt eiusmodi criterium, vti initio meminimus: ſcilicet etiamſi caſu $AA > \frac{2n}{n-1}B$, nihil pro realitate radicum ſequatur, at ſi ſit $AA < \frac{2n}{n-1}B$, hoc tamen certum ſit ſignum duarum ſaltem radicum imaginariarum. Sic vt omnes radices ſint reales, ſucceſſiue pro n numero 2, 3, 4, 5, &c. ſubſtituendo requiritur, vt ſequitur:

$x^2 - Ax + B = 0 \quad \ldots\ldots\ldots \quad A^2 > 4B$

$x^3 - Ax^2 + Bx - C = 0 \quad \ldots\ldots \quad A^2 > \frac{6}{2}B$

$x^4 - Ax^3 + Bx^2 - Cx + D = 0 \quad \ldots \quad A^2 > \frac{8}{3}B$

$x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0 \quad .. \quad A^2 > \frac{10}{4}B$

Hinc ſi terminus ſecundus deſit, tertiique coefficiens B ſit affirmatiuus, vt aequatio ſit huiusmodi:

$$x^n + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \&c. = 0,$$

haec omnes radices reales habere nequit, ſed ad minimum duae erunt imaginariae.

319. Huiusmodi vero criteria pro coefficientibus ſequentium terminorum erui poſſunt, ſi perpendamus aequationem hanc:

$$1 - Ay + By^2 - Cy^3 + Dy^4 - \&c. = 0$$

totidem habere radices tam reales quam imaginarias, quot ipſa aequatio propoſita contineat. Haec enim aequatio ex illa oritur, ſi ponatur $x = \frac{1}{y}$, ita vt ex radicibus huius aequationis ſimul radices illius habeantur. Quare ſi aequatio propoſita omnes radices habeat reales, tum quoque reciprocae iſtius differentialis, ſcilicet huius $-A + 2By - 3Cy^2 + 4Dy^3 - \&c. = 0$ radices omnes erunt reales. Subſtituatur in hac iterum x pro $\frac{1}{y}$, atque emerget iſta aequatio:

$$Ax^{n-1} - 2Bx^{n-2} + 3Cx^{n-3} - 4Dx^{n-4} + \&c. = 0,$$

cuius radices propterea omnes erunt reales, ſi radices aequationis propoſitae fuerint tales. Hinc iam patet, ſi fuerit $n = 3$, neceſſe eſſe vt ſit $BB > 3AC$.

320. Differentietur autem iſta aequatio vlterius, atque prodibunt:

$$Ax^{n-2} - \frac{2(n-2)}{n-1}Bx^{n-3} + \frac{3(n-2)(n-3)}{(n-1)(n-2)}Cx^{n-4} - \&c. = 0$$

$$Ax^{n-3} - \frac{2(n-3)}{n-1}Bx^{n-4} + \frac{3(n-3)(n-4)}{(n-1)(n-2)}Cx^{n-5} - \&c. = 0$$

$$Ax^{n-4} - \frac{2(n-4)}{n-1}Bx^{n-5} + \frac{3(n-4)(n-5)}{(n-1)(n-2)}Cx^{n-6} - \&c. = 0$$

&c.

Gene-

Generaliter ergo, ſi m ſit numerus minor quam n, erit:

$$Ax^m - \frac{2m}{n-1}Bx^{m-1} + \frac{3m(m-1)}{(n-1)(n-2)}Cx^{m-2} - \&c. = 0.$$

Si iam ponatur $m = 2$, habebitur iſta aequatio:

$$Ax^2 - \frac{4}{n-1}Bx + \frac{6}{(n-1)(n-2)}C = 0,$$

cuius radices vt ſint reales, oportet eſſe $\frac{4BB}{(n-1)^2} > \frac{6AC}{(n-1)(n-2)}$. Quare ſi aequatio propoſita omnes habeat radices reales erit $BB > \frac{3(n-1)}{2(n-2)}AC$. Atque ſi fuerit $BB < \frac{3(n-1)}{2(n-2)}AC$, hoc certum eſt ſignum, aequationem propoſitam ad minimum duas habere radices imaginarias. Si igitur ſit $n = 3$, criterium erit $BB > 3AC$; ſi ſit $n = 4$; erit $BB > \frac{3.3}{2.2}AC$; ſi $n = 5$, erit $BB > \frac{3.4}{2.3}AC$, & ita porro.

321. Vt haec criteria ad ſequentes coefficientes transferamus, reſumamus aequationem differentialem in y inuentam:

$$-A + 2By - 3Cy^2 + 4Dy^3 - 5Ey^4 + \&c. = 0$$

hancque denuo differentiemus, vt habeamus:

$$2B - 6Cy + 12Dy^2 - 20Ey^3 + \&c. = 0$$

quae reſtituto $\frac{1}{x}$ loco y dabit:

$$Bx^{n-2} - 3Cx^{n-3} + 6Dx^{n-4} - 10Ex^{n-5} + \&c. = 0$$

ex cuius vlterioris differentiatione ſequuntur hae aequationes:

Bx^{n-3}

$$Bx^{n-3} - \frac{3(n-3)}{n-2} Cx^{n-n} + \frac{6(n-3)(n-4)}{(n-2)(n-3)} Dx^{n-5} - \&c. = 0$$

& generaliter

$$Bx^m - \frac{3m}{n-2} Cx^{m-1} + \frac{6m(m-1)}{(n-2)(n-3)} Dx^{m-2} - \&c. = 0$$

Quodſi igitur ponamus $m = 2$ prodibit aequatio quadrata:

$$Bx^2 - \frac{2.\ 3}{n-2} Cx + \frac{6.\ 2}{(n-2)(n-3)} D = 0$$

cuius radices erunt reales, ſi fuerit $\frac{9CC}{(n-2)^2} > \frac{6.2\,BD}{(n-2)(n-3)}$ ſeu $CC > \frac{4(n-2)}{3(n-3)} BD$. Quare ſi aequatio propoſita omnes radices habeat reales, erit $CC > \frac{4(n-2)}{3(n-3)} BD$, atque ſi haec conditio deficiat, aequatio certo duas ad minimum habebit radices imaginarias.

322. Si aequationem ſuperiorem $2B - 6Cy + 12Dy^2 - \&c. = 0$ denuo differentiemus, prodibit:

$$-6C + 24Dy - 60Ey^2 + \&c. = 0, \quad \text{ſiue}$$

$$C - 4Dy + 10Ey^2 - 20Fy^3 + \&c. = 0,$$

quae reſtituto x loco $\frac{1}{y}$ abibit in hanc:

$$Cx^{n-3} - 4Dx^{n-4} + 10Ex^{n-5} - 20Fx^{n-6} + \&c. = 0$$

ex cuius vlteriori differentiatione ſequuntur:

$$Cx^{n-4} - \frac{4(n-4)}{(n-3)} Dx^{n-5} + \frac{10(n-4)(n-5)}{(n-3)(n-4)} Ex^{n-6} - \&c. = 0$$

Cx^{n-5}

$$Cx^{n-5} - \frac{4(n-5)Dx^{n-6}}{n-3} + \frac{10(n-5)(n-6)}{(n-3)(n-4)}Ex^{n-7} - \&c.$$

& generaliter

$$Cx^m - \frac{4mD}{n-3}x^{m-1} + \frac{10m(m-1)}{(n-3)(n-4)}Ex^{m-2} - \&c. = 0.$$

Ponamus $m = 2$, eritque $Cx^2 - \frac{2.\,4}{n-3}Dx + \frac{2.\,10}{(n-3)(n-4)}E = 0$ ex qua ſi eius radices ſint reales ſequitur fore:

$$\frac{4.\,4}{(n-3)^2}DD > \frac{2.\,10}{(n-3)(n-4)}CE \text{ ſeu } DD > \frac{5(n-3)}{4(n-4)}CE.$$

323. Ex his iam ſatis perſpicitur relatio omnium coefficientium. Generatim ergo ſi aequatio haec:

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - Ex^{n-5} + \&c. = 0$$

omnes radices habeat reales; erit

$$AA > \frac{2n}{1(n-1)}B$$

$$BB > \frac{3(n-1)}{2(n-2)}AC$$

$$CC > \frac{4(n-2)}{3(n-3)}BD$$

$$DD > \frac{5(n-3)}{4(n-4)}CE$$

$$EE > \frac{6(n-4)}{5(n-5)}DF$$

&c.

 Quarum

Quarum conditionum ſi vna deſit, aequatio ad minimum duas habebit radices imaginarias. Atque ſi iſta criteria a ſe inuicem non pendeant, facile perſpicitur, quotquot eorum non conueniant, totidem dari paria radicum imaginariarum. Quamuis autem hae conditiones omnes in quapiam aequatione locum habeant, tamen inde non ſequitur, nullas dari radices imaginarias; quin potius euenire poteſt, vt hoc non obſtante omnes radices ſint imaginariae. Cauendum ergo eſt, ne his criteriis plus tribuatur, quam ipſis vi principiorum, vnde ſunt deducta, tribui poteſt.

324. Facile autem apparet non ſingula criteria, quae deficiunt, binas radices imaginarias indicare poſſe; in aequatione enim n dimenſionum, quia habentur $n+1$ termini, atque ex ſingulis praeter primum & vltimum criterium deſumi poteſt, omnino criteria habebuntur $n-1$; neque tamen ſi ſingula deficiant, aequatio $2n-2$ radices imaginarias habere poterit, propterea quod omnino tantum n habeat radices. Vnum autem criterium ſemper duas radices imaginarias patefacit, & quia fieri poteſt, vt duo criteria huiusmodi radicum non plures oſtendant, videndum eſt vtrum haec duo criteria ſint contigua nec ne: priori caſu numerus radicum imaginariarum non augebitur, poſteriori vero, quia criteria litteras prorſus diuerſas inuoluunt, vnum quodque binas radices imaginarias monſtrabit. Ita etiamſi fuerit $AA < \frac{2n}{1(n-1)}B$ & $BB < \frac{3(n-1)}{2(n-2)}AB$, tamen

men hinc non neceſſario quatuor radices imaginariae indicantur, ſed vtrumque fortaſſe easdem binas indicat. Quodſi vero fuerit $AA < \frac{2n}{1(n-1)}B$ & $CC < \frac{4(n-2)}{3(n-3)}BD$ exiſtente $BB > \frac{3(n-1)}{2(n-2)}AC$, quatuor radices imaginariae indicabuntur.

325. Ex criteriis ergo radicum imaginariarum ſe immediate inſequentibus plus non ſequitur, quam ex vno; ſin autem ea ordine interrupto procedant, vt inter bina quaeque criterium vnum vel plura contraria interiaceant, tum ex vnoquoque binae radices imaginariae concludi poterunt. Quae conſideratio ſequentem regulam ſuppeditat. Aequationis propoſitae ſingulis terminis, praeter primum & vltimum, inſcribantur coefficientes criteriorum ante inuenti, hoc modo:

$$\begin{array}{cccccc} & \frac{2n}{1(n-1)} & \frac{3(n-1)}{2(n-2)} & \frac{4(n-2)}{3(n-3)} & \frac{5(n-3)}{4(n-4)} & \&c. \\ x^n - & Ax^{n-1} + & Bx^{n-2} - & Cx^{n-3} + & Dx^{n-4} - & \&c. = 0 \\ + & .. & .. & .. & .. & \&c. \end{array}$$

Tum examinetur quadratum cuiusque coefficientis, vtrum ſit maius an minus, quam fractio inſcripta per productum adiacentium coefficientium multiplicata, priori caſu termino ſubſcribatur ſignum +, poſteriori ſinum —; primo vero termino & vltimo perpetuo ſignum + ſubſcribatur. Quo facto, quot ſignorum

 horum

horum ſubſcriptorum variationes occurrunt, totidem radices imaginarias aequatio ad minimum habere cenſenda erit.

326. Haec eſt regula a *Neutono* inuenta ad radices imaginarias cuiusque aequationis explorandas; de qua autem probe tenendum eſt, quod iam annotauimus, ſaepenumero fieri poſſe, vt aequatio plures habeat radices imaginarias, quam hac methodo deteguntur. Hinc alii operam dederunt, vt ſimiles regulas alias inuenirent, quae numerum radicum imaginariarum exactius praeberent, ita vt verus iſtiusmodi radicum numerus minus ſaepe eum, quem regula oſtendat, excederet. In hoc genere imprimis proſtat regula *Campbelli* Arithmeticae *Neutoni* vniuerſali ſubiuncta, quam propterea hic explicari conueniet, etiamſi non ſit perfecta. Nititur autem hoc lemmate: Si fuerint α, β, γ, δ, ϵ, &c. quantitates, earumque numerus ſit m, ponatur ſumma harum quantitatum $\alpha + \beta + \gamma + \delta +$ &c. $=$ S, ſumma quadratorum $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 +$ &c. $=$ V, erit vtique V $>$ o. Sed cum ſit productum ex binis $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta +$ &c. $= \frac{SS - V}{2}$; erit $(m - 1)V > SS - V$ ſeu $mV > SS$. Nam ſi differentiarum inter binas quantitates quadrata ſumantur, erit eorum ſumma

$$= (\alpha-\beta)^2 + (\alpha-\gamma)^2 + (\alpha-\delta)^2 + (\beta-\gamma)^2 + (\beta-\delta)^2 + \&c.$$

$$= (m-1)(\alpha^2+\beta^2+\gamma^2+\delta^2+\&c.) - 2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\&c.)$$

$$= (m-1)V - 2\,\frac{(SS-V)}{2} = mV - SS.$$ Cum

igitur

igitur ſumma quadratorum realium ſit ſemper affirmatiua, erit $mV - SS > 0$ ideoque $mV > SS$.

327. Hoc lemmate praemiſſo ſi habeatur haec aequatio:

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - Ex^{n-5} + Fx^{n-6} - \&c. = 0,$$

eiusque omnes radices fuerint reales numero n, quae ſint a, b, c, d, e, &c. erit vti conſtat ex natura aequationum:

aequationum:	numerus termin.
$A = a + b + c + d + \&c.$	n
$B = ab + ac + ad + bc + bd + \&c.$	$\frac{n(n-1)}{1.\,2}$
$C = abc + abd + abe + acd + bcd + \&c.$	$\frac{n(n-1)(n-2)}{1.\,2.\,3}$
$D = abcd + abce + abde + \&c.$	$\frac{n(n-1)(n-2)(n-3)}{1.\,2.\,3.\,4}$
&c.	

Sumantur iam ſingulorum harum ſerierum terminorum quadrata, ac ponatur:

$$P = a^2 + b^2 + c^2 + d^2 + \&c.$$

$$Q = a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + \&c.$$

$$R = a^2b^2c^2 + a^2b^2d^2 + a^2b^2e^2 + a^2c^2d^2 + \&c.$$

$$S = a^2b^2c^2d^2 + a^2b^2c^2e^2 + a^2b^2d^2e^2 + \&c.$$

&c.

 erit

erit ex natura combinationum:

$$P = A^2 - 2B$$
$$Q = B^2 - 2AC + 2D$$
$$R = C^2 - 2BD + 2AE - 2F$$
$$S = D^2 - 2CE + 2BF - 2AG + 2H \quad \&c.$$

328. Vi igitur lemmatis praemiſſi habebimus:

$$n\, P > AA$$
$$\frac{n(n-1)}{1.\ 2}\, Q > BB$$
$$\frac{n(n-1)(n-2)}{1.\ 2.\ 3}\, R > CC$$
$$\frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4}\, S > DD \quad \&c.$$

Quodſi ergo loco P, Q, R, &c. valores ante inuenti ſubſtituantur, obtinebimus ſequentes radicum realium proprietates:

$$nAA - 2nB > AA \quad \text{ſeu} \quad AA > \frac{2n}{n-1}B$$

$$\frac{n(n-1)}{1.\ 2}BB - \frac{2n(n-1)}{1.\ 2}AC + \frac{2n(n-1)}{1.\ 2}D > BB,$$

ſiue

$$BB > \frac{\frac{2n(n-1)}{1.\ 2.}}{\frac{n(n-1)}{1.\ 2.} - 1}(AC - D)$$

ſimi-

ſimilique modo aequationes ſequentes praebent:

$$CC > \frac{\frac{2n(n-1)(n-2)}{1.\;2.\;3.}}{\frac{n(n-1)(n-2)}{1.\;2.\;3.}-1}(BD-AE+F)$$

$$DD > \frac{\frac{2n(n-1)(n-2)(n-3)}{1.\;2.\;3.\;4.}}{\frac{n(n-1)(n-2)(n-3)}{1.\;2.\;3.\;4.}-1}(CE-BF+AG-H)$$

Hinc ergo cuiusque coefficientis quadratum non ſolum cum producto proxime adiacentium comparatur, ſed etiam cum rectangulis binorum quorumque vtrinque aeque diſtantium; ita tamen vt horum rectangulorum ſigna alternatim mutentur.

329. Singulis igitur aequationis terminis praeter primum & vltimum inſcribi debent fractiones, quarum numeratores ſint vnciae binomii ad ſimilem dignitatem eleuati duplicatae, denominatores vero eaedem vnciae vnitate minutae. Ita conſiderando aequationes quadratas, cubicas, biquadratas &c. ſi earum radices omnes fuerint reales, erit:

$$x^2 - \overset{\frac{4}{1}}{A}x + B = 0 \quad ; \quad A^2 > 4B$$

Pro aequatione cubica:

$$x^3 - \overset{\frac{6}{2}}{A}x^2 + \overset{\frac{6}{2}}{B}x - C = 0$$

erit $A^2 > 3B$ & $B^2 > 3AC$.

Pro

Pro aequatione biquadrata:

$$x^4 - \overset{\frac{8}{3}}{A}x^3 + \overset{\frac{12}{5}}{B}x^2 - \overset{\frac{8}{3}}{C}x + D = 0$$

erit $A^2 > \frac{8}{3}B$; $B^2 > \frac{12}{5}(AC - D)$; $C^2 > \frac{8}{3}BD$

Pro aequatione poteſtatis quintae:

$$x^5 - \overset{\frac{10}{4}}{A}x^4 + \overset{\frac{20}{9}}{B}x^3 - \overset{\frac{20}{9}}{C}x^2 + \overset{\frac{10}{4}}{D}x - E = 0$$

erit $AA > \frac{10}{4}B$; $B^2 > \frac{20}{9}(AC-D)$; $C^2 > \frac{20}{9}(BD-AE)$
& $D^2 > \frac{10}{9}CE$.

Pro aequatione poteſtatis ſextae:

$$x^6 - \overset{\frac{12}{5}}{A}x^5 + \overset{\frac{30}{14}}{B}x^4 - \overset{\frac{40}{19}}{C}x^3 + \overset{\frac{30}{14}}{D}x^2 - \overset{\frac{12}{5}}{E}x + F = 0$$

erit $A^2 > \frac{12}{5}B$; $B^2 > \frac{30}{14}(AC-D)$; $C^2 > \frac{40}{19}(BD-AE+F)$;
$D^2 > \frac{30}{14}(CE-BF)$; $E^2 > \frac{12}{5}DF$. &c.

330. Si igitur quodpiam criterium fallat, id erit indicium duas ad minimum ineſſe radices imaginarias in aequatione propoſita. Cum autem ſi ſingula fallant, aequatio ideo non duplo plures habere queat radices imaginarias, ſimili modo iudicium his caſibus erit abſolvendum, quem ante pro Neutoniana regula indicauimus. Scilicet ſi cuiusque termini quadratum maius fuerit quam fractio inſcripta per producta terminorum adiacentium & vtrinque aequidiſtantium multiplicata, tum iſti termino ſubſcribatur ſignum $+$, contra vero ſignum $-$; primo vero & vltimo termino conſtanter ſubſcribatur ſignum $+$. Quo facto inſpiciatur ordo ſignorum

rum horum ſubſcriptorum, & quoties occurrit variatio, toties radix imaginaria indicabitur. Quoties ergo haec regula plures radices imaginarias indicat, quam Neutoniana, toties quoque ad veritatém magis accedit. Interim tamen fieri poteſt, vt aequatio plures habeat radices imaginarias, quam per vtramque regulam indicantur.

331. Falleremur ergo, ſi his criteriis tanquam perfectis ſignis radicum realium & imaginarium vti vellemus; propterea quod fieri poteſt, vt aequatio plures habeat radices imaginarias, quam haec criteria indicant: error autem eo maior eſſe poſſet, quo altioris gradus fuerit aequatio propoſita. Nam in aequatione quadrata haec criteria ita veritati ſunt conſentanea, vt ſi nullas radices imaginarias indicent, etiam aequatio nullas ſit habitura. Aequatio autem cubica duas radices imaginarias habere poteſt, etiamſi neutra regula, (ambae autem hoc caſu adhuc conueniunt) eas exhibeat. Hos igitur caſus inueſtigaturi, ſit propoſita haec aequatio cubica generalis:

$$x^3 - \overset{3}{A}x^2 + \overset{3}{B}x - C = 0$$

in qua ſi fuerit $AA > 3B$ & $BB > 3AC$ neutra regula radices imaginarias indicat. Supra autem (306) vidimus ad id, vt nullae radices imaginariae adſint requiri primo vt ſit $B < \frac{1}{3}AA$, quam conditionem quoque ambae regulae requirunt. Sit igitur $B = \frac{1}{3}AA - \frac{1}{3}ff$, atque neceſſe eſt vt C contineatur intra hos limites: $\frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3$ & $\frac{1}{27}A^3 - \frac{1}{9}Aff + \frac{2}{27}f^3$.

 Vtra-

Vtraque autem regula tantum poſtulat, vt ſit $C < \frac{BB}{3A}$, hoc eſt $C < \frac{1}{27}A^3 - \frac{2}{27}Aff + \frac{f^4}{27A}$. Quae conditio locum habere poteſt, etiamſi C non intra dictos limites contineatur.

332. Sit enim $C = \frac{1}{27}A^3 - \frac{2}{27}Aff + \frac{f^4}{27A} - gg$: atque regulae nullas radices imaginarias indicabunt. Interim tamen in erunt duae radices imaginariae, ſi fuerit vel

$$\frac{1}{27}A^3 - \frac{2}{27}Aff + \frac{f^4}{27A} - gg < \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3$$

vel

$$\frac{1}{27}A^3 - \frac{2}{27}Aff + \frac{f^4}{27A} - gg > \frac{1}{27}A^3 - \frac{1}{9}Aff + \frac{2}{27}f^3.$$

Si igitur fuerit vel $gg > \frac{(ff + Af)^2}{27A}$ vel $gg < \frac{(Af - ff)^2}{27A}$, aequatio cubica duas habebit radices imaginarias, etiamſi neutra regula eas indicet. Sumimus autem hic eſſe A quantitatem affirmatiuam, ſi enim eſſet negatiua, ponendo $x = -y$ aequatio in eiusmodi formam transmutaretur, in qua A eſſet affirmatiua. Hinc infinitae aequationes cubicae formari poſſunt, quae habeant duas radices imaginarias, etiamſi per regulam non indicentur. Sit enim $gg = \frac{(ff + Af)^2}{27A} + hh$, erit $C = \frac{(ff - AA)^2}{27A} - gg$ $= \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3 - hh$, & $B = \frac{1}{3}AA - \frac{1}{3}ff$. Vel ſit $gg = \frac{(Af - ff)^2}{27A} - hh$ exiſtente $hh < \frac{(Af - ff)^2}{27A}$; erit

C =

$C = \frac{1}{27}A^3 - \frac{1}{9}Aff + \frac{2}{27}f^3 + hh$ & $B = \frac{1}{3}AA - \frac{1}{3}ff$. Vtroque caſu prodibit aequatio duas habens radices imaginarias, neutra regula indicandas. Ponamus verbi gratia $A = 4$, $f = 1$, erit $B = 5$; & ob $gg = \frac{25}{108} + hh$; erit $C = \frac{225}{108} - \frac{25}{108} - hh = \frac{50}{27} - hh$. Quare ſi ſit $C < \frac{50}{27}$, aequatio $x^3 - 4x^2 + 5x - C = 0$ ſemper habebit duas radices imaginarias. At ſumto $gg = \frac{1}{12} - hh$ debebit eſſe $hh < \frac{1}{12}$, fietque $C = \frac{25}{12} - \frac{1}{12} + hh = 2 + hh$. Sit $hh = \frac{1}{16}$; atque aequatio $x^3 - 4xx + 5x - \frac{33}{16} = 0$ duas habebit radices imaginarias, etiamſi nulla regulis prodatur.

333. Quin etiam eiusmodi aequationes generales formari poſſunt, in quibus neutra regula radices imaginarias exhibeat, etiamſi tamen ſaepiſſime duae pluresue inſint. Euenit hoc ſi perpetuo duo ſigna ſimilia ſe mutuo excipiant, vti:

$x^n - Ax^{n-1} - Bx^{n-2} + Cx^{n-3} + Dx^{n-4} - Ex^{n-5} - Fx^{n-6} + \&c. = 0$

vel $x^n + Ax^{n-1} - Bx^{n-2} - Cx^{n-3} + Dx^{n-4} + Ex^{n-5} - \&c. = 0$,

hic vtraque regula nullam vnquam radicem imaginariam prodit. Quod autem ſaepiſſime huiusmodi radices continere queant, vel ex aequatione cubica elucet $x^3 - Ax^2 - Bx + C = 0$, quae poſito $ff = AA + 3B$ ſemper habet duas radices imaginarias, ſi fuerit vel $-C < \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3$ vel $-C > \frac{1}{27}A^3 - \frac{1}{9}Aff + \frac{2}{27}f^3$. Interim tamen & hos caſus ex regulis elicere licet, ſi aequatio ope ſubſtitutionis in aliam formam transformetur. Ponatur $x = y + k$, fietque:

$$\left.\begin{array}{llll} y^3 + 3ky^2 & + 3kky & + k^3 \\ - Ayy & - 2Aky & - Akk \\ & - By & - Bk \\ & & + C \end{array}\right\} = 0$$

quae ſecundum regulas examinata dabit primo quidem ſponte $(3k - A)^2 > 3(3kk - 2Ak - B)$; at quo ſit $(3kk - 2Ak - B)^2 > 3(3k - A)(k^3 - Akk - Bk + C)$, quod eſt alterum criterium, neceſſe eſt vt ſit: $BB + 3AC + (AB - 9C)k + (AA + 3B)kk > 0$, quicunque valor ipſi k tribuatur. Sumatur ergo k ita, vt haec expreſſio minimum valorem adipiſcatur, quod fiet ponendo $k = \frac{9C - AB}{2(AA + 3B)}$, & ſi iſta expreſſio adhuc fuerit > 0, probabile erit aequationem propoſitam nullas habere radices imaginarias. Fiet autem $BB + 3AC - \frac{(AB - 9C)^2}{2(AA + 3B)} + \frac{(AB - 9C)^2}{4(AA + 3B)} > 0$ ſeu $BB + 3AC > \frac{(AB - 9C)^2}{4(AA + 3B)}$. Cum ergo ſit $B = \frac{1}{3}ff - \frac{1}{3}AA$, erit $4ff(\frac{1}{9}f^4 - \frac{2}{9}AAff + \frac{1}{9}A^4 + 3AC) > (\frac{1}{3}Aff - \frac{1}{3}A^3 - 9C)^2$ ſeu

$$4f^6 - A8^2f^4 + 4A^4ff + 108ACff > A^2f^4 - 2A^4f^2 - 54ACff + A^6 + 54A^3C + 729CC$$

vel $4f^6 > 9A^2f^4 - 6A^4ff - 162ACff + A^6 + 54A^3C + 729CC$

vnde factoribus ſumtis eſſe debebit:

$$(2f^3 + A^3 - 3Af + 27C)(2f^3 - A^3 + 3Af - 27C) > 0.$$

Hincque regulae radices imaginarias oſtendent, ſi fuerit vel $C > -\frac{1}{27}A^3 + \frac{1}{9}Af - \frac{2}{27}f^3$ & $C > -\frac{1}{27}A^3 + \frac{1}{9}Af + \frac{2}{27}f^3$ vel $C < -\frac{1}{27}A^3 + \frac{1}{9}Af - \frac{2}{27}f^3$ & $C < -\frac{1}{27}A^2 + \frac{1}{9}Af + \frac{2}{27}f^3$.

Quae

Quae ſunt eaedem conditiones, quas ſupra inuenimus. Patet ergo idonea aequationis propoſitae transmutatione regulas hoc capite traditas ita perfici poſſe, vt a veritate non diffideant, etiamſi conuertantur.

334. Ex his principiis quoque regula Harriotti, qua quaelibet aequatio tot radices affirmatiuas habere praedicatur, quot dentur ſignorum variationes, tot vero negatiuas, quot dentur eiusdem ſigni ſucceſſiones, demonſtrari poteſt, quae quidem regula pro radicibus tantum realibus valet. Ponamus ergo aequationem $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \&c. = 0$ omnes radices habere reales atque affirmatiuas, atque eius differentialis $nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - \&c. = 0$ non ſolum omnes ſuas radices quoque habebit reales & affirmatiuas, ſed etiam huius radices conſtituent limites radicum illius aequationis. Praeterea vero poſito $x = \frac{1}{y}$ haec aequatio $1 - Ay + By^2 - Cy^3 + Dy^4 - \&c. = 0$ omnes quoque radices habebit reales affirmatiuas, ſed reciprocas illius, ita vt quae radices in illa aequatione ſint maximae, hae in iſta fiant minimae. His poſitis ſi illa aequatio propoſita continuo differentietur, donec ad aequationem primi ordinis perueniatur, quae erit $x - \frac{1}{n}A = 0$, (317) huius radix adhuc erit affirmatiua, ideoque coefficiens ſecundi termini habebit ſignum $-$ vti aſſumſimus. Sin autem iſte coefficiens haberet ſignum $+$, tum certo ſequeretur, aequationem

propoſitam non omnes radices habere affirmatiuas, ſed vnam ad minimum fore negatiuam, & quidem eam, quae limitibus hucusque perductis reſpondeat.

335. Si aequatio propoſita in ſui reciprocam convertatur & differentietur, tum vero iterum x reſtituatur, atque differentiationes continuentur, donec perueniatur ad aequationem ſimplicem, quae ex §. 320. erit huiusmodi $Ax - \frac{2}{n-1}B = 0$, cuius propterea radix quoque debet eſſe affirmatiua, ſi quidem propoſita omnes ſuas radices habeat reales affirmatiuas, hincque ſecundus & tertius terminus diuerſa ſigna habebunt. Quodſi ergo hi duo termini ſimilia habeant ſigna, ad minimum vna radix negatiua indicabitur, reſpondens limiti hac aequatione ſignato, qui diuerſus erit a limite praecedente aequatione indicato, propterea quod hic radices ſemel ſunt in ſuas reciprocas conuerſae: vnde concluditur, ſi tres termini aequationis initiales paria habuerint ſigna, tum duas radices negatiuas indicari.

336. Simili modo ſi conuerſiones & differentiationes ſecundum §. 321. inſtituantur, atque eousque continuentur, donec ad aequationem ſimplicem $Bx - \frac{3}{n-2}C = 0$ perueniatur, & huius aequationis radix eſſe debet affirmatiua, ſi quidem propoſitae aequationis omnes radices fuerint tales; vnde ſi termini tertius & quartus paria habeant ſigna, indicabitur vna radix negatiua. Sicque perpetuo, ſi duo quicunque termini contigui aequalibus

ſignis

ſignis fuerint affecti, vna radix negatiua proditur; ideoque quotcunque fuerint eiusdem ſigni ſucceſſiones, totidem ad minimum aequatio propoſita habebit radices negatiuas, quoniam haec ſingula criteria ad diuerſos limites referuntur. Quod ſi autem aequatio propoſita omnes radices negatiuas habere ponatur, tum quia radices omnium aequationum differentialium ex ea deductarum debent eſſe pariter negatiuae, omnes termini aequalibus ſignis affecti eſſe debebunt. Quare ſi duo termini contigui diuerſa habeant ſigna, ex iis vna ad minimum radix affirmatiua concludetur. Atque ſimili modo, quotcunque in aequatione occurrant binorum terminorum variationes ſignorum, totidem ad minimum radices affirmatiuae ineſſe dicendae ſunt. Cum igitur aequatio omnis tot habeat radices, quot dantur duorum ſignorum contiguorum combinationes, neque plures, ſequitur quamuis aequationem, cuius omnes radices ſint reales, tot habere radices affirmatiuas, quot fuerint ſignorum contiguorum variationes, tot vero negatiuas, quot fuerint eiusdem ſigni ſucceſſiones.

CAPUT

CAPUT XIV.

DE DIFFERENTIALIBUS FUNCTIONUM IN CERTIS TANTUM CASIBUS.

337.

Si y fuerit functio quaecunque ipſius x, atque haec quantitas variabilis x augeatur incremento ω, vt x abeat in $x + \omega$, tum functio y induet hunc valorem:

$$y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} + \&c.$$

ideoque capiet hoc incrementum:

$$\frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} + \&c.$$

vti ſupra demonſtrauimus. Quare ſi fiat $\omega = dx$, ita vt x ſuo differentiali dx creſcat, tum functio y incrementum accipiet $= dy + \frac{1}{2} ddy + \frac{1}{6} d^3y + \frac{1}{24} d^4y + \&c.$ quod erit verum differentiale ipſius y. Quoniam vero huius ſeriei quilibet terminus ad ſequentes habet rationem infinitam, prae primo omnes euaneſcunt, ita vt dy more conſueto ſumtum praebeat verum differentiale ipſius y. Simili modo vera differentialia ſecunda, tertia, quarta, &c. ipſius y ita ſe habebunt:

$$dd.y = ddy + \frac{3}{3}d^3y + \frac{7}{3.4}d^4y + \frac{15}{3.4.5}d^5y + \frac{31}{3.4.5.6}d^6y + \&c.$$

$$d^3.y = d^3y + \frac{6}{4}d^4y + \frac{25}{4.5}d^5y + \frac{90}{4.5.6}d^6y + \frac{301}{4.5.6.7}d^7y + \&c.$$

$$d^4.y = d^4y + \frac{10}{5}d^5y + \frac{65}{5.6}d^6y + \frac{350}{5.6.7}d^7y + \&c.$$

$$d^5.y = d^5y + \frac{15}{6}d^6y + \frac{140}{6.7}d^7y + \&c.$$

$$d^6.y = d^6y + \frac{21}{7}d^7y + \&c.$$

quae ſequuntur ex §. 56. ſi loco ω ponatur dx. Erunt ergo haec differentialia ipſius y completa, quippe in quibus ne ii quidem termini, qui reſpectu primi euaneſcunt, negliguntur. Inueniuntur autem ſinguli iſti termini, ſi functio y continuo differentietur, ponendo dx conſtans. Sic poſito $y = ax - xx$ ob $dy = adx - 2xdx$ & $ddy = -2dx^2$; erunt ipſius y differentialia completa: $dy = adx - 2xdx - dx^2$; $ddy = -2dx^2$; ſequentia autem ſunt nulla.

338. Quanquam autem generatim in his expreſſionibus differentialium ſequentes termini prae primis pro nihilo reputantur; tamen in caſibus ſpecialibus, quibus ipſe terminus primus euaneſcit, haec ratio ceſſat, neque terminus ſecundus amplius negligi poterit. Sic in exemplo praecedente etiamſi formulae $y = ax - xx$ differentiale in genere eſt $= (a - 2x)dx$ reiecto termino $-dx^2$, quippe qui eſt infinities minor quam pri-

 mus

mus $(a-2x)dx$: hic tamen iſta conditio manifeſto ſubintelligitur, niſi primus terminus per ſe euaneſcat. Quocirca ſi ipſius $y = ax - xx$ quaeratur differentiale, caſu quo $x = \frac{1}{2}a$, tum id dicendum erit eſſe $= -dx^2$; ſcilicet ſi variabilis x differentiali dx creſcat, tum functionis y caſu $x = \frac{1}{2}a$ decrementum erit dx^2. Hoc autem ſolo caſu excepto perpetuo functionis y differentiale erit $= (a - 2x)dx$; niſi enim ſit $x = \frac{1}{2}a$, terminus ſecundus $-dx^2$ prae primo ſemper recte negligitur. Neque vero neglectio termini dx^2 etiam in caſu $x = \frac{1}{2}a$ in errorem inducere poteſt: comparari enim differentialia prima inter ſe ſolent; vnde quia $dy = -dx^2$ caſu $x = \frac{1}{2}a$, prae differentialibus primis dx euaneſcit, perinde eſt ſiue hoc caſu habeamus $dy = 0$ ſiue $dy = -dx^2$.

339. Denotante y functionem quamcunque ipſius x, ſit differentialibus continuis ſumtis:

$$dy = pdx;\ dp = qdx;\ dq = rdx;\ dr = sdx;\ \&c.$$

Hinc ergo differentialia completa, in quibus nihil negligatur, ipſius y erunt:

$$d.y = pdx + \tfrac{1}{2}qdx^2 + \tfrac{1}{6}rdx^3 + \tfrac{1}{24}sdx^4 + \&c.$$
$$d^2.y = qdx^2 + rdx^3 + \tfrac{7}{12}sdx^4 + \tfrac{1}{4}tdx^5 + \&c.$$
$$d^3.y = rdx^3 + \tfrac{3}{2}sdx^4 + \tfrac{5}{4}tdx^5 + \&c.$$
$$d^4.y = sdx^4 + 2tdx^5 + \&c.$$
$$d^5.y = tdx^5 + \&c.$$

Niſi ergo primi termini harum expreſſionum euaneſcant, ii ſoli differentialia ipſius y exhibebunt; ſin autem quopiam

piam caſu primus terminus fiat $= 0$, tum ſequens differentiale quaeſitum exprimet. Atque ſi etiam ſecundus terminus euaneſcat, tum tertius terminus valorem differentialis quaeſiti praebebit, ſin autem & hic euaneſcat, quartus & ita deinceps. Vnde intelligitur nullius functionis ipſius x differentiale primum vnquam penitus euaneſcere; etiamſi enim fiat $p = 0$, quo caſu vulgo dy euaneſcere cenſetur, tum hoc differentiale per altiorem ipſius dx poteſtatem exprimetur. Vti vel per $\frac{1}{2} q dx^2$, vel ſi etiam ſit $q = 0$, per $\frac{1}{6} r dx^3$, & ita porro.

340. Quanquam autem his caſibus differentiale ipſius y reſpectu aliorum differentialium primorum, quibuscum comparatur, recte negligitur, atque pro nihilo reputatur; tamen ſaepenumero eius veram expreſſionem noſſe iuuat. Ex completa enim differentialis forma ſtatim perſpici poteſt, quibus caſibus data functio fiat maximum vel minimum. Si enim fuerit:

$$d.y = p dx + \frac{1}{2} q dx^2 + \frac{1}{6} r dx^3 + \&c.$$

quo y nanciſcatur maximum minimumue valorem, neceſſe eſt vt ſit $p = 0$; erit ergo hoc caſu $dy = \frac{1}{2} q dx^2$, & functio y, ſi loco x ponatur $x \pm dx$, abit in $y + \frac{1}{2} q dx^2$, eritque propterea minima, ſi q habeat valorem affirmatiuum, at maxima ſi q habeat valorem negatiuum. At ſi ſimul fiat $q = 0$, erit $dy = \frac{1}{6} r dx^3$, & functio y ponendo $x \pm dx$ loco x abibit in $y \pm \frac{1}{6} r dx^3$, neque hoc caſu maximum neque minimum prodit; ſin autem fiat & $r = 0$, tum poſito $x \pm dx$ loco x functio y euadet $= y + \frac{1}{24} s dx^4$, quae maximum exhibet, ſi s fuerit quan-

 titas

titas negatiua, minimum vero, ſi s ſit quantitas affirmatiua. Aliae occaſiones, quibus differentialium completa expreſſio vſum habet, infra occurrent.

341. Ponamus p euaneſcere caſu $x = a$, quod euenit ſi fuerit $p = (x-a)P$. Talis autem valor prodit, ſi fuerit $y = (x-a)^2 P + C$, denotante C quantitatem conſtantem quamcunque. Cum enim ſit $pdx = (x-a)^2 dP + 2(x-a)Pdx$, erit vtique $p = 0$, poſito $x = a$. Tum ergo ob $dpdx = qdx^2 = (x-a)^2 ddP + 4(x-a)dPdx + 2Pdx^2$, poſito $x = a$, fiet $qdx^2 = 2Pdx^2$, atque differentiale completum hoc caſu $x = a$, erit $d.y = Pdx^2$, niſi forte & P euaneſcat poſito $x = a$, quos caſus poſtea contemplabor. Praeſens autem caſus generalius hoc modo exhiberi poteſt. Sit $z = (x-a)^2 P + C$, atque y ſit functio quaecunque ipſius z, ita vt fiat $dy = Zdz$, denotante Z functionem quamcunque ipſius $z = (x-a)^2 P + C$. Erit ergo $dz = (x-a)^2 dP + 2(x-a)Pdx$, & $pdx = Z(x-a)^2 dP + 2Z(x-a)Pdx$, quod membrum fit $= 0$ ſi $x = a$; eodemque caſu neglectis terminis, qui continent factorem $x-a$, erit $qdx^2 = 2PZdx^2$, ideoque caſu $x = a$, fiet $dy = PZdx^2$; poſtquam in PZ vbique loco x poſitum fuerit a. Quare ſi fuerit y functio quaecunque ipſius $z = (x-a)^2 P + C$, ita vt ſit $dy = Zdz$, erit caſu $x = a$, differentiale $dy = PZdx^2$. Fiet ergo haec functio y maxima caſu $x = a$, ſi eodem caſu fiat PZ quantitas negatiua, minima vero, ſi PZ fiat quantitas affirmatiua.

342. Si fuerit $p = (x-a)^2 P$, caſu $x = a$ quoque q euaneſcit, talis autem expreſſio pro p oritur, ſi fuerit $y = (x-a)^3 P + C$. Erit ergo $p dx = (x-a)^3 dP + 3(x-a)^2 P dx$; $q dx^2 = (x-a)^3 ddP + 6(x-a)^2 dP dx + 6(x-a) P dx^2$, quorum vtrumque membrum caſu $x = a$ euaneſcit; at vero ſequens erit $r dx^3 = (x-a)^3 d^3 P + 9(x-a)^2 ddP dx + 18(x-a) dP dx^2 + 6 P dx^3 = 6 P dx^3$, poſito $x = a$. Quare cum & p & q caſu $x = a$ euaneſcat, fiet $dy = \frac{1}{6} r dx^3 = P dx^3$. Simili modo ſi ponatur $z = (x-a)^3 P + C$, fueritque y functio quaecunque ipſius z, ita vt ſit $dy = Z dx$, ob $dz = (x-a)^3 dP + 3(x-a)^2 P dx$, fiet quoque $p = 0$ & $q = 0$, eritque $r dx^3 = 6 PZ dx^3$; vnde caſu $x = a$, erit $dy = PZ dx^3$. Quare iſta functio y, etiamſi caſu $x = a$, fiat $p = 0$, tamen neque maximum neque minimum valorem recipit.

343. Haec differentialia facilius inueniri poſſunt ex ipſa differentialium natura. Cum enim differentiale ipſius y oriatur, ſi y a ſtatu ſequenti proximo ſubtrahatur, qui prodit, ſi loco x ponatur $x + dx$; ponamus caſu primo quo erat, $y = (x-a)^2 P + C$, $x + dx$ loco x, eritque $y^I = (x-a+dx)^2 P^I + C$, vnde fiet $dy = (x-a+dx)^2 P^I - (x-a)^2 P$. Caſu igitur quo $x = a$, erit $dy = P^I dx^2$, & cum P^I ad P rationem aequalitatis habeat, erit $dy = P dx$. Simili modo ſi fuerit $z = (x-a)^2 P + C$, erit $dz = P dx^2$; quare ſi ſit y functio quaecunque ipſius z, ita vt ſit $dy = Z dz$, erit $dy = PZ dx^2$ caſu, quo ponitur $x = a$. Deinde ſi ſit $z = (x-a)^3 P + C$, erit $z^I = (x-a+dx)^3 P^I + C$, & propterea caſu $x = a$, fiet $z^I - z = dz = P dx^3$. Hinc

ſi fuerit y functio quaecunque ipſius z, atque $dy = Z dz$, erit quoque caſu $x = a$, differentiale $dy = PZ dx^3$, ſiquidem in functionibus P & Z loco x vbique ſubſtituatur a. Quoniam vero hoc caſu fit $z = C$, atque Z eſt functio ipſius z, euadet Z quantitas conſtans, talis ſcilicet functio ipſius C, qualis ante erat ipſius z.

344. Si igitur generaliter fuerit $y = (x-a)^n P + C$, quia eſt $y^I = (x - a + dx)^n P^I + C$, caſu $x = a$, fiet $dy = P dx^n$; vnde ſi fuerit $n > 1$, hoc differentiale reſpectu aliorum differentialium primorum, quae ipſi dx ſunt homogenea, euaneſcet. Ex praecedentibus ergo manifeſtum eſt, functionem y fieri caſu $x = a$, vel maximam vel minimam, ſi fuerit n numerus par: tum enim ſi poſito $x = a$ fiat P quantitas affirmatiua, fiet y minimum, ſin autem P fit quantitas negatiua, fiet y maximum. Hocque ergo modo ratio maximorum & minimorum multo facilius inuenitur, quam methodo ſupra expoſita, quia non opus eſt ad differentialia altiora progredi. Quod ſi vero ſit $z = (x - a)^n P + C$, atque y fuerit functio quaecunque ipſius z, vt ſit $dy = Z dz$, erit caſu $x = a$ differentiale $dy = PZ dx^n$. Notandum autem eſt, hic n ſumi pro numero affirmatiuo ſeu o maiore, ſi enim n eſſet numerus negatiuus, tum poſito $x = o$, non euaneſceret $(x - a)^n$, vti aſſumſimus, ſed adeo fieret infinite magnum.

345. Iam vidimus hoc pacto differentiale multo expeditius inueniri, quam ope ſeriei, qua ante differentiale

tiale completum expreſſimus; ſi enim ſit n numerus integer, tot ſeriei illius termini perluſtrari deberent, quot n contineat vnitates. Verum ſi n ſit numerus fractus, tum ſeries iſta nequidem verum differentiale vnquam exhibebit. Ponamus enim eſſe $y = (x-a)^{\frac{3}{2}} + aVa$, ſi ſeriem $dy = p\,dx + \frac{1}{2}q\,dx^2 + \frac{1}{6}r\,dx^3 + \frac{1}{24}s\,dx^4 + \&c.$ ſpectemus, fiet $p = \frac{3}{2}V(x-a)$, $q = \frac{3}{4V(x-a)}$, $r = \frac{-3}{8(x-a)V(x-a)}$, $s = \frac{9}{16(x-a)^2V(x-a)}$, &c. Quare ſi ponatur $x=a$ fiet quidem $p=0$, at ſequentes termini omnes q, r, s, &c. euadent infiniti; vnde valor differentialis dy hoc caſu omnino definiri non poteſt. At vero methodus ex ipſa differentialium natura deducta nullum dubium relinquit. Cum enim ſit $y = (x-a)^{\frac{3}{2}} + aVa$, poſito $x+dx$ loco x fiet $y^{\mathrm{I}} = (x-a+dx)^{\frac{3}{2}} + aVa$, eritque, ſi $x=a$ ponatur, $dy = dxVdx$. Euaneſcit ergo hoc differentiale prae dx, at vero differentialia ſecunda cum dx^2 homogenea prae eo euaneſcent.

346. Euoluamus hos caſus, quibus exponens n eſt numerus fractus aliquanto accuratius, ſitque $y = PV(x-a) + C$, ob $y^{\mathrm{I}} = P^{\mathrm{I}}V(x-a+dx) + C$, fiet $dy = PVdx$ caſu $x=a$; vnde hoc differentiale ad dx, & ad differentialia cum dx homogenea rationem tenebit infinitam. Hinc etiam patet, quid hoc caſu de ratione maximi ac minimi ſit tenendum. Cum enim poſito $x+dx$ loco x, abeat y in $PV(x-a) + C + PVdx$, ob Vdx am-

ambiguum, functio y geminum induet valorem, alterum maiorem quam C, quem recipit poſito $x = a$, alterum minorem; vnde caſu $x = a$ neque maximum neque minimum fiet. Praeterea ſi dx capiatur negatiue tum valor ipſius y adeo fiet imaginarius. Idem tenendum eſt ſi ſit $z = P\surd(x-a) + C$, & y functio quaecunque ipſius z, vt ſit $dy = Zdz$, tum enim erit $dy = PZ\surd dx$ caſu $x = a$.

347. Si propoſita fuerit iſta functio $y = (x-a)^{\frac{m}{n}}P + C$, cuius differentiale quaeritur caſu $x = a$, erit vti ex antecedentibus colligitur $dy = Pdx^{\frac{m}{n}}$. Quocirca ſi fuerit $m > n$ hoc differentiale prae dx euaneſcat, ſin autem ſit $m < n$, ratio $\frac{dy}{dx}$ erit infinite magna. Praeterea vero ſi n ſit numerus par, differentiale dy geminum habebit valorem, alterum affirmatiuum, alterum negatiuum; ſicque functio y, quae caſu $x = a$ fit $= C$, ſi ponatur $x = a + dx$ binos habebit valores alterum maiorem quam C alterum vero minorem; ſin autem poneretur $x = a - dx$, tum y adeo fieret imaginarium; vnde hoc caſu y neque maximum fit neque minimum. Ponamus nunc denominatorem n eſſe numerum imparem, erit numerator m vel par vel impar. Sit primo m numerus par; quia dy eundem valorem retinet, ſiue dx ſumatur affirmatiue ſiue negatiue, perſpicuum eſt, functionem y caſu $x = a$ fieri ſiue maximam ſiue minimam, prout hoc caſu fuerit P vel quantitas negatiua vel affirmati-

matiua. Sin autem vterque numerus m & n fuerit impar, differentiale dy in ſui negatiuum abibit, poſito dx negatiuo; hocque ergo caſu functio y neque maximum erit neque minimum, ſi ponatur $x = a$.

348. Si functio y ex pluribus huiusmodi terminis, quorum ſinguli ſint diuiſibiles per $x - a$, conſtet, ita vt ſit $y = (x - a)^m P + (x - a)^n Q + C$, tum eius differentiale caſu $x = a$ erit $dy = P dx^m + Q dx^n$; in qua expreſſione, ſi fuerit $n > m$, terminus ſecundus prae primo euaneſcit, ita vt tantum prodeat $dy = P dx^m$. Sin autem n ſit fractio denominatorem habens parem, tum etiamſi $Q dx^n$ prae $P dx^m$ euaneſcat, tamen omnino negligi non poteſt. Ex eo enim apparet, ſi capiatur dx negatiue, valorem ipſius dy fieri imaginarium, quod ex ſolo termino primo $P dx^m$ non patet. Cum ergo ſi n ſit fractio denominatorem habens parem, dx negatiue accipi nequeat, ſin autem affirmatiue capiatur, terminus $Q dx^n$ geminum praebeat valorem: functio $y = (x - a)^m P + (x - a)^n Q + C$ quae caſu $x = a$ fit $= C$, ſi ponatur $x = a + dx$, erit $y = C + P dx^m \pm Q dx^n$, quorum valorum vterque cum vel maior ſit vel minor quam C, prout P fuerit quantitas vel affirmatiua vel negatiua, erit functio y caſu $x = a$ vel minimum vel maximum ſecundae ſpeciei.

349. His igitur caſibus differentialia functionum vera non per regulas differentiationis conſuetas inueniri poſſunt; quippe quae tantum valent, quamdiu differen-

tiale functionis eſt homogeneum cum dx. Sin autem caſu quopiam ſingulari differentiale functionis exprimatur per eius poteſtatem dx^n, tum regula praebet pro hoc differentiali o, ſi n fuerit numerus vnitate maior; at vero differentiale exhibet infinite magnum, ſi n ſit exponens vnitate minor. Sic ſi ipſius $y = \surd(a - x)$ differentiale quaeratur caſu $x = a$, quia eſt $dy = - \frac{dx}{\surd(a-x)}$, facto $x = a$ prodit $dy = - \frac{dx}{0}$. Atque ſi differentialia ſequentia in ſubſidium vocare velimus, omnia pariter ob denominatores $= 0$ in infinitum excreſcunt, ita vt inde nihil concludi poſſit. At vero hoc caſu vidimus eſſe $dy = \surd - dx$, atque adeo imaginarium. Sin autem loco x ponatur $x - dx$, erit $dy = \surd dx$, atque adeo erit infinities maius quam dx, ita vt dx prae dy euaneſcat. Quare regula conſueta etiam hoc caſu in errorem non inducit, cum valorem ipſius dy infinitum exhibeat.

350. A regula ergo conſueta differentiationis recedendum eſt, quoties in ſerie $pdx + \frac{1}{2}qdx^2 + \frac{1}{6}rdx^3 +$ &c. qua differentiale completum functionis y exprimitur, primus terminus p vel fit $= 0$ vel in infinitum excreſcit, eoque caſu differentiale ex primis principiis deriuari debet. Quoties ergo functionis y differentiale quaeritur dato ipſius x valori reſpondens, quo littera p vel infinite parua euadit vel infinite magna, toties recurrendum eſt ad ipſa prima differentiationis principia. Omnibus vero

vero reliquis caſibus, quibus ſit neque $p = 0$ neque $p = \infty$, conſueta regula veros differentialis valores praebebit. Interim tamen caſus ante (348) memoratus non eſt negligendus, ſi functio y contineat huiusmodi membrum $(x - a)^n Q$ exiſtente n fractione denominatorem parem habente; etiamſi enim adſint differentialia inferiora quam $Q dx^n$, prae quibus hoc euaneſcat; tamen quoniam $Q dx^n$ ſi ſit dx negatiuum, ſit imaginarium, hoc membrum $Q dx^n$ reliqua omnia, prae quibus euaneſcit, quoque transmutat in imaginaria: cuius circumſtantiae ratio potiſſimum in lineis erit habenda. Huiusmodi ergo caſus particulares, quibus verum differentiale communi regula non indicatur, in adiunctis exemplis explicabo.

EXEMPLUM I.

Quaeratur differentiale functionis

$$y = a + x - \sqrt{[xx + ax - x\sqrt{(2ax - xx)}]}$$

caſu quo ponitur $x = a$.

Differentiali iſtius functionis caſu $x = a$ per regulam receptam non reperiri, ex differentiatione patet, ſit enim:

$$dy = dx - \frac{xdx - \frac{1}{2}adx + \frac{1}{2}dx\sqrt{(2ax - xx)} + (axdx - xxdx) : \sqrt{(2ax - xx)}}{\sqrt{(xx + ax - x\sqrt{(2ax - xx)})}}$$

poſito enim $x = a$ erit $dy = dx - \frac{adx}{a} = 0$. Ordiamur ergo a principiis differentiationis, ac primo quidem poſito $x + dx$ loco x fiet:

$$y^{I} = a + x + dx - \sqrt{[xx + 2xdx + dx^2 + ax + adx - (x + dx)\sqrt{(2ax - xx + 2adx - 2xdx - dx^2)}]}$$

 Poſito

Poſito autem $x = a$ erit:

$y^{I} = 2a + dx - \sqrt{[2aa + 3adx + dx^2 - (a+dx)\sqrt{(aa-dx^2)}]}$

Jam cum ſit $\sqrt{(aa - dx^2)} = a - \frac{dx^2}{2a}$, ſequentes enim termini tuto negligi poterunt, quia non omnes, qui ſunt infinities maiores, deſtruentur, vt mox patebit: erit $y^{I} = 2a + dx - \sqrt{(aa + 2adx + \frac{3}{2}dx^2)}$, porroque radicem extrahendo fiet $y^{I} = 2a + dx - \left(a + dx + \frac{dx^2}{4a}\right) = a - \frac{dx^2}{4a}$. At caſu $x = a$, erit $y = a$; vnde cum ſit $y' = y + dy$ obtinebitur $dy = -\frac{dx^2}{4a}$: ex quo ſimul perſpicitur functionem propoſitam y fieri maximum, ſi ponatur $x = a$.

EXEMPLUM II.

Inuenire differentiale huius functionis:

$$y = 2ax - xx + a\sqrt{(aa - xx)}$$

caſu, quo ponitur $x = a$.

Facta differentiatione more conſueto fit $dy = 2adx - 2xdx - \frac{axdx}{\sqrt{(aa - xx)}}$, quod poſito $x = a$ in infinitum abit, neque ergo hoc modo indicatur. Differentialia vero ſequentium ordinum pariter omnia fient infinita, ita vt ex iis nequidem ex ſerie $pdx + \frac{1}{2}qdx^2 + \frac{1}{6}rdx^3 +$ &c. verus valor differentialis inueniri queat. Ponamus ergo $x + dx$ loco x, atque habebimus

$y' = 2ax - xx + 2adx - 2xdx - dx^2 + a\sqrt{(aa - xx - 2xdx - dx^2)}$

&

& poſito $x = a$ erit:

$$y' = aa - dx^2 + a\sqrt{(-2adx - dx^2)}$$

At eodem caſu fit $y = aa$; vnde erit $dy = -dx^2 + a\sqrt{-2adx}$, & cum dx^2 prae $\sqrt{-2adx}$ euaneſcat, erit $dy = a\sqrt{-2adx}$. Quare ſi differentiale dx affirmatiue capiatur, erit dy imaginarium; ſin autem pro x ſcribatur $x - dx$, erit $dy = a\sqrt{2adx}$, cuius cum duplex ſit valor alter affirmatiuus, alter negatiuus, functio y caſu $x = a$ neque maxima fiet neque minima.

EXEMPLUM III.

Inuenire differentiale functionis:

$$y = 3aax - 3axx + x^3 + (a - x)^2 \sqrt[3]{(a^3 - x^3)}$$

caſu quo ponitur $x = a$.

Quoniam haec functio in iſtam formam transformatur $y = a^3 - (a - x)^3 + (a - x)^{\frac{7}{3}} \sqrt[3]{(aa + ax + xx)}$, poſito $x = a + dx$ fit $y' = a^3 + dx^3 - dx^{\frac{7}{3}} \sqrt[3]{3aa}$, eodemque caſu eſt $y = a^3$. Erit ergo $dy = dx^3 - dx^{\frac{7}{3}} \sqrt[3]{3aa}$, & cum dx^3 euaneſcat prae $dx^{\frac{7}{3}}$, erit $dy = -dx^{\frac{7}{3}} \sqrt[3]{3aa}$. caſu ergo $x = a$ functio y neque maximum fit neque minimum.

EXEMPLUM. IV.

Inuenire differentiale functionis:

$$y = \sqrt{x} + \sqrt[4]{x^3} = (1 + \sqrt[4]{x})\sqrt{x}$$

caſu $x = 0$.

Quoniam caſus $x = 0$ proponitur, eoque ſit $y = 0$, loco x

 tantum

tantum dx ſcribatur, & habebitur $dy = dx^{\frac{1}{2}} + dx^{\frac{3}{4}}$, ſeu $dy = (1 + \sqrt[4]{dx})\sqrt{dx}$; vnde primum patet dx negatiue accipi non poſſe. Tum vero etiamſi alias $\sqrt{dx}$ geminum valorem prae ſe ferat, alterum affirmatiuum alterum negatiuum, tamen hoc caſu, quia eius radix $\sqrt[4]{dx}$ occurrit, non niſi affirmatiue accipi poteſt. At vero $\sqrt[4]{dx}$ vtrumque ſignificatum recipit, eritque $dy = \sqrt{dx} \pm \sqrt[4]{dx^3}$ & $y' = 0 + \sqrt{dx} \pm \sqrt[4]{dx^3}$, ob $y = 0$. Cum igitur vterqne ipſius y' valor maior ſit, quam ipſius y, ſequitur caſu $x = 0$ fieri y minimum. Quod autem functio $y = \sqrt{x} + \sqrt[4]{x^3}$ non complectatur hanc $y = -\sqrt{x} + \sqrt[4]{x^3}$, vtramque ad rationalitatem perducendo patebit. Prior enim fuſa in hanc formam $y - \sqrt{x} = \sqrt[4]{x^3}$, & quadrata dat, $y^2 - 2y\sqrt{x} + x = x\sqrt{x}$ ſeu $y^2 + x = (x + 2y)\sqrt{x}$, quae denuo quadrata praebet $y^4 - 2y^2x - 4xxy + xx - x^3 = 0$. Altera vero $y + \sqrt{x} = \sqrt[4]{x^3}$ dabit $y^2 + x = (x - 2y)\sqrt{x}$ & porro $y^4 - 2yyx + 4xxy + xx - x^3 = 0$ quae ab illa eſt diverſa. At vero alterum membrum $\sqrt[4]{x^3}$ ambiguitatem ſigni retinet. Quamobrem iſta circumſtantia probe eſt notanda, quod etiamſi communiter radices poteſtatum parium vtrumque ſignum $+$ & $-$ includant, tamen haec ambiguitas ceſſet, ſi in eadem expreſſione earumdem radicum vlteriores radices poteſtatum

tum parium occurrant; quippe quae fierent imaginariae, ſi radices priores negatiue acciperentur. Atque ex hoc fonte maxima & minima ſecundae ſpeciei ſequuntur, quando talia non locum habere videantur.

EXEMPLUM V.

Inuenire differentiale functionis:

$$y = a + \sqrt{(x-f)} + (x-f)\sqrt[4]{(x-f)} + (x-f)^2 \sqrt[8]{(x-f)}$$

caſu quo ponitur $x = f$.

Ponamus $x-f=t$, & cum ſit $y=a+\sqrt{t}+t\sqrt[4]{t}+tt\sqrt[8]{t}$ huius differentiale quaeritur caſu $t=0$, quo fit $y=a$. Poſito ergo $t+dt$ ſeu $0+dt$ loco t fiet $y' = y+dy = a+\sqrt{dt}+dt\sqrt[4]{dt}+dt^2\sqrt[8]{dt}$, ideoque habebitur $dy=\sqrt{dt}+dt\sqrt[4]{dt}+dt^2\sqrt[8]{dt}$. Vbi primo patet differentiale dt negatiue accipi non poſſe, quin dy fiat imaginarium. Tum verum non ſolum $\sqrt{dt}$, ſed nequidem $\sqrt[4]{dt}$ negatiue accipi poteſt; fieret enim $\sqrt[8]{dt}$ imaginarium: vnde differentiale dy geminum tantum habet valorem, $dy=\sqrt{dt}+dt\sqrt[4]{dt}\pm dt^2\sqrt[8]{dt}$, quorum cum vterque maior ſit nihilo, ſequitur functionem y fieri minimum ſecundae ſpeciei poſito $t=0$ ſeu $x=f$. Quanquam ergo his caſibus termini $dt\sqrt[4]{dt}$ & $dt^2\sqrt[8]{dt}$ prae primo $\sqrt{dt}$ euaneſcant; tamen eorum ratio eſt habenda, ſi multiplicitas valorum ſpectetur, vt imaginaria euitentur.

EXEM-

EXEMPLUM VI.

Inuenire differentiale functionis:

$$y = ax + bxx + (x - f)^n + (x - f)^{m+\frac{1}{2}n}$$

caſu $x = f$.

Si ponatur $x = f$ fiet $y = af + bff$, & ſi loco x ponatur $x + dx$ ſeu $f + dx$, prodibit valor proximus $y' = af + bff + adx + 2bfdx + bdx^2 + dx^n + dx^{m+\frac{1}{2}n}$, ita vt ſit $dy = adx + 2bfdx + bdx^2 + dx^n + dx^m \sqrt{dx^n}$. Niſi ergo ſit n numerus par, differentiale dx negatiue ſumi nequit. Vltimus autem terminus $dx^m \sqrt{dx^n}$ ſignum habet ambiguum; vnde valor ipſius y' erit duplex vterque maior quam ipſius y, ſi quidem $a + 2bf$ fuerit quantitas affirmatiua, atque exponentes n & $m + \frac{1}{2}n$ vnitate fuerint maiores. Fiet ergo valor functionis y caſu $x = f$ minimus: hocque euenit ſive n ſit numerus integer ſiue fractus, dummodo numerator hoc caſu, & ipſe numerus illo caſu non fuerit par.

351. Imprimis autem haec methodus differentialia ex ipſis principiis deducendi vſum habet in functionibus tranſcendentibus, cum quibusdam caſibus differentiale more conſueto inuentum vel euaneſcit, vel in infinitum exceſcere videtur. Occurrunt autem hic eiusmodi infinitorum & infinite paruorum ſpecies, quae in algebraicis nunquam inueniuntur. Cum enim ſi i denotet numerum infinitum, li ſit quoque infinitus quidem, ſed tamen ad ipſum numerum i, eiusque adeo poteſtatem quamcunque i^n, quamtumuis exiguus ſtatuatur exponens n,

ratio-

rationem tenens infinite paruam, erit fractio $\frac{li}{i^n}$ infinite parua, neque ante finita eſſe poterit, quam exponens n fiat infinite paruus. Erit ergo li homogeneum cum i^n, ſi exponens n fuerit infinite paruus. Ponamus nunc $i = \frac{1}{\omega}$, exiſtente ω quantitate infinite parua, erit $-l\omega$ homogeneum cum $\frac{1}{\omega^n}$, ſi exponens n ſit infinite paruus, ideoque $-\frac{1}{l\omega}$ homogeneum erit cum ω^n; hincque $-\frac{1}{ldx}$, erit infinite paruum comparandum cum dx^n, exiſtente n fractione infinite parua. Ita ſi fuerit $y = -\frac{1}{lx}$ differentiale ipſius y caſu $x = 0$, erit $= -\frac{1}{ldx} = dx^n$ ideoque dy ad dx atque ad quamcunque ipſius dx poteſtatem tenebit rationem infinitam: atque prae $-\frac{1}{ldx}$ euaneſcunt omnes omnino poteſtates ipſius dx, quantumuis exigui fuerint earum exponentes.

352. Deinde quoque vidimus, ſi a fuerit numerus vnitate maior, & i infinitus, tum a^i fore infinitum tam excelſi gradus, vt prae eo non ſolum i, ſed etiam quaeuis ipſius i poteſtas euaneſcat; neque i^n ante homogeneum cum a^i euadet, quam exponens n in infinitum fuerit auctus. Sit nunc $i = \frac{1}{\omega}$, ita vt ω infinite par-

vum denotet, erit $a^{\frac{1}{\omega}}$ homogeneum cum $\frac{1}{\omega^n}$, exiſtente n numero infinite magno: ideoque $a^{\frac{-1}{\omega}}$ ſeu $\frac{1}{a^{1:\omega}}$, erit infinite paruum comparandum cum ω^n. Hinc $\frac{1}{a^{1:dx}}$, erit infinite paruum, quod autem prae omnibus ipſius dx poteſtatibus euaneſcit; cum homogeneum ſit cum poteſtate dx^n exiſtente n numero infinite magno. Quare ſi quaeratur differentiale ipſius $y = \frac{1}{a^{1:x}}$ caſu $x = 0$; quoniam fit $y = 0$, erit $dy = \frac{1}{a^{1:dx}}$, ideoque infinities minus eſt quam poteſtas quantumuis alta ipſius dx.

353. Sin autem a ſit numerus vnitate minor, tum quia $\frac{1}{a}$ fit vnitate maior, quaeſtio ad caſum praecedentem reducitur. Scilicet ſi habeatur expreſſio $a^{\frac{1}{\omega}}$, ea ponendo $a = 1:b$ transmutabitur in $b^{-\frac{1}{\omega}}$, ſeu $\frac{1}{b^{1:\omega}}$, quae homogenea erit ob $b > 1$ cum ω^n, exiſtente n numero infinite magno. His igitur praemiſſis ſequentia exempla reſoluere poterimus.

EXEM-

EXEMPLUM I.

Inuenire differentiale functionis: $y = xx - \frac{1}{lx}$, *caſu* $x = 0$.

Quoniam poſito $x = 0$ fit $y = 0$, ſi ponamus $x + dx$, ſeu $0 + dx$ loco x, fiet $y' = dy = dx^2 - \frac{1}{ldx}$. Cum autem $- \frac{1}{ldx}$ homogeneum ſit cum dx^n, denotante n numerum infinite paruum, prae eo dx^2 euaneſcet, eritque $dy = - \frac{1}{ldx} = dx^n$. At vero quia logarithmi numerorum negatiuorum ſunt imaginarii, dx negatiue accipi non poterit; eritque adeo caſu $x = 0$ functio y minimum, ſed neque ad primam neque ad ſecundam ſpeciem pertinens. Ad primam ſcilicet ſpeciem non pertinet, quia y nullos habet valores antecedentes proximos, ſed tantum minus eſt valoribus ſequentibus, ſi x nihilo maius ſtatuatur. Ad ſecundam autem ſpeciem ideo non pertinet, quia valores ſequentes, quibuscum comparatur, non ſunt gemini: ſic itaque prodit tertia ſpecies maximorum minimorumue, quae in functionibus logarithmicis & tranſcendentibus tantum locum habet, in algebraicis autem nunquam occurrit; de qua in ſequente parte de lineis curuis fuſius agetur.

EXEMPLUM II.

Inuenire differentiale functionis: $y = (a-x)^n - x^n(la - lx)^n$ *caſu quo* $x = a$.

Differentiale hoc ſi n ſit numerus integer, ex formula generali $dy = pdx + \frac{1}{2}qdx^2 + \frac{1}{6}rdx^3 +$ &c. inueniri poteſt, erit enim:

$$pdx = -n(a-x)^{n-1}dx - nx^{n-1}dx(la-lx)^n + nx^{n-1}(la-lx)^{n-1}dx$$

qui valor poſito $x = a$ vtique euaneſcit: nam etiamſi ſit $n = 1$, erit $pdx = -dx + dx = 0$. Si igitur vlterius progrediamur, erit: $\frac{1}{2}qdx^2 =$

$$\frac{n(n-1)}{1.2}(a-x)^{n-2}dx^2 - \frac{n(n-1)}{1.2}x^{n-2}dx^2(la-lx)^n + \frac{n^2}{2}x^{n-2}dx^2(la-lx)^{n-1}$$

$$+ \frac{n(n-1)}{1.2}x^{n-2}dx^2(la-lx)^{n-1} - \frac{n(n-1)}{1.2}x^{n-2}dx^2(la-lx)^{n-2}$$

Hinc ergo ſi fuerit $n = 1$, erit $\frac{1}{2}qdx^2 = \frac{dx^2}{2a}$ poſito $x = a$

Simili modo ſi ſit $n = 2$, ad terminum tertium $\frac{1}{6}rdx^3$ eſſet pergendum, & ita porro. Facilius ergo vtemur ipſis differentiationis principiis, & cum poſito $x = a$ fiat $y = 0$, ſi ponamus $x + dx$ ſeu $a + dx$ loco x, erit $y' = (-dx)^n - (a+dx)^n[la - l(a+dx)]^n = y + dy = dy$ ob $y = 0$. Eſt vero $l(a+dx) = la + \frac{dx}{a} - \frac{dx^2}{2a^2} + \frac{dx^3}{3a^3} -$ &c.

vnde fit $dy = (-dx)^n -$

$$-\left(a^n + na^{n-1}dx + \frac{n(n-1)}{1.2}a^{n-2}dx^2\right)\left(-\frac{dx}{a} + \frac{dx^2}{2a^2} - \frac{dx^3}{3a^3}\right)^n = \frac{n}{2a}(-dx)^{n+1}$$

Caſu

Caſu igitur $x = a$ erit formulae propoſitae differentiale quaeſitum dy, vt ſequitur:

ſi $n = 1$	$dy = \frac{dx^2}{2a}$ vt ante inuenimus
ſi $n = 2$	$dy = -\frac{2dx^3}{2a}$
ſi $n = 3$	$dy = \frac{3dx^4}{2a}$
ſi $n = 4$	$dy = -\frac{4dx^5}{2a}$
&c.	&c.

Si ergo n fuerit numerus impar, functio y caſu $x = a$ fit minimum, ſin autem n ſit numerus par, neque maximum neque minimum: quod idem valet, ſi n fuerit fractio denominatorem habens imparem. Sin autem n fuerit fractio denominatorem habens parem, tum dx negatiue accipi debet, ne in imaginaria incidamus; & ob ambiguitatem ſignificationis functio quoque neque maxima neque minima euadet.

EXEMPLUM III.

Inuenire differentiale functionis: $y = x^x$ *caſu* $x = \frac{1}{e}$ *denotante* e *numerum, cuius logarithmus hyperbolicus eſt* $= 1$.

Quia fit in genere $dy = x^x dx(lx + 1)$, hoc differentiale caſu $x = \frac{1}{e}$ ſeu $lx = -1$ euaneſcit. Compa-

 retur

retur ergo hoc differentiale cum forma generali $p\,dx + \frac{1}{2}q\,dx^2 +$ &c. erit $p = x^x(lx+1)$ & $q = x^x(lx+1)^2 + x^{x-1}$, & poſito $lx = -1$ ſeu $x = \frac{1}{e}$, erit $q = \left(\frac{1}{e}\right)^{\frac{1-e}{e}} = e^{\frac{e-1}{e}}$. Quare differentiale quaeſitum erit $dy = \frac{1}{2}e^{(e-1):e}dx^2$, euaditque ergo functio $y = x^x$ minimum caſu $x = \frac{1}{e}$.

EXEMPLUM IV.

Inuenire differentiale functionis huius: $y = x^n + e^{-1:x}$ *caſu quo* $x = 0$.

Quia facto $x = 0$ fit $y = 0$, ſi ponatur $x = 0 + dx$, erit $y' = dy = dx^n + \frac{1}{e^{1:dx}}$. Vidimus autem $\frac{1}{e^{1:dx}}$ homogeneum eſſe cum poteſtate ipſius dx infinita, ſeu cum dx^∞, ideoque prae dx^n euaneſcet; ita vt ſit $dy = dx^n$.

354. Quod in differentialibus primis certis caſibus vſu venit, vt conſueta differentiationis regula non prodeant, idem quoque in differentialibus ſecundi ac tertii ſuperiorumque ordinum euenit, iis caſibus, quibus in forma differentiali completa:

$$d.y = p\,dx + \tfrac{1}{2}q\,dx^2 + \tfrac{1}{6}r\,dx^3 + \tfrac{1}{24}s\,dx^4 + \&c.$$

quantitatum q, r, s, &c. nonnullae vel euaneſcunt, vel in infinitum abeunt. Scilicet cum ſit:

$$dd.y = q\,dx^2 + r\,dx^3 + \tfrac{7}{12}s\,dx^4 + \&c.$$

ſi

ſi quo caſu fiat $q = 0$, tum erit $ddy = r dx^3$; ſin autem eodem caſu & r euaneſcat, tum erit $ddy = \frac{7}{12} s dx^4$, & ita porro. Sin autem vel q vel r vel s &c. fiat infinitum, tum ex iſta ſerie differentiale ſecundum prorſus inueniri nequit, ſed confugiendum erit ad principia differentialium: ſcilicet ponendo $x + dx$ loco x quaeratur valor y', & ponendo $x + 2dx$ loco x valor ipſius y'', quo facto erit verus valor differentialis ſecundi $ddy = dy' - dy = y'' - 2y' + y$. Simili modo ſi de differentiali tertio quaeſtio proponatur, tum praeterea in y loco x ſcribatur $x + 3dx$, inuentoque valore y''' erit $d^3y = y''' - 3y'' + 3y' - y$, ſicque deinceps. Quos caſus ſequentibus exemplis illuſtrabimus.

EXEMPLUM I.

Inuenire differentiale ſecundum functionis $y = \frac{aa - xx}{aa + xx}$ *caſu quo ponitur* $x = \frac{a}{\sqrt{3}}$.

Quaerendo differentiale completum ipſius y, ex forma $dy = p dx + \frac{1}{2} q dx^2 + \frac{1}{6} r dx^3 + \frac{1}{24} s dx^4 +$ &c. prodibunt pro p, q, r, s, &c. ſequentes valores:

$$p = -\frac{4aax}{(aa+xx)^2}; \quad q = -\frac{4a^4 + 12aaxx}{(aa+xx)^3}; \text{ atque}$$

$$r = \frac{48a^4x - 48aax^3}{(aa+xx)^4}.$$

Cum nunc ſit $ddy = q dx^2 + r dx^3 + \frac{7}{12} s dx^4 +$ &c. ob

ob $q = 0$ caſu $x = \frac{a}{\sqrt{3}}$, eodemque caſu ſit $r = \frac{27\sqrt{3}}{8a^3}$, fiet differentiale ſecundum quaeſitum $ddy = \frac{27dx^3\sqrt{3}}{8a^3}$.

EXEMPLUM II.

Inuenire differentiale tertium functionis $y = \frac{aa - xx}{aa + xx}$ *caſu* $x = a$.

Quaerendo vt ante differentiale completum
$dy = \frac{1}{2}qdx^2 + \frac{1}{6}rdx^3 + \frac{1}{24}sdx^4 + \&c.$
quia eſt differentiale tertium $d^3y = rdx^3 + \frac{3}{2}sdx^4$, ob $r = \frac{48a^4x - 48aax^3}{(aa+xx)^4}$, fiet $r = 0$ caſu $x = a$; quare ad valorem s eſt progrediendum, qui erit:

$$s = \frac{48a^4 - 144aaxx}{(aa+xx)^4} - \frac{8x(48a^4x - 48aax^3)}{(aa+xx)^5}$$

facto ergo $x = a$, erit $s = -\frac{96a^4}{2^4a^8} = -\frac{6}{a^4}$; vnde hoc caſu erit $d^3y = -\frac{9dx^4}{a^4}$.

EXEMPLUM III.

Inuenire differentialia cuiusque gradus functionis $y = ax^m + bx^n$ *caſu* $x = 0$.

Ponendo ſucceſſiue $x + dx$; $x + 2dx$; $x + 3dx$; &c. loco x valores ſequentes functionis y erunt:

$$y' = a(x + dx)^m + b(x + dx)^n$$
$$y'' = a(x + 2dx)^m + b(x + 2dx)^n$$
$$y''' = a(x + 3dx)^m + b(x + 3dx)^n \quad \&c.$$

Poſito

Poſito ergo $x = 0$, erit $y = 0$, eiusque differentialia erunt:

$$dy = adx^m + bdx^n$$

$$ddy = (2^m - 2)adx^m + (2^n - 2)bdx^n$$

$$d^3y = (3^m - 3.2^m + 3)adx^m + (3^n - 3.2^n - 3.2^n + 3)bdx^n$$

$$d^4y = (4^m - 4.3^m + 6.2^m - 4)adx^m + (4^n - 4.3^n + 6.2^n - 4)bdx^n$$

&c.

Si igitur exponens n fuerit maior quam m, termini ſecundi in his expreſſionibus euaneſcunt prae primis. Interim tamen eorum ratio erit habenda, ſi n fuerit numerus fractus, vt caſus, quibus haec differentialia vel fiunt imaginaria, vel ambigua, diiudicari queant. Vlteriorem vero horum caſuum euolutionem in doctrinam de lineis curuis reſeruari conuenit.

CAPUT XV.

DE VALORIBUS FUNCTIONUM, QUI CERTIS CASIBUS VIDENTUR INDETERMINATI.

355.

Si functio ipſius x quaecunque y fuerit fractio $\frac{P}{Q}$, cuius numerator ac denominator poſito loco x certo quodam valore ſimul euaneſcant; tum iſto caſu fractio $\frac{P}{Q}$ valorem functionis y exprimens euadet $= \frac{0}{0}$, quae expreſſio cum cuique quantitati ſiue finitae ſiue infinitae ſiue infinite paruae poſſit eſſe aequalis, ex ea prorſus valor ipſius y hoc caſu colligi nequit, atque ideo videtur indeterminatus. Interim tamen facile perſpicitur, quia praeter hunc caſum functio y perpetuo valorem determinatum recipit, quicquid pro x ſubſtituatur, etiam hoc caſu valorem ipſius y indeterminatum eſſe non poſſe. Manifeſtum hoc fiet vel ex hoc exemplo, ſi fuerit $y = \frac{aa - xx}{a - x}$, quo facto $x = a$ fit vtique $y = \frac{0}{0}$. Cum autem numeratore per denominatorem diuiſo fiat $y = a + x$, euidens eſt ſi ponatur $x = a$ fore $y = 2a$, ita vt hoc caſu fractio illa $\frac{0}{0}$ aequiualeat quantitati $2a$.

356. Quoniam ergo ſupra oſtendimus, inter cyphras rationem quamcunque intercedere poſſe, in huiusmodi

exem-

exemplis ratio determinata, quam numerator ad denominatorem teneat, inueſtigari debet. Cum autem in cyphris abſolutis iſta diuerſitas perſpici nequeat, earum loco quantitas infinite paruae introduci debent, quae etſi ratione ſignificationis a cyphra non differunt, tamen ex diuerſis earum functionibus, quae numeratorem & denominatorem conſtituunt, valor fractionis ſponte elucet. Sic ſi habeatur iſta fractio $\frac{a\,dx}{b\,dx}$, etiamſi reuera numerator & denominator ſit $= 0$, tamen patet valorem huius fractionis eſſe determinatum nempe $= \frac{a}{b}$. Sin autem habeatur haec fractio $\frac{a\,dx^2}{b\,dx}$, huius valor erit nullus, quemadmodum huius valor $\frac{a\,dx}{b\,dx^2}$ eſt infinite magnus. Si igitur loco nihilorum, quae ſaepenumero in calculum ingrediuntur, infinite parua introducamus, hunc inde fructum percipiemus, vt rationem, quam illa nihila inter ſe tenent, mox cognoſcamus, nullumque amplius dubium circa ſignificationem huiusmodi expreſſionum ſuperſit.

357. Quo haec planiora reddantur, ponamus fractionis $y = \frac{P}{Q}$ tam numeratorem quam denominatorem euaneſcere, ſi ſtatuatur $x = a$. Ad haec autem nihila, quae inter ſe comparari non poſſunt, euitanda, ponamus $x = a + dx$, quae poſitio reuera in priorem $x = a$ recidit ob $dx = 0$. Cum vero, ſi loco x ponatur $x + dx$, functiones P & Q abeant in $P + dP$ & $Q + dQ$; po-

 ſitio-

ſitioni $x = a + dx$ ſatisfiet, ſi in his valoribus vbique ſtatuatur $x = a$, quo quidem caſu P & Q euaneſcere aſſumuntur. Hinc ſi loco x ponatur $a + dx$, fractio $\frac{P}{Q}$ transmutabitur in hanc $\frac{dP}{dQ}$, quae propterea valorem functionis $y = \frac{P}{Q}$ exprimit caſu $x = a$. Haecque expreſſio indeterminata amplius eſſe non poterit, ſiquidem functionum P & Q differentialia vera ſumantur, vti in capite praecedente docuimus. Hoc enim pacto differentialia dP & dQ nunquam in nihilum abſolutum abeunt, ſed niſi per differentiale dx ipſum exprimantur, ſaltem per eius poteſtates exhibebuntur. Quodſi igitur reperiatur $dP = R dx^m$ & $dQ = S dx^n$, erit functionis $y = \frac{P}{Q}$ caſu $x = a$ valor $= \frac{R dx^m}{S dx^n}$, qui propterea erit finitus & $= \frac{R}{S}$, ſi fuerit $m = n$; ſin autem ſit $m > n$, tum valor fractionis propoſitae reuera erit $= 0$: at ſi ſit $m < n$, iſte valor in infinitum excreſcit.

358. Quoties ergo huiusmodi fractio occurrit $\frac{P}{Q}$, cuius numerator & denominator certo caſu puta $x = a$ ſimul euaneſcant, valor iſtius fractionis hoc caſu $x = a$ per ſequentem regulam inuenietur:

Quaerantur quantitatum P *&* Q *differentialia caſu* x = a, *eaque loco ipſarum* P *&* Q *ſubſtituantur, quo facto* *frac-*

fractio $\frac{dP}{dQ}$ *exhibebit valorem fractionis* $\frac{P}{Q}$ *quaeſitum.* Si differentialia dP & dQ methodo conſueta inuenta neque infinita fiant neque euaneſcant caſu $x = a$, tum ea retineri poterunt; ſin autem ambo vel $= 0$ fiant vel $= \infty$, tum modo in praecedente Capite expoſito haec differentialia completa caſu $x = a$ inueſtigari debent. Plerumque etiam calculus mirifice contrahitur, ſi antea ponatur $x - a = t$ ſeu $x = a - t$, quo prodeat fractio $\frac{P}{Q}$, cuius numerator ac denominator euaneſcunt caſu $t = 0$; tum enim differentialia dP & dQ habebuntur, ſi vbique dt loco t ſubſtituatur.

EXEMPLUM I.

Quaeratur valor fractionis huius $\frac{b - \sqrt{(bb - tt)}}{tt}$ *caſu* $t = 0$.

Quoniam hoc caſu $t = 0$ & numerator & denominator euaneſcit, loco t tantum ſcribatur dt, atque valor quaeſitus exprimetur hac fractione $\frac{b - \sqrt{(bb - dt^2)}}{dt^2}$. Cum vero ſit $\sqrt{(bb - dt^2)} = b - \frac{dt^2}{2b}$, iſta fractio abit in hanc $\frac{dt^2}{2bdt^2} = \frac{1}{2b}$. Hinc fractio propoſita $\frac{b - \sqrt{(bb - tt)}}{tt}$ caſu $t = 0$ recipit hunc valorem $\frac{1}{2b}$.

EXEMPLUM II.

Quaeratur valor huius fractionis:

$$\frac{\sqrt{(aa+ax+xx)}-\sqrt{(aa-ax+xx)}}{\sqrt{(a+x)}-\sqrt{(a-x)}}$$

caſu $x=0$.

Hic iterum ſtatim dx loco x ſubſtitui poteſt; quo facto cum ſit:

$$\sqrt{(aa+adx+dx^2)}=a+\tfrac{1}{2}dx+\frac{3dx^2}{8a}$$

$$\sqrt{(aa-adx+dx^2)}=a-\tfrac{1}{2}dx+\frac{3dx^2}{8a}$$

atque $$\sqrt{(a+dx)}=\sqrt{a}+\frac{dx}{2\sqrt{a}}$$

$$\sqrt{(a-dx)}=\sqrt{a}-\frac{dx}{2\sqrt{a}}$$

fiet numerator $=dx$ & denominator $=\frac{dx}{\sqrt{a}}$, ex quo fractionis propoſitae valor quaeſitus erit $=\sqrt{a}$.

EXEMPLUM III.

Quaeratur valor huius fractionis:

$$\frac{x^3-4ax^2+7a^2x-2a^3-2a^2\sqrt{(2ax-aa)}}{xx-2ax-aa+2a\sqrt{(2ax-xx)}}$$

caſu $x=a$.

Si more conſueto differentialia ſumantur & in loca numeratoris ac denominatoris ſubſtituantur, habebitur: $\frac{3xx-8ax+7a^2-2a^3:\sqrt{(2ax-aa)}}{2x-2a+2a(a-x):\sqrt{(2ax-xx)}}$, cuius fractio-

nis

nis numerator ac denominator denuo euenescunt, ſi ponatur $x = a$. Quare ob eandem rationem eorum loco denuo ipſorum differentialia ſubſtituantur, prodibitque:

$\frac{6x - 8a + 2a^4 : (2ax - aa)^{\frac{3}{2}}}{2 - 2a^3 : (2ax - xx)^{\frac{3}{2}}}$; cuius numerator ac denominator iterum caſu $x = a$ euaneſcunt. Pergamus ergo eorum loco ipſorum differentialia ſubſtituere:

$$\frac{6 - 6a^5 : (2ax - xx)^{\frac{5}{2}}}{6a^3(a-x) : (2ax - xx)^{\frac{5}{2}}} = \frac{1 - a^5 : (2ax - xx)^{\frac{5}{2}}}{a^3(a-x) : (2ax - xx)^{\frac{5}{2}}}$$

Verum & hic poſito $x = a$ denuo tam numerator quam denominator euaneſcunt. Porro igitur differentialibus ipſorum loco ſubſtitutis, orietur:

$$\frac{5a^6 : (2ax - aa)^{\frac{7}{2}}}{-(5a^5 - 8a^4x + 4a^3xx) : (2ax - xx)^{\frac{7}{2}}}.$$

Nunc denique loco x ponatur, a prodibitque haec fractio determinata $\frac{5 : a}{-1 : a^2} = -5a$, qui eſt valor quaeſitus fractionis propoſitae:

Quodſi autem antequam haec inueſtigatio ſuſcipiatur, ponatur $x = a + t$, fractio propoſita transmutabitur in hanc:

$$\frac{2a^3 + 2a^2t - att + t^3 - 2a^2 \sqrt{(aa + 2at)}}{-2aa + tt + 2a\sqrt{(aa - tt)}}$$

quae cum recipiat formam $\frac{0}{0}$, ſi ponatur $t = 0$, ponatur dt loco t, & erit:

$2a^3 +$

$$\frac{2a^3 + 2a^2dt - adt^2 + dt^3 - 2a^2\sqrt{(aa + 2adt)}}{-2aa + dt^2 + 2a\sqrt{(aa - dt^2)}}$$

Conuertantur iam formulae irrationales in ſeries, quae eousque continuentur, quoad termini a membro rationali non amplius deſtruantur:

$$\sqrt{(aa + 2adt)} = a + dt - \frac{dt^2}{2a} + \frac{dt^3}{2aa} - \frac{5dt^4}{8a^3}$$

$$\sqrt{(aa - dt^2)} = a - \frac{dt^2}{2a} - \frac{dt^4}{8a^3}.$$

quibus valoribus ſubſtitutis prodibit fractio haec:

$$\frac{5dt^4 : 4a}{-dt^4 : 4aa} = -5a,$$

qui eſt valor fractionis propoſitae iam ante inuentus.

EXEMPLUM IV.

Inuenire valorem huius fractionis:

$$\frac{a + \sqrt{(2aa - 2ax)} - \sqrt{(2ax - xx)}}{a - x + \sqrt{(aa - xx)}}$$

caſu $x = a$.

Subſtitutis in loca numeratoris & denominatoris eorum differentialibus prodibit haec fractio, quae caſu $x = a$ ipſi propoſitae erit aequalis:

$$-\frac{a : \sqrt{(aa - 2ax)} - (a - x) : \sqrt{(2ax - xx)}}{-1 - x : \sqrt{(aa - xx)}}$$

cuius numerator ac denominator caſu $x = a$ fiunt infiniti. Verum ſi vterque per $-\sqrt{(a - x)}$ multiplicetur, habebitur

$a :$

$$\frac{a:\sqrt{2a}+(a-x)^{\frac{3}{2}}:\sqrt{(2ax-xx)}}{\sqrt{(a-x)}+x:\sqrt{(a+x)}}$$

quae poſito $x=a$ dabit hunc valorem determinatum, $\frac{a:\sqrt{2a}}{a:\sqrt{2a}}=1$, qui propterea aequalis eſt fractioni propoſitae caſu $x=a$.

359. Si igitur habeatur fractio $\frac{P}{Q}$, cuius numerator & denominator caſu $x=a$ euaneſcat, eius valor per conſuetas differentiandi regulas aſſignari poterit, neque opus erit ad differentialia, quae capite praecedente tractauimus, recurrere. Sumtis enim differentialibus fractio propoſita $\frac{P}{Q}$ caſu $x=a$ aequalis erit fractioni $\frac{dP}{dQ}$; cuius ſi numerator & denominator poſito $x=a$ induant valores finitos, cognoſcetur valor fractionis propoſitae; ſin autem alter fiat $=0$, manente altero finito, tum fractio erit vel $=0$ vel $=\infty$, prout vel numerator euaneſcat vel denominator. At ſi alteruter vel vterque fiat $=\infty$, quod euenit, ſi diuidantur per quantitates caſu $x=a$ euaneſcentes, tum multiplicando vtrumque per hos diuiſores, iſtud incommodum tolletur, vti in exemplo poſtremo euenit. Quodſi vero tam numerator quam denominator caſu $x=a$ denuo euaneſcat, tum iterum, vti initio factum eſt, differentialia erunt capienda, ita vt haec fractio $\frac{ddP}{ddQ}$ prodeat, quae caſu $x=a$ propoſitae adhuc erit aequalis; &, ſi idem rurſus in hac fractione

 vſu

vſu veniat, vt fiat $= \frac{0}{0}$, tum in eius locum ſurrogetur haec $\frac{d^3 P}{d^3 Q}$, atque ita porro, donec ad fractionem perveniatur, quae valorem determinatum exhibeat, ſiue finitum ſiue infinite magnum ſiue infinite paruum. Sic in exemplo tertio oportebat ad fractionem $\frac{d^4 P}{d^4 Q}$ progredi, antequam valorem fractionis propoſitae $\frac{P}{Q}$ aſſignari licuerit.

360. Vſus huius inueſtigationis elucet in definiendis ſummis ſerierum, quas ſupra capite II. §. 22. eruimus, ſi ponatur $x = 1$. Ex iis enim, quae ibi tradita ſunt, ſequitur fore:

$$x + x^2 + x^3 + \ldots + x^n = \frac{x - x^{n+1}}{1 - x}$$

$$x + x^3 + x^5 + \ldots + x^{2n-1} = \frac{x - x^{2n+1}}{1 - xx}$$

$$x + 2x^2 + 3x^3 + \ldots + nx^n = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}$$

$$x + 3x^3 + 5x^5 + \ldots + (2n-1)x^{2n-1} = \frac{x + x^3 - (2n+1)x^{2n+1} + (2n-1)x^{2n+3}}{(1 - xx)^2}$$

$$x + 4x^2 + 9x^3 + \ldots + n^2 x^n = \frac{x + x^2 - (n+1)^2 x^{n+1} + (2nn + 2n - 1)x^{n+2} - nnx^{n+3}}{(1-x)^3}$$

&c.

Quod ſi nunc harum ſerierum ſummae deſiderentur caſu

ſu quo $x = 1$, in expreſſionibus iſtis tam numerator quam denominator euaneſcunt. Valores ergo harum ſummarum caſu $x = 1$ methodo hic expoſita definiri poterunt. Quoniam vero eaedem ſummae aliunde conſtant, ex conſenſu veritas huius methodi magis elucebit.

EXEMPLUM I.

Definire valorem huius fractionis $\frac{x - x^{n+1}}{1 - x}$ *caſu* $x = 1$, *qui exhibebit ſummam ſeriei* $1 + 1 + 1 + \ldots + 1$ *ex* n *terminis conſtantis, quae propterea erit* $= n$.

Quoniam caſu $x = 1$ numerator ac denominator euaneſcit, ſubſtituantur differentialia in eorum locum, habebiturque $\frac{1 - (n+1)x^n}{-1}$, quae poſito $x = 1$ dat n pro ſumma ſeriei quaeſita.

EXEMPLUM II.

Definire valorem fractionis $\frac{x - x^{2n+1}}{1 - xx}$ *caſu* $x = 1$, *qui exhibebit ſummam ſeriei* $1 + 1 + 1 + \ldots + 1$ *ex* n *terminis conſtantis, quae propterea erit* $= n$.

Sumtis differentialibus fractio propoſita transmutatur in hanc: $\frac{1 - (2n+1)x^{2n}}{-2n}$, cuius valor poſito $x = 1$, erit $= n$.

EXEMPLUM III.

Inuenire valorem huius fractionis: $\frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}$

caſu $x=1$, *qui exprimet ſummam ſeriei* $1+2+3+\ldots+n$,

quam conſtat eſſe $=\frac{nn+n}{2}$.

Sumtis differentialibus peruenietur ad hanc fractionem $\frac{1-(n+1)^2x^n+n(n+2)x^{n+1}}{-2(1-x)}$, cuius adhuc tam numerator quam denominator caſu $x=1$ euaneſcit. Hinc denuo differentialia ſumantur, vt prodeat haec fractio: $-\frac{n(n+1)^2x^{n-1}+n(n+1)(n+2)x^n}{2}$, quae poſito $x=1$ abit in $\frac{n(n+1)}{2}=\frac{nn+n}{2}$ ſummam ſeriei propoſitae.

EXEXPLUM IV.

Inuenire valorem huius fractionis:

$$\frac{x+x^3-(2n+1)x^{2n+1}+(2n-1)x^{2n+3}}{(1-xx)^2}$$

caſu $x=1$, *qui exprimet ſummam ſeriei*

$1+3+5+\ldots+(2n-1)$

quam conſtat eſſe $=nn$.

Subſtitutis differentialibus in loca numeratoris & denominatoris prouenit haec fractio:

$$\frac{1+3xx-(2n+1)^2x^{2n}+(2n-1)(2n+3)x^{2n+2}}{-4x(1-xx)}$$

quae

quae cum adhuc idem incommodum habeat, vt poſito $x = 1$ abeat in $\frac{0}{0}$, denuo differentialia ſumantur,

$$\frac{6x - 2n(2n+1)^2 x^{2n-1} + (2n-1)(2n+2)(2n+3)x^{2n+1}}{-4+12xx}$$

quae poſito $x = 1$ abit in:

$$\frac{6 - 2n(2n+1)^2 + (2n-1)(2n+2)(2n+2)}{8} = nn.$$

EXEMPLUM. V.

Inuenire valorem huius fractionis:

$$\frac{x + x^2 - (n+1)^2 x^{n+1} + (2nn+2n-1)x^{n+2} - nnx^{n+3}}{(1-x)^3}$$

caſu $x = 1$, *qui dabit ſummam ſeriei* $1 + 4 + 9 + \ldots + n^2$, *quam conſtat eſſe* $= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$.

Sumtis numeratoris ac denominatoris differentialibus, fiet

$$\frac{1 + 2x - (n+1)^3 x^n + (n+2)(2nn+2n-1)x^{n+1} - nn(n+3)x^{n+2}}{-3(1-x)^2}$$

in qua cum numerator ac denominator poſito $x = 1$ denuo euaneſcat, differentialia ſecunda ſumantur:

$$\frac{2 - n(n+1)x^{n-1} + (n+1)(n+2)(2nn+3n-1)x^n - n^2(n+2)(n+3)x^{n+1}}{6(1-x)}$$

Eodem vero adhuc ſubſiſtente incommodo, ad differentialia tertia procedatur, vt prodeat haec fractio:

$$\frac{-n(n-1)(n+1)^3 x^{n-2} + n(n+1)(n+2)(2nn+2n-1)x^{n-1} - n^2(n+1)(n+2)(n+3)x^n}{-6}$$

 quae

quae tandem poſito $x = 1$ abit in hanc formam determinatam:

$$\frac{-n(n-1)(n+1)^3 + n(n+1)(n+2)(nn-n-1)}{-6} = \frac{n(n+1)(2n+1)}{6} =$$

$= \frac{1}{3}n^2 + \frac{1}{2}n^2 + \frac{1}{6}n$; qui eſt ille ipſe valor, quo ſeriem memoratam exprimi inuenimus.

EXEMPLM VI.

Sit propoſita iſta fractio $\frac{x^m - x^{m+n}}{1 - x^{2p}}$, *cuius valorem caſu* $x = 1$ *aſſignari oporteat.*

Quoniam haec fractio eſt productum ex his duabus: $\frac{x^m}{1+x^p} \cdot \frac{1-x^n}{1-x^p}$, prioris autem factoris caſu $x = 1$ valor eſt $= \frac{1}{2}$, tantum opus eſt vt alterius factoris $\frac{1-x^n}{1-x^p}$ valor eodem caſu quaeratur, qui ſumtis differentialibus erit $= \frac{nx^{n-1}}{px^{p-1}} = \frac{n}{p}$: vnde fractionis propoſitae valor caſu $x = 1$, erit $= \frac{n}{2p}$. Idem valor prodit, ſi immediate differentialia in fractione propoſita capiantur: fiet enim $\frac{mx^{m-1} - (m+n)x^{m+n-1}}{-2px^{2p-1}}$, cuius valor poſito $x = 1$, erit $= \frac{-n}{-2p} = \frac{n}{2p}$, vt ante.

361. Eadem methodo erit vtendum, ſi in fractione propoſita $\frac{P}{Q}$ vel numerator vel denominator vel

vterque

vterque fuerit quantitas tranſcendens. Quae operationes, quo clarius explicentur, ſequentia exempla adiicere viſum eſt.

EXEMPLUM I.

Sit propoſita iſta fractio $\frac{a^n - x^n}{la - lx}$, *cuius valor quaeratur caſu* $x = a$.

Sumtis differentialibus ſtatim peruenitur ad hanc fractionem $-\frac{nx^{n-1}}{-1:x} = nx^n$, cuius valor poſito $x = a$, erit na^n.

EXEMPLUM II.

Sit propoſita iſta fractio $\frac{lx}{\sqrt{(1-x)}}$, *cuius valor quaeritur caſu* $x = 1$.

Sumtis differentialibus numeratoris & denominatoris prodit $\frac{1:x}{-1:2\sqrt{(1-x)}} = \frac{-2\sqrt{(1-x)}}{x}$, cuius valor poſito $x = 1$, cum ſit $= 0$, ſequitur fractionem $\frac{lx}{\sqrt{(1-x)}}$ caſu $x = 1$ euaneſcere.

EXEMPLUM III.

Sit propoſita iſta fractio $\frac{a - x - ala + alx}{a - \sqrt{(2ax - xx)}}$, *cuius valor quaeratur poſito* $x = a$, *quo caſu numerator & denominator euaneſcunt.*

Differentiatis ſecundum regulam numeratore ac denominatore

minatore erit $\frac{-1+a:x}{-(a-x):\sqrt{(2ax-xx)}} = \frac{(a-x)\sqrt{(2ax-xx)}}{-x(a-x)}$; vbi etſi numerator ac denominator caſu $x = a$ adhuc euaneſcit, tamen quia vterque diuiſibilis eſt per $a - x$, habebitur ita fractio $-\sqrt{\frac{2a-x}{x}}$, cuius valor caſu $x = a$ eſt determinatus atque $= -1$; abitque igitur fractio propoſita in -1, ſi ponatur $x = a$.

EXEMPLUM IV.

Sit propoſita iſta fractio $\frac{e^x - e^{-x}}{l(1+x)}$, *cuius valor quaeritur poſito* $x = 0$.

Sumtis differentialibus habebitur iſta functio $\frac{e^x + e^{-x}}{1:(1+x)}$, quae poſito $x = 0$ dat 2 pro valore quaeſito.

EXEMPLUM V.

Inuenire valorem huius fractionis $\frac{e^x - 1 - l(1+x)}{xx}$, *caſu quo ponitur* $x = 0$.

Si loco numeratoris ac denominatoris eorum differentialia ſubſtituantur, orietur haec fractio $\frac{e^x - 1 : (1+x)}{2x}$, quae cum adhuc abeat in $\frac{0}{0}$, ſi ponatur $x = 0$, denuo differentialia ſumantur, vt habeatur $\frac{e^x + 1 : (1+x)^2}{2}$, quae poſito $x = 0$ praebet $\frac{1+1}{2} = 1$. Quod idem patet ſi loco

loco x ſtatim $0 + dx$ ſubſtituatur: cum enim ſit $e^{dx} = 1 + dx + \frac{1}{2} dx^2 + \&c.$ & $l(1+dx) = dx - \frac{1}{2} dx^2 + \&c.$ $\frac{e^{dx} - 1 - l(1+dx)}{dx^2} = \frac{dx^2}{dx^2} = 1.$

EXEMPLUM VI.

Quaeratur valor fractionis $\frac{x^n}{lx}$, *caſu quo ponitur* $x = \infty$.

Quo iſta fractio ad formam, quae hoc caſu tranſeat in $\frac{0}{0}$ reducatur, ita repraeſentetur $\frac{1:lx}{1:x^n}$: ſic enim caſu $x = \infty$, tam numerator quam denominator euaneſcet. Ponatur vero porro $x = \frac{1}{y}$, ita vt caſu $x = \infty$, fiat $y = 0$, atque proponetur iſta fractio $-\frac{1:ly}{y^n}$, cuius valor caſu $y = 0$ inueſtigari debet. Sumtis autem differentialibus erit $\frac{1:y(ly)^2}{ny^{n-1}} = \frac{1:(ly)^2}{ny^n}$, quae poſito $y = 0$, cum abeat in $\frac{0}{0}$, ſumantur denuo differentialia, eritque $\frac{-2:(ly)^3}{n^2y^n}$; vbi quia idem incommodum adeſt, ſi porro differentialia ſumantur prodibit $\frac{6:(ly)^4}{n^3y^n}$, ſicque quousque procedamus, perpetuo idem incommodum occurret. Quamobrem vt hoc non obſtante valorem quaeſitum eruamus, ſit s valor fractionis $-\frac{1:ly}{y^n}$ caſu, quo poni-

ponitur $y = 0$, & cum eodem caſu ſit quoque $s = \frac{1:(ly)^2}{ny^n}$; erit ex illa aequatione $ss = \frac{1:(ly)^2}{y^{2n}}$, quae per iſtam diuiſa dabit $s = \frac{ny^n}{y^{2n}} = \frac{n}{y^n}$, ex qua perſpicitur caſu $y = 0$ fieri s infinitum. Fit ergo fractionis $-\frac{1:ly}{y^n}$ valor caſu $y = 0$ infinitus, ideoque poſito $y = dx$, habebit $\frac{1}{ldx}$ ad dx^n rationem infinitam, vti iam ſupra innuimus.

EXEMPLUM VII.

Quaeratur valor fractionis $\frac{x^n}{e^{-1:x}}$ *caſu* $x = 0$, *quo tam numerator quam denominator euaneſcit.*

Sit hoc caſu $\frac{x^n}{e^{-1:x}} = s$, erit ſumtis differentialibus quoque $s = \frac{nx^{n-1}}{e^{-1:x}:xx} = \frac{nx^{n+1}}{e^{-1:x}}$, & quia hic idem incommodum occurrit, perpetuoque recurrit, quousque differentiationes continuentur, remedio ante adhibito vtamur. Prior aequatio dat $x^n = e^{-1:x}s$, & $x^{n(n+1)} = e^{-(n+1):x}s^{n+1}$ altera aequatio dat $x^{n+1} = e^{-1:x}s:n$, vnde fit $x^{n(n+1)} = e^{-n:x}s^n:n^n$, qui valor illi aequatus dabit $e^{-1:x}sn^n = 1$ ideoque $s = \frac{1}{n^n e^{-1:x}} = \infty$, ſi $x = 0$. Quare poſito x infinite paruo habebit dx^n ad $e^{-1:dx}$ rationem infinite

finite magnam, quicunque numerus finitus pro n ſtatuatur: vnde ſequitur $e^{-1:dx}$ eſſe infinite paruum homogeneum cum dx^m, ſi m fuerit numerus infinite magnus.

EXEMPLUM VIII.

Quaeratur valor fractionis $\frac{1 - \sin x + \cos x}{\sin x + \cos x - 1}$ *caſu, quo ponitur* $x = \frac{\pi}{2}$ *ſeu arcui* 90 *graduum.*

Sumtis differentialibus obtinebitur haec fractio $-\frac{\cos x - \sin x}{\cos x - \sin x}$, quae poſito $x = \frac{\pi}{2}$ ob ſin $x = 1$ & cos $x = 0$ abit in 1: ita vt vnitas ſit valor quaeſitus fractionis propoſitae. Quod idem patet ſine differentiatione: cum enim ſit $\cos x = \sqrt{(1 + \sin x)(1 - \sin x)}$ fractio propoſita abit in hanc $\frac{\sqrt{(1 - \sin x)} + \sqrt{(1 + \sin x)}}{\sqrt{(1 + \sin x)} - \sqrt{(1 - \sin x)}}$, quae ſit euidenter $= 1$, ſi fiat ſin $x = 1$.

EXEMPLUM IX.

Inuenire valorem huius expreſſionis $\frac{x^x - x}{1 - x + lx}$ *caſu quo ponitur* $x = 1$.

Loco numeratoris & denominatoris eorum differentialibus ſubſtitutis prodibit iſta fractio: $\frac{x^x(1 + lx) - 1}{-1 + 1 : x}$; quae cum etiam nunc fiat $= \frac{0}{0}$ poſito $x = 1$, ſuman-

 tur

tur denuo differentialia, vt prodeat $\frac{x^x(1+lx)^2 + x^x : x}{-1 : xx}$, quae poſito $x = 1$ abit in -2, qui eſt valor fractionis propoſitae caſu $x = 1$.

362. Quoniam hic omnes expreſſiones, quae quibusdam caſibus indeterminatos valores recipere videntur, pertractare conſtituimus, huc non ſolum pertinent eae fractiones $\frac{P}{Q}$, quarum numerator ac denominator certo caſu euaneſcunt; ſed etiam eiusmodi fractiones, quarum numerator ac denominator certo caſu fiunt infiniti, huc ſunt referendae: propterea quod earum valores aeque indeterminati videntur. Si ſcilicet P & Q eiusmodi fuerint functiones ipſius x, vt caſu quopiam $x = a$, ambae fiant infinitae, fractioque $\frac{P}{Q}$ induat hanc formam $\frac{\infty}{\infty}$; quoniam infinita aeque ac cyphrae inter ſe rationem quamcunque tenere poſſunt, hinc valor verus minime cognoſci poteſt. Hic quidem caſus ad praecedentem reuocari poteſt, fractionem $\frac{P}{Q}$ in hanc formam $\frac{1 : Q}{1 : P}$ transmutando, cuius fractionis nunc numerator ac denominator caſu $x = a$ euaneſcunt; ideoque eius valor modo ante tradito inueniri poteſt. At vero quoque ſine hac transformatione valor inuenietur, ſi loco x non a, ſed $a + dx$ ſubſtituatur, quo facto non eiusmodi infinita abſoluta ∞ prouenient; ſed ita erunt expreſſa

 1

$\frac{1}{dx}$ vel $\frac{A}{dx^n}$; quae expreſſiones etſi ſunt aeque infinitae ac ∞, tamen comparatione inter dx eiusue poteſtates inſtituta, valor quaeſitus facile colligetur.

363. Ad eandem claſſem quoque pertinent producta ex duobus factoribus conſtantia, quorum alter certo caſu $x = a$ euaneſcit, alter vero in infinitum abit: cum enim quaeuis quantitas per huiusmodi productum $0.\infty$ repraeſentari poſſit, eius valor indefinitus videtur. Sit PQ huiusmodi productum, in quo, ſi ponatur $x = a$, fiat $P = 0$ & $Q = \infty$, eius valor per praecepta ante tradita inuenietur, ſi ponatur $Q = \frac{1}{R}$, tum enim productum PQ transmutabitur in fractionem $\frac{P}{R}$: cuius numerator ac denominator ante caſu $x = a$ euaneſcunt, ideoque eius valor methodo ambo expoſita inueſtigari poterit. Sic ſi quaeratur valor huius producti $(1-x)\tang\frac{\pi x}{2}$ caſu $x = 1$, quo fit $1 - x = 0$ & $\tang\frac{\pi x}{2} = \infty$, convertatur id in hanc fractionem $\frac{1-x}{\cot\frac{1}{2}\pi x}$, cuius numerator ac denominator caſu $x = 1$ euaneſcunt. Cum igitur ſit differentiale numeratoris $(1-x) = -dx$, & differentiale denominatoris $\cot\frac{\pi x}{2} = -\frac{\pi dx : 2}{(\sin\frac{1}{2}\pi x)^2}$, caſu $x = 1$

 valor

valor fractionis propoſitae erit $= \frac{2}{\pi} \sin \frac{\pi x}{2} . \sin \frac{\pi x}{2} = \frac{2}{\pi}$, ob $\sin \frac{\pi}{2} = 1$.

364. Imprimis autem huc ſunt referendae eiusmodi expreſſiones, quae dum ipſi x certus quidam valor tribuitur, abeunt in huiusmodi formam $\infty - \infty$: quoniam enim duo infinita quauis quantitate finita inter ſe diſcrepare poſſunt, manifeſtum eſt hoc caſu valorem expreſſionis non determinari, niſi differentia inter illa duo infinita aſſignari poſſit. Iſte ergo caſus occurrit, ſi proponatur huiusmodi functio $P - Q$, in qua poſito $x = a$ fiat tam $P = \infty$ quam $Q = \infty$, quo caſu ope regulae ante traditae valor quaeſitus non tam facile aſſignari poteſt. Etſi enim poſito hoc caſu fieri $P - Q = f$, ſtatuatur $e^{P-Q} = e^f$, ita vt ſit $e^f = \frac{e^{-Q}}{e^{-P}}$, vbi caſu $x = a$ tam numerator e^{-Q} quam denominator e^{-P} euaneſcit; tamen ſi regula ante tradita huc transferatur, fiet $e^f = \frac{e^{-Q} dQ}{e^{-P} dP}$, vnde ob $e^f = \frac{e^{-Q}}{e^{-P}}$, fieret $1 = \frac{dQ}{dP}$, ideoque valor quaeſitus ipſius f hinc non innoteſcit. Quoties quidem P & Q ſunt quantitates algebraicae, quoniam hae infinitae fieri nequeunt, niſi ſint fractiones, quarum denominatores euaneſcunt; tum $P - Q$ in vnicam fractionem colligi poterit, cuius denominator

 pari-

pariter euaneſcet. Quo facto ſi etiam numerator euaneſcat, valor modo ſupra explicato definietur: ſin autem numerator non euaneſcat, tum eius valor reuera erit infinitus. Sic ſi huius expreſſionis $\frac{1}{1-x}-\frac{2}{1-xx}$ valor deſideretur caſu $x=1$, quia ea abit in $\frac{1-x}{1-xx}=\frac{-1}{1+x}$, patet valorem quaeſitum eſſe $=-\frac{1}{2}$.

365. Verum ſi functiones P & Q fuerint transcendentes, tum plerumque haec transformatio ad calculum moleſtiſſimum perduceret. Expediet ergo his caſibus methodo directa vti, atque loco $x=a$, quo ambae quantitates P & Q in infinitum abeunt, poni $x=a+\omega$, exiſtente ω quantitate infinite parua, pro qua dx accipi poterit. Quo facto ſi fiat $P=\frac{A}{\omega}+B$ & $Q=\frac{A}{\omega}+C$, manifeſtum eſt functionem $P-Q$ abituram eſſe in $B-C$, qui erit valor finitus. Rationem igitur huiusmodi functionum valores inueſtigandi ſequentibus exemplis illuſtrabimus.

EXEMPLUM I.

Quaeratur valor huius expreſſionis $\frac{x}{x-1}-\frac{1}{lx}$ *caſu, quo ponitur* $x=1$.

Quoniam tam $\frac{x}{x-1}$ quam $\frac{1}{lx}$ fit infinitum poſito $x=1$, ſtatuatur $x=1+\omega$, atque expreſſio propoſita transformabitur in $\frac{1+\omega}{\omega}-\frac{1}{l(1+\omega)}$. Cum igitur ſit

$l(1+\omega)$

$$l(1+\omega) = \omega - \tfrac{1}{2}\omega^2 + \tfrac{1}{3}\omega^3 - \&c. = \omega(1 - \tfrac{1}{2}\omega + \tfrac{1}{3}\omega^2 - \&c.)$$

habebitur

$$\frac{(1+\omega)(1-\frac{1}{2}\omega+\frac{1}{3}\omega^2-\&c.)-1}{\omega(1-\frac{1}{2}\omega+\frac{1}{3}\omega^2-\&c.)} = \frac{\frac{1}{2}\omega-\frac{1}{6}\omega^2+\&c.}{\omega(1-\frac{1}{2}\omega+\frac{1}{3}\omega^2-\&c.)}$$

$$= \frac{\frac{1}{2}-\frac{1}{6}\omega+\&c.}{1-\frac{1}{2}\omega+\frac{1}{3}\omega^2-\&c.}$$

Poſito nunc ω infinite paruo ſeu $\omega = 0$, manifeſtum eſt valorem quaeſitum eſſe $= \frac{1}{2}$.

EXEMPLUM II.

Denotantibus e *numerum, cuius logarithmus hyperbolicus eſt* $= 1$, *&* π *ſemicircumferentiam circuli, cuius radius eſt* $= 1$, *inueſtigare valorem huius expreſſionis:*

$$\frac{\pi x - 1}{2xx} + \frac{\pi}{x(e^{2\pi x}-1)},\ \textit{caſu}\ x = 0.$$

Expreſſio iſta propoſita exhibet ſummam huius ſeriei:

$$\frac{1}{1+xx} + \frac{1}{4+xx} + \frac{1}{9+xx} + \frac{1}{16+xx} + \frac{1}{25+xx} + \&c.$$

vnde ſi ponatur $x = 0$, prodire debet ſumma ſeriei huius $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \&c.$ quam conſtat eſſe $= \frac{\pi\pi}{6}$. Facto autem $x = 0$ expreſſionis propoſitae $\frac{\pi x - 1}{2xx} + \frac{\pi}{x(e^{2\pi x}-1)}$ valor maxime videtur indeterminatus, ob omnes terminos infinitos. Ponatur ergo $x = \omega$, exiſtente ω quantitate infinite parua, atque membrum prius $\frac{\pi x - 1}{2xx}$ abit in $-\frac{1}{2\omega^2} + \frac{\pi}{2\omega}$. Cum deinde

ſit

ſit $e^{2\pi\omega} - 1 = 2\pi\omega + 2\pi^2\omega^2 + \frac{4}{3}\pi^3\omega^3 + \&c.$ alterum membrum $\frac{\pi}{x(e^{2\pi x} - 1)}$ abit in

$$\frac{\pi}{\omega(2\pi\omega + 2\pi^2\omega^2 + \frac{4}{3}\pi^3\omega^3 + \&c.)} = \frac{1}{2\omega^2(1 + \pi\omega + \frac{2}{3}\pi^2\omega^2 + \&c.)}$$

At eſt $\frac{1}{1 + \pi\omega + \frac{2}{3}\pi^2\omega^2 + \&c.} = 1 - \pi\omega + \frac{1}{3}\pi^2\omega^2 - \&c.$ vnde poſterius membrum fit $= \frac{1}{2\omega^2} - \frac{\pi}{2\omega} + \frac{1}{6}\pi^2 - \&c.$ ad quod ſi prius addatur prodit $\frac{1}{6}\pi^2$, qui eſt valor quaeſitus expreſſionis propoſitae caſu $x = 0$.

Idem quoque per methodum fractionum, quarum numerator ac denominator certo caſu euaneſcunt, praeſtari poteſt: expreſſio enim propoſita in hanc fractionem transmutatur: $\frac{\pi x e^{2\pi x} - e^{2\pi x} + \pi x + 1}{2xxe^{2\pi x} - 2xx}$, cuius numerator ac denominator caſu $x = 0$ euaneſcunt. Sumtis ergo differentialibus oritur:

$$\frac{\pi e^{2\pi x} + 2\pi\pi x e^{2\pi x} - 2\pi e^{2\pi x} + \pi}{4xe^{2\pi x} + 4\pi xxe^{2\pi x} - 4x}$$

ſiue haec $\frac{\pi - \pi e^{2\pi x} + 2\pi\pi x e^{2\pi x}}{4xe^{2\pi x} + 4\pi xxe^{2\pi x} - 4x}$

cuius, ſi ponatur $x = 0$, adhuc numerator ac denominator euaneſcunt. Quare ſumtis denuo differentialibus habebitur:

$$\frac{-2\pi\pi e^{2\pi x} + 2\pi\pi e^{2\pi x} + 4\pi^3 x e^{2\pi x}}{4e^{2\pi x} + 8\pi xe^{2\pi x} + 8\pi xe^{2\pi x} + 8\pi^2 xxe^{2\pi x} - 4}$$

 ſeu

ſeu $$\frac{\pi^3 x e^{2\pi x}}{e^{2\pi x} + 4\pi x e^{2\pi x} + 2\pi^2 x^2 e^{2\pi x} - 1}$$

ſeu $$\frac{\pi^3 x}{1 + 4\pi x + 2\pi^2\pi^2 - e^{-2\pi x}}$$

cuius numerator ac denominator adhuc euaneſcunt caſu $x = 0$. Quocirca iterum differentialia ſumantur

$$\frac{\pi^3}{4\pi + 4\pi^2 x + 2\pi e^{-2\pi x}}$$

quae fractio poſito $x = 0$ abit in $\frac{\pi^2}{6}$, vt ante.

EXEMPLUM III.

Retinentibus e *&* π *eosdem valores, quaeratur valor expreſſionis huius caſu* $x = 0$

$$\frac{\pi}{4x} - \frac{\pi}{2x(e^{\pi x}+1)}.$$

Expreſſio haec transmutatur in hanc: $\frac{\pi e^{\pi x} - \pi}{4x e^{\pi x} + 4x}$ cuius numerator ac denominator caſu $x = 0$ euaneſcunt. Ponatur ergo $x = \omega$, & cum ſit

$$e^{\pi\omega} = 1 + \pi\omega + \tfrac{1}{2}\pi^2\omega^2 + \tfrac{1}{6}\pi^3\omega^3 + \&c.$$

formula propoſita transmutatur in hanc:

$$\frac{\pi^2\omega + \tfrac{1}{2}\pi^3\omega^2 + \tfrac{1}{6}\pi^4\omega^3 + \&c.}{8\omega + 4\pi\omega^2 + 2\pi^2\omega^3 + \&c.}$$

quae poſito ω infinite paruo ſtatim dat $\frac{1}{8}\pi^2$, qui eſt valor quaeſitus expreſſionis propoſitae caſu $x = 0$. At vero expreſſio propoſita $\frac{\pi}{4x} - \frac{\pi}{2x(e^{\pi x}+1)}$, exhibet ſum-

mam

mam huius ſeriei $\frac{1}{1+xx}+\frac{1}{9+xx}+\frac{1}{25+xx}+\frac{1}{49+xx}+$ &c. cuius ſumma poſito $x=0$ vtique fit $=\frac{1}{8}\pi^2$.

EXEMPLUM IV.

Quaeratur valor huius expreſſionis $\frac{1}{2xx}-\frac{\pi}{2x\,tang\,\pi x}$ *caſu* $x=0$.

Formula haec propoſita $\frac{1}{2xx}-\frac{\pi}{2x\,\text{tang}\,\pi x}$ exprimit ſummam huius ſeriei infinitae

$$\frac{1}{1-xx}+\frac{1}{4-xx}+\frac{1}{9-xx}+\frac{1}{16-xx}+\text{\&c.}$$

Si igitur ponatur $x=0$, prodire debet ſumma ſeriei $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+$ &c. quae eſt $=\frac{1}{6}\pi\pi$.

Quoniam eſt tang $\pi x=\frac{\text{ſin}\,\pi x}{\text{coſ}\,\pi x}$, expreſſio propoſita induet hanc formam: $\frac{1}{2xx}-\frac{\pi\,\text{coſ}\,\pi x}{2x\,\text{ſin}\,\pi x}=\frac{\text{ſin}\,\pi x-\pi x\,\text{cſ}\,\pi x}{2xx\,\text{ſin}\,\pi x}$ cuius numerator ac denominator euaneſcit poſito $x=0$. Ponatur ergo $x=\omega$ & cum ſit

$$\text{ſin}\,\pi x=\pi\omega-\tfrac{1}{6}\pi^3\omega^3+\text{\&c.}$$
$$\text{coſ}\,\pi x=1-\tfrac{1}{2}\pi^2\omega^2+\text{\&c.}$$

expreſſio propoſita fiet:

$$\frac{\pi\omega-\frac{1}{6}\pi^3\omega^3+\text{\&c.}-\pi\omega+\frac{1}{2}\pi^3\omega^3-\text{\&c.}}{2\pi\omega^3-\frac{1}{3}\pi^3\omega^5+\text{\&c.}}=\frac{\frac{1}{3}\pi^3\omega^3-\text{\&c.}}{2\pi\omega^3-\text{\&c.}}$$

quae ob ω infinite paruum dat $\frac{1}{6}\pi^2$.

EXEMPLUM V.

Cum ſit ſumma huius ſeriei infinitae

$$\frac{1}{1-xx}+\frac{1}{9-xx}+\frac{1}{25-xx}+\frac{1}{49-xx}+\&c.=\frac{\pi\,\mathit{ſin}\,\frac{1}{2}\pi x}{4x\,\mathit{cſ}\,\frac{1}{2}\pi x}$$

inuenire eius ſummam, ſi fuerit $x=0$.

Quia eſt ſin $\frac{1}{2}\pi x = \frac{1}{2}\pi x - \frac{1}{48}\pi^3 x^3 + \&c.$ & coſ $\frac{1}{2}\pi x = 1 - \frac{1}{8}\pi^2 x^2 + \&c.$ erit expreſſio propoſita

$$=\frac{\frac{1}{2}\pi^2 x - \frac{1}{48}\pi^4 x^3 + \&c.}{4x - \frac{1}{2}\pi^2 x^3 + \&c.} = \frac{\frac{1}{2}\pi^2 - \frac{1}{48}\pi^4 x^2 + \&c.}{4 - \frac{1}{2}\pi^2 x^2 + \&c.}$$

in qua ſi fiat $x=0$, valor erit manifeſto $=\frac{1}{8}\pi^2$, quam eſſe ſummam ſeriei $1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\&c.$ ſupra pluribus modis eſt demonſtratum. Sin autem pro x ſumatur numerus par quicunque, ſumma ſeriei propoſitae ſemper eſt $=0$.

366. In his ſeriebus, quas binis vltimis exemplis tractauimus, aliisque litteram variabilem x continentibus, ipſi x eiusmodi valores tribui poſſunt, vt quidam termini in infinitum excreſcant, quibus quidem caſibus ſumma totius ſeriei fiet infinita. Sic ſeries:

$$\frac{1}{1-xx}+\frac{1}{4-xx}+\frac{1}{9-xx}+\frac{1}{16-xx}+\&c.$$

ſi pro x ponatur numerus quicunque integer, vnus perpetuo terminus ob denominatorem euaneſcentem fit infinitus; hancque ob cauſam ipſa ſeriei ſumma infinita euadet. Quodſi autem iſte terminus infinitus ex ſerie tolla-

tollatur, tum ſumma reliqua ſine dubio erit finita, exprimeturque ſumma priori infinita termino iſto infinito mulctata, hoc modo $\infty - \infty$: quemnam ergo habitura ſit valorem determinatum modo hic expoſito inueniri poterit; id quod clarius ex ſubiunctis exemplis perſpicietur.

EXEMPLUM I.

Inuenire ſummam ſeriei

$$\frac{1}{1-xx} + \frac{1}{4-xx} + \frac{1}{9-xx} + \frac{1}{16-xx} + \&c.$$

caſu $x = 1$, *& demto termino primo, qui hoc caſu in infinitum augetur.*

Quia in genere ſumma eſt $= \frac{1}{2xx} - \frac{\pi}{2x\,\text{tang.}\,\pi x}$, erit ſumma quaeſita $= \frac{1}{2xx} - \frac{\pi}{2x\,\text{tang}\,\pi x} - \frac{1}{1-xx}$ poſito $x = 1$. Sit $x = 1 + \omega$, & habebitur pro ſumma quaeſita $\frac{1}{2(1+2\omega+\omega\omega)} - \frac{\pi}{2(1+\omega)\,\text{tang}(\pi+\omega\pi)} + \frac{1}{2\omega+\omega\omega}$. At eſt $\text{tang}(\pi+\omega\pi) = \text{tang}\,\omega\pi = \pi\omega + \frac{1}{3}\pi^3\omega^3 + \&c.$ Vnde cum primus terminus $\frac{1}{2xx}$ poſito $x = 1$ determinatum habeat valorem $\frac{1}{2}$, duo reliqui tantum termini ſunt ſpectandi, qui erunt

$$\frac{1}{\omega(2+\omega)} - \frac{\pi}{2\omega(1+\omega)(\pi+\frac{1}{3}\pi^3\omega^2)} = \frac{1}{\omega(2+\omega)} - \frac{1}{\omega(2+2\omega)(1+\frac{1}{3}\pi^2\omega^2)}$$

ſi quidem ω ſit infinite paruum, quo caſu etiam ter-

 mi-

minus $\frac{1}{3}\pi^2\omega^2$ negligi poterit. Proueniet autem $\frac{\omega}{\omega(2+\omega)\,(2+2\omega)} = \frac{1}{4}$ poſito $\omega = 0$, eſtque ergo $\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$ ſumma ſeriei: $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+$ &c. vti aliunde conſtat.

EXEMPLUM II.

Inuenire ſummam ſeriei

$$\frac{1}{1-xx}+\frac{1}{4-xx}+\frac{1}{9-xx}+\frac{1}{16-xx}+\text{\&c.}$$

caſu quo pro x *ponitur numerus quicunque integer* n *& demto ex ſerie termino illo* $\frac{1}{nn-xx}$, *qui ſit infinitus.*

Summa ergo haec, quae quaeritur, ita erit expreſſa $\frac{1}{2xx}-\frac{\pi}{2x\,\mathrm{tang}\,\pi x}-\frac{1}{nn-xx}$, ſi quidem ſtatuatur $x = n$, quo quidem caſu primus terminus $\frac{1}{2xx}$ abit in $\frac{1}{2nn}$, bini vero reliqui ambo fiunt infiniti. Ponatur ergo $x = n+\omega$, & cum ſit $\mathrm{tang}(\pi n+\pi\omega) = \mathrm{tang}\,\pi\omega = \pi\omega$, poſito ω infinite paruo, habebimus pro ſumma quaeſita:

$$\frac{1}{2nn}-\frac{\pi}{2(n+\omega)\pi\omega}+\frac{1}{2n\omega+\omega\omega} \quad \text{ſeu}$$

$$\frac{1}{2nn}-\frac{1}{\omega(2n+2\omega)}+\frac{1}{\omega(2n+\omega)}=\frac{1}{2nn}+\frac{1}{(2n+2\omega)(2n+\omega)}$$

vnde ſi fiat $\omega = 0$, prodibit ſumma quaeſita $= \frac{1}{2nn}+\frac{1}{4nn}=\frac{3}{4nn}$. Quocirca erit $\frac{3}{4nn} =$

1–

$$\frac{1}{1-nn}+\frac{1}{4-nn}+\frac{1}{9-nn}+\ldots\ldots+\frac{1}{(n-1)^2-nn}$$

$$+\frac{1}{(n+1)^2-nn}+\frac{1}{(n+2)^2-nn}+\&c.$$

in infinitum, ſiue erit iſtius ſeriei infinitae ſumma:

$$\frac{1}{(n+1)^2-nn}+\frac{1}{(n+2)^2-nn}+\frac{1}{(n+3)^2-nn}+\&c.$$

$$=\frac{3}{4nn}+\frac{1}{nn-1}+\frac{1}{nn-4}+\frac{1}{nn-9}+\ldots+\frac{1}{nn-(n-1)^2}.$$

EXEMPLUM III.

Inuenire ſummam huius ſeriei

$$\frac{1}{1-xx}+\frac{1}{9-xx}+\frac{1}{25-xx}+\frac{1}{49-xx}+\&c.$$

ſi ponatur $x=1$, *atque terminus primus* $\frac{1}{1-xx}$, *qui hoc caſu fit infinitus, auferatur.*

Cum huius ſeriei ſumma ſit in genere $=\frac{\pi\,\text{ſin}\,\frac{1}{2}\pi x}{4x\,\text{coſ}\,\frac{1}{2}\pi x}$ erit ſumma quaeſita $=\frac{\pi\,\text{ſin.}\,\frac{1}{2}\pi x}{4\pi x\,\text{coſ.}\,\frac{1}{2}\pi x}-\frac{1}{1-xx}$, ſi ponatur $x=1$. Quia vero vterque terminus fit infinitus, ponatur $x=1-\omega$, & cum ſit $\text{ſin}\,(\frac{1}{2}\pi-\frac{1}{2}\pi\omega)=\text{coſ}\,\frac{1}{2}\pi\omega=1-\frac{1}{8}\pi^2\omega^2$, & $\text{coſ}(\frac{1}{2}\pi-\frac{1}{2}\pi\omega)=\text{ſin}\,\frac{1}{2}\pi\omega=\frac{1}{2}\pi\omega$ ob ω infinite paruum, habebitur iſta expreſſio:

$$\frac{\pi(1-\frac{1}{8}\pi^2\omega^2)}{4(1-\omega)\frac{1}{2}\pi\omega}-\frac{1}{2\omega-\omega\omega}=\frac{1}{\omega(2-2\omega)}-\frac{1}{\omega(2-\omega)}$$

quae fit $=\frac{1}{4}$ poſito $\omega=0$, eſtque propterea

$$\frac{1}{4}=\frac{1}{8}+\frac{1}{24}+\frac{1}{48}+\frac{1}{80}+\frac{1}{120}+\&c.$$

EXEM-

EXEMPLM IV.

Inuenire ſummam ſeriei huius:

$$\frac{1}{1-xx}+\frac{1}{9-xx}+\frac{1}{25-xx}+\frac{1}{49-xx}+\&c.$$

ſi pro x *ponatur numerus quicunque integer impar* 2n−1 *isque terminus* $\frac{1}{(2n-1)^2-xx}$, *qui hoc caſu ſit infinitus, e medio tollatur.*

Erit ergo ſumma, quae quaeritur, $= \frac{\pi \sin \frac{1}{2}\pi x}{4x\cos \frac{1}{2}\pi x} - \frac{1}{(2n-1)^2-xx}$ poſito $x = 2n-1$. Statuamus ergo $x = 2n-1-\omega$, exiſtente ω infinite paruo, fietque $\sin \frac{1}{2}\pi x = \sin\left(\frac{2n-1}{2}\pi - \frac{1}{2}\pi\omega\right) = \pm \cos \frac{1}{2}\pi\omega$, vbi ſignum ſuperius valet, ſi ſit n numerus impar, inferius vero ſi ſit par. Simili modo erit $\cos \frac{1}{2}\pi x = \cos\left(\frac{2n-1}{2}\pi - \frac{1}{2}\pi\omega\right) = \pm \sin \frac{1}{2}\pi\omega$; ideoque ſiue n ſit par ſiue impar, erit $\frac{\sin \frac{1}{2}\pi x}{\cos \frac{1}{2}\pi x} = \frac{1}{\text{tang.} \frac{1}{2}\pi\omega} = \frac{1}{\frac{1}{2}\pi\omega}$. Hinc ſumma quaeſita ita exprimetur:

$$\frac{1}{2\omega(2n-1-\omega)} - \frac{1}{\omega[2(2(2n-1)-\omega]},$$ eritque propterea $= \frac{1}{4(2n-1)^2}$. Sic ſi ſit $n = 2$, erit

$$\frac{1}{36} = -\frac{1}{8}+\frac{1}{16}+\frac{1}{40}+\frac{1}{72}+\frac{1}{112}+\&c.$$

cuius ſummationis veritas aliunde conſtat.

CAPUT

CAPUT XVI.

DE DIFFERENTIATIONE FUNCTIONUM INEXPLICABILIUM.

367.

Functiones inexplicabiles hic voco, quae neque expreſſionibus determinatis, neque per aequationum radices explicari poſſunt; ita vt non ſolum non ſint algebraicae, ſed etiam plerumque incertum ſit, ad quod genus tranſcendentium pertineant. Huiusmodi functio inexplicabilis eſt $1 + \frac{1}{2} + \frac{1}{3} \ldots\ldots + \frac{1}{x}$, quae vtique ab x pendet, at niſi x ſit numerus integer nullo modo explicari poteſt. Simili modo haec expreſſio $1 \cdot 2 \cdot 3 \cdot 4 \ldots\ldots x$, erit functio inexplicabilis ipſius x, quoniam ſi x ſit numerus quicunque, eius valor non ſolum non algebraice, ſed ne quidem per vllum certum quantitatum tranſcendentium genus exprimi poteſt. Generatim ergo talium functionum inexplicabilium notio ex ſeriebus deriuari poteſt. Sit enim propoſita ſeries quaecunque

$$\begin{array}{ccccccccc} 1 & & 2 & & 3 & & 4 & \ldots & x \\ A & + & B & + & C & + & D & + \ldots + & X \end{array}$$

cuius ſumma ſi formula finita exprimi nequeat, praebebit functionem inexplicabilem ipſius x, nempe

$$S = A + B + C + D + \ldots + X.$$

Similiter continua producta ex terminis ſerierum vti

$$P = A\,B\,C\,D \ldots . X$$

exhibebunt functiones inexplicabiles ipſius x, quae autem ope logarithmorum ad formam priorem reuocari poſſunt, erit enim:

$$lP = lA + lB + lC + lD + \ldots + lX$$

368. Hoc igitur capite methodum explicare conſtitui, huiusmodi functionum inexplicabilium differentialia inueſtigandi. Quod argumentum, quamuis ad primam huius operis partem, vbi praecepta calculi differentialis ſunt tradita, pertinere videatur; tamen quoniam vberiorem doctrinae ſerierum cognitionem poſtulat, ad quam in hac altera parte peruenire licuit, ordinem naturalem relinquere coacti hoc loco attingamus. Cum autem haec inueſtigatio prorſus ſit noua, neque a quoquam adhuc tractata, tantum abeſt vt hanc calculi differentialis partem abſoluere queamus, vt potius prima tantum eius elementa adumbrare conemur. Praeterea vero nonnullas quaeſtiones proponam, quarum enodatio differentiationem huiusmodi functionum inexplicabilium requirat, quo ſimul vſus huius tractationis, qui autem in poſterum ſine dubio multo amplior erit, clarius perſpiciatur.

369. Ad huiusmodi functiones inexplicabiles differentiandas ante omnia neceſſe eſt, vt earum valores inveſtigemus, quos induunt, ſi pro x ponatur $x + \omega$.

Sit

Sit igitur

$$S = \overset{1}{A} + \overset{2}{B} + \overset{3}{C} + \overset{4}{D} + \ldots + \overset{x}{X}$$

atque ponatur Σ valor ipſius S, quem recipit, ſi pro x ponatur $x+\omega$, ſitque Z terminus ſeriei reſpondens indici $x+\omega$. Iam igitur termini, qui reſpondent indicibus $x+1$, $x+2$, $x+3$, &c. indicentur per X', X'', X''', X^{IV}, &c. atque is, qui conuenit indici infinito $x+\infty$ per $X^{|\infty|}$. Similique modo termini competentes indicibus $x+\omega+1$, $x+\omega+2$, $x+\omega+3$ &c. indicentur per Z', Z'', Z''', &c. & ſit $Z^{|\infty|}$ terminus reſpondens indici $x+\omega+\infty$. Quibus poſitis erit

$$S' = S + X'$$
$$S'' = S + X' + X''$$
$$S''' = S + X' + X'' + X'''$$
&c.
$$S^{|\infty|} = S + X' + X'' + X''' + \ldots + X^{|\infty|}$$

Simili modo cum etiam Σ ſucceſſiue terminis Z', Z'' &c. augeatur, erit

$$\Sigma' = \Sigma + Z'$$
$$\Sigma'' = \Sigma + Z' + Z''$$
$$\Sigma''' = \Sigma + Z' + Z'' + Z'''$$
&c.
$$\Sigma^{|\infty|} = \Sigma + Z' + Z'' + Z''' + \ldots + Z^{|\infty|}$$

370. Nunc natura ſeriei S, S', S'', S''', &c. eſt perpendenda, qualis futura ſit, ſi in infinitum continue-

tur: quae ſi in infinito cum progreſſione arithmetica confundatur; quod fit ſi termini ſeriei X, X', X'', X''', &c. in infinito ad aequalitatem conuergant, ita vt differentiae ſeriei S, S', S'', &c. tandem fiant aequales: hoc caſu quantitates $S^{|\omega|}$, $S^{|\omega+2|}$, $S^{|\omega+1|}$ &c. erunt in arithmetica progreſſione, & cum ſit $\Sigma^{|\omega|} = S^{|\omega+\omega|}$ ob $S^{|\omega+\omega|}$ $= S^{|\omega|} + \omega(S^{|\omega+1|} - S^{|\omega|}) = \omega S^{|\omega+1|} + (1-\omega) S^{|\omega|}$ erit $\Sigma^{|\omega|} = \omega S^{|\omega+1|} + (1-\omega) S^{|\omega|}$. At eſt $S^{|\omega+1|} =$ $S^{|\omega|} + X^{|\omega+1|}$, vnde fit $\Sigma^{|\omega|} = S^{|\omega|} + \omega X^{|\omega+1|}$, ex quo obtinebitur haec aequatio

$$\Sigma + Z' + Z'' + Z''' + \ldots + Z^{|\omega|} =$$
$$S + X' + X'' + X''' + \ldots + X^{|\omega|} + \omega X^{|\omega+1|}$$

ex qua definitur valor quaeſitus Σ, quem induit functio S, dum in ea $x + \omega$ loco x ſubſtituitur; eritque

$$\Sigma = S + \omega X^{|\omega+1|} + X' + X'' + X''' + \text{\&c. in infinitum}$$
$$- Z' - Z'' - Z''' - \text{\&c. in infinitum}$$

Quare ſi ſeriei A, B, C, D, &c. termini infiniteſimi euaneſcant, terminus $\omega X^{|\omega+1|}$ euaneſcit, & omitti poteſt.

371. Exprimitur ergo valor ipſius Σ per nouam ſeriem infinitam, quae exhiberi poteſt, ſi ſeriei $A + B + C +$ &c. habeatur terminus generalis, ex quo valores terminorum Z', Z'', Z''', &c. definiri queant. Poſito ergo ω infinite paruo, cum ſit $\Sigma - S$ differentiale functionis S, hoc differentiale dS per ſeriem infinitam exprimetur. Atque ſi nequidem altiores poteſtates ipſius

ω

ω negligantur, habebitur differentiale completum functionis huius inexplicabilis S, cuius natura, quo clarius ob oculos ponatur, ſequentibus exemplis hoc negotium illuſtrabimus.

EXEMPLUM I.

Inuenire differentiale huius functionis inexplicabilis

$$S = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \dots + \frac{1}{x}.$$

Quoniam huius ſeriei terminus generalis X eſt $= \frac{1}{x}$ ac propterea

$X' = \frac{1}{x+1}$	$Z' = \frac{1}{x+1+\omega}$
$X'' = \frac{1}{x+2}$	$Z'' = \frac{1}{x+2+\omega}$
$X''' = \frac{1}{x+3}$	$Z''' = \frac{1}{x+3+\omega}$
&c.	&c.

ob $X^{|\infty+1|} = \frac{1}{x+\infty+1} = 0$, ſi loco x ponatur $x+\omega$ functio S abibit in Σ, vt ſit

$$\Sigma = S + \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \&c.$$
$$- \frac{1}{x+1+\omega} - \frac{1}{x+2+\omega} - \frac{1}{x+3+\omega} - \&c.$$

ſiue binis his terminis in ſingulos colligendis, erit

$$\Sigma = S + \frac{\omega}{(x+1)(x+1+\omega)} + \frac{\omega}{(x+2)(x+2+\omega)} + \frac{\omega}{(x+3)(x+3+\omega)} + \&c.$$

 ſeu

ſeu cum ſit

$$\frac{1}{x+1+\omega} = \frac{1}{x+1} - \frac{\omega}{(x+1)^2} + \frac{\omega^2}{(x+1)^3} - \frac{\omega^3}{(x+1)^4} + \&c.$$

$$\frac{1}{x+2+\omega} = \frac{1}{x+2} - \frac{\omega}{(x+2)^2} + \frac{\omega^2}{(x+2)^3} - \frac{\omega^3}{(x+2)^4} + \&c.$$

&c.

erit ſeriebus ſecundum poteſtates ipſius ω diſpoſitis

$$\Sigma = S + \omega\left(\frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \frac{1}{(x+4)^2} + \&c.\right)$$
$$-\omega^2\left(\frac{1}{(x+1)^3} + \frac{1}{(x+2)^3} + \frac{1}{(x+3)^3} + \frac{1}{(x+4)^3} + \&c.\right)$$
$$+\omega^3\left(\frac{1}{(x+1)^4} + \frac{1}{(x+2)^4} + \frac{1}{(x+3)^4} + \frac{1}{(x+4)^4} + \&c.\right)$$
$$-\omega^4\left(\frac{1}{(x+1)^5} + \frac{1}{(x+2)^5} + \frac{1}{(x+3)^5} + \frac{1}{(x+4)^5} + \&c.\right)$$

&c.

Poſito ergo dx pro ω obtinebimus functionis propoſitae S differentiale completum

$$dS = dx\left(\frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \frac{1}{(x+4)^2} + \&c.\right)$$
$$-dx^2\left(\frac{1}{(x+1)^3} + \frac{1}{(x+2)^3} + \frac{1}{(x+3)^3} + \frac{1}{(x+4)^3} + \&c.\right)$$
$$+dx^3\left(\frac{1}{(x+1)^4} + \frac{1}{(x+2)^4} + \frac{1}{(x+3)^4} + \frac{1}{(x+4)^4} + \&c.\right)$$
$$-dx^4\left(\frac{1}{(x+1)^5} + \frac{1}{(x+2)^5} + \frac{1}{(x+3)^5} + \frac{1}{(x+4)^5} + \&c.\right)$$

&c.

EXEM-

EXEMPLUM II.

Inuenire differentiale huius functionis inexplicabilis ipſius S:

$$S = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots + \frac{1}{2x-1}.$$

Quia huius ſeriei terminus generalis eſt $X = \frac{1}{2x-1}$; erit:

$X' = \frac{1}{2x+1}$	$Z' = \frac{1}{2x+1+2\omega}$
$X'' = \frac{1}{2x+3}$	$Z'' = \frac{1}{2x+3+2\omega}$
$X''' = \frac{1}{2x+5}$	$Z''' = \frac{1}{2x+5+2\omega}$
&c.	&c.

ob terminos huius ſeriei infiniteſimos euaneſcentes & aequales, prodibit valor ipſius S, ſi loco x ponatur $x+1$:

$$\Sigma = S + \frac{1}{2x+1} + \frac{1}{2x+3} + \frac{1}{2x+5} + \&c.$$
$$- \frac{1}{2x+1+2\omega} - \frac{1}{2x+3+2\omega} - \frac{1}{2x+5+2\omega} - \&c.$$

ſeu

$$\Sigma = S + \frac{2\omega}{(2x+1)(2x+1+2\omega)} + \frac{2\omega}{(2x+3)(2x+3+2\omega)} + \&c.$$

Verum ſi ſinguli termini in ſeries ſecundum dimenſiones ipſius ω reſoluantur, erit:

$\Sigma =$

$$\Sigma = S + 2\omega\left(\frac{1}{(2x+1)^2} + \frac{1}{(2x+3)^2} + \frac{1}{(2x+5)^2} + \&c.\right)$$
$$- 4\omega^2\left(\frac{1}{(2x+1)^3} + \frac{1}{(2x+3)^3} + \frac{1}{(2x+5)^3} + \&c.\right)$$
$$+ 8\omega^3\left(\frac{1}{(2x+1)^4} + \frac{1}{(2x+3)^4} + \frac{1}{(2x+5)^4} + \&c.\right)$$
$$- 16\omega^4\left(\frac{1}{(2x+1)^5} + \frac{1}{(2x+3)^5} + \frac{1}{(2x+5)^5} + \&c.\right)$$
&c.

Ponátur nunc dx pro ω, atque prodibit differentiale completum functionis inexplicabilis S propoſitae:

$$dS = 2dx\left(\frac{1}{(2x+1)^2} + \frac{1}{(2x+3)^2} + \frac{1}{(2x+5)^2} + \&c.\right)$$
$$- 4dx^2\left(\frac{1}{(2x+1)^3} + \frac{1}{(2x+3)^3} + \frac{1}{(2x+5)^3} + \&c.\right)$$
$$+ 8dx^3\left(\frac{1}{(2x+1)^4} + \frac{1}{(2x+3)^4} + \frac{1}{(2x+5)^4} + \&c.\right)$$
$$- 16dx^4\left(\frac{1}{(2x+1)^5} + \frac{1}{(2x+3)^5} + \frac{1}{(2x+5)^5} + \&c.\right)$$
&c.

EXEM-

EXEMPLUM III.

Inuenire differentiale completum functionis huius inexplicabilis ipſius S:

$$S = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \quad . \; . \; . \; . \; . \; . \quad + \frac{1}{x^n}.$$

Cum huius ſeriei terminus generalis ſit $= \frac{1}{x^n}$, erunt termini infiniteſimi euaneſcentes & inter ſe aequales. Hincque ob

$X' = \frac{1}{(x+1)^n}$	$Z' = \frac{1}{(x+1+\omega)^n}$
$X'' = \frac{1}{(x+2)^n}$	$Z'' = \frac{1}{(x+2+\omega)^n}$
$X''' = \frac{1}{(x+3)^n}$	$Z''' = \frac{1}{(x+3+\omega)^n}$
&c.	&c.

erit:

$$X' - Z' = \frac{n\omega}{(x+1)^{n+1}} - \frac{n(n+1)\omega^2}{2(x+1)^{n+2}} + \frac{n(n+1)(n+2)\omega^3}{6(x+1)^{n+3}} - \&c.$$

$$X'' - Z'' = \frac{n\omega}{(x+2)^{n+1}} - \frac{n(n+1)\omega^2}{2(x+2)^{n+2}} + \frac{n(n+1)(n+2)\omega^3}{6(x+2)^{n+3}} - \&c.$$

&c.

ex quibus inuenitur:

$$\Sigma - S = n\omega \left(\frac{1}{(x+1)^{n+1}} + \frac{1}{(x+2)^{n+1}} + \frac{1}{(x+3)^{n+1}} + \&c. \right)$$

$$- \frac{n(n+1)}{1.\ 2}\omega^2 \left(\frac{1}{(x+1)^{n+2}} + \frac{1}{(x+2)^{n+2}} + \frac{3}{(x+3)^{n+2}} + \&c. \right)$$

$$+ \frac{n(n+1)(n+2)}{1.\ 2.\ 3}\omega^3 \left(\frac{1}{(x+1)^{n+3}} + \frac{1}{(x+2)^{n+3}} + \frac{1}{(x+3)^{n+3}} + \&c. \right)$$

&c.

 Qua-

Quare poſito $\omega = dx$ prodibit differentiale completum functionis S quaeſitum:

$$dS = +\, n\,dx \left(\frac{1}{(x+1)^{n+1}} + \frac{1}{(x+2)^{n+1}} + \frac{1}{(x+3)^{n+1}} + \&c.\right)$$
$$- \frac{n(n+1)}{1.\,2} dx^2 \left(\frac{1}{(x+1)^{n+2}} + \frac{1}{(x+2)^{n+2}} + \frac{1}{(x+3)^{n+2}} + \&c.\right)$$
$$+ \frac{n(n+1)(n+2)}{1.\,2.\,3} dx^3 \left(\frac{1}{(x+1)^{n+3}} + \frac{1}{(x+2)^{n+3}} + \frac{1}{(x+3)^{n+3}} + \&c.\right)$$
&c.

362. Ex his quoque ſummae iſtarum ſerierum interpolari, ſeu valores terminorum ſummatoriorum exhiberi poſſunt, quando numerus terminorum non eſt numerus integer. Si enim ponatur $x = 0$, erit quoque $S = 0$, atque Σ exprimet ſummam tot terminorum, quot numerus ω continet vnitates, etiamſi iſte numerus ω non ſit integer. Ita in exemplo primo ſi ponatur

$$\Sigma = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots + \frac{1}{\omega}$$

erit:

$$\Sigma = \frac{\omega}{1(1+\omega)} + \frac{\omega}{2(2+\omega)} + \frac{\omega}{3(3+\omega)} + \frac{\omega}{4(4+\omega)} + \&c.$$

ſiue

$$\Sigma = \omega\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \&c.\right)$$
$$- \omega^2\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \&c.\right)$$
$$+ \omega^3\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \&c.\right)$$
&c.

In

In exemplo vero tertio erit:

$$\Sigma = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \ldots + \frac{1}{\omega^n}.$$

Valorque ipſius Σ, ſiue ω ſit numerus integer ſiue fractus, per ſeries ſequenti modo exprimetur:

$$\Sigma = n\omega\left(1 + \frac{1}{2^{n+1}} + \frac{2}{3^{n+1}} + \frac{1}{4^{n+1}} + \&c.\right)$$
$$- \frac{n(n+1)}{1.\ 2}\omega^2\left(1 + \frac{1}{2^{n+2}} + \frac{1}{3^{n+2}} + \frac{1}{4^{n+2}} + \&c.\right)$$
$$+ \frac{n(n+1)(n+2)}{1.\ 2.\ 3}\omega^3\left(1 + \frac{1}{2^{n+3}} + \frac{1}{3^{n+3}} + \frac{1}{4^{n+3}} + \&c.\right)$$
$$\&c.$$

373. Haec eadem quoque ad ſeriem generalem accommodari poſſunt, cum enim ſit

$$S = \overset{1}{A} + \overset{2}{B} + \overset{3}{C} + \overset{4}{D} + \overset{\cdots}{\ldots} + \overset{x}{X}$$

atque poſito $x+\omega$ loco x, abeat X in Z, & S in Σ, erit:

$$Z = X + \frac{\omega dX}{dx} + \frac{\omega^2 ddX}{1.2\, dx^2} + \frac{\omega^3 d^3X}{1.2.3\, dx^3} + \&c.$$

& quia ſimili modo Z', Z'', Z''', &c. per X', X'', X''', &c. exprimuntur, erit:

$$\Sigma = S + \omega X^{|\omega+1|} - \frac{\omega}{dx} d.\,[X' + X'' + X''' + X'''' + \&c.]$$
$$- \frac{\omega^2}{1.2\, dx^2} dd.\,[X' + X'' + X''' + X'''' + \&c.]$$
$$- \frac{\omega^3}{1.2.3\, dx^3} d^3.\,[X' + X'' + X''' + X'''' + \&c.]$$
$$\&c.$$

&

& niſi $X^{|\omega+1|}$ ſit $=0$, hoc modo exprimi poterit, vt conſideratio infiniti tollatur:

$$X^{|\omega+1|} = X' + (X''-X') + (X'''-X'') + (X''''-X''') + \&c.$$

eritque ergo:

$$\Sigma = S + \omega X' + \omega[(X''-X') + (X'''-X'') + (X''''-X''') + \&c.]$$
$$-\frac{\omega}{dx}\, d.\, [X' + X'' + X''' + X'''' + X''''' + \&c.]$$
$$-\frac{\omega^2}{2dx^2}\, dd.\, [X' + X'' + X''' + X'''' + \&c.]$$
$$-\frac{\omega^3}{6dx^3}\, d^3.\, [X' + X'' + X''' + X'''' + \&c.]$$

&c.

Si ergo ponatur $\omega = dx$, orietur differentiale completum ipſius $S = A + B + C + \ldots + X$, ita expreſſum:

$$dS = X'dx + dx[(X''-X') + (X'''-X'') + (X''''-X''') + \&c.]$$
$$-\; d.\, [X' + X'' + X''' + X'''' + \&c.]$$
$$-\tfrac{1}{2}\, dd.\, [X' + X'' + X''' + X'''' + \&c.]$$
$$-\tfrac{1}{6}\, d^3.\, [X' + X'' + X''' + X'''' + \&c.]$$

&c.

374. Ponamus eſſe $x = 0$, fiet $X' = A$, $X'' = B$, &c. ideoque $X' + X'' + X''' + \&c.$ erit ſeries infinita cuius terminus generalis eſt $= X$. Formentur deinde ſeries ex his terminis generalibus:

$$\frac{dX}{dx};\quad \frac{ddX}{2dx^2};\quad \frac{d^3X}{6dx^3};\quad \frac{d^4X}{24dx^4};\quad \&c.$$

qua-

quarum ſerierum in infinitum continuatarum ſummae ſint:

$$\int . X = \mathfrak{A}$$

$$\int . \frac{dX}{dx} = \mathfrak{B}$$

$$\int . \frac{ddX}{2\,dx^2} = \mathfrak{C}$$

$$\int . \frac{d^3X}{6\,dx^3} = \mathfrak{D} \quad \&c.$$

& quia poſito $x = 0$, fit quoque $S = 0$, & Σ erit ſumma ſeriei $A + B + C + D + \ldots + Z$ continentis ω terminos; eſt enim Z terminus indicis ω, ſiue ω ſit numerus integer ſiue fractus. Quare habebitur

$$\Sigma = \omega A + \omega [(B-A) + (C-B) + (D-C) + \&c.]$$
$$- \omega \mathfrak{B} - \omega^2 \mathfrak{C} - \omega^3 \mathfrak{D} - \omega^4 \mathfrak{E} - \&c.$$

vbi prima ſeries praetermitti poteſt, ſi ſeriei propoſitae termini tandem euaneſcant.

375. Scribamus nunc x loco ω, abibitque Σ in S ita vt ſit

$$\begin{matrix} & 1 & & 2 & & 3 & & 4 & \ldots\ldots & x \\ S = & A & + & B & + & C & + & D & + \ldots + & X \end{matrix}$$

atque idem ipſius S valor iam per ſeriem infinitam exprimetur hoc modo:

$$S = Ax + x[(B-A) + (C-B) + (D-C) + \&c.]$$
$$- \mathfrak{B}x - \mathfrak{C}x^2 - \mathfrak{D}x^3 - \mathfrak{E}x^4 - \mathfrak{F}x^5 - \&c.$$

cuius valor cum aeque diſtincte exprimatur, ſiue x ſit numerus integer ſiue fractus, differentialia ipſius S cuiusque ordinis hinc facile exhiberi poſſunt:

$$\frac{dS}{dx} = A + (B-A) + (C-B) + (D-C) + \&c.$$
$$- \mathfrak{B} - 2\mathfrak{C}x - 3\mathfrak{D}x^2 - 4\mathfrak{E}x^3 - \&c.$$
$$\frac{ddS}{2dx^2} = - \mathfrak{C} - 3\mathfrak{D}x - 6\mathfrak{E}x^2 - 10\mathfrak{F}x^3 - \&c.$$
$$\frac{d^3S}{6dx^3} = - \mathfrak{D} - 4\mathfrak{E}x - 10\mathfrak{F}x^2 - 20\mathfrak{G}x^3 - \&c.$$
$$\frac{d^4S}{24dx^4} = - \mathfrak{E} - 5\mathfrak{F}x - 15\mathfrak{G}x^2 - \&c.$$

Quare cum differentiale completum ſit

$$= dS + \tfrac{1}{2}ddS + \tfrac{1}{6}d^3S + \tfrac{1}{24}d^4S + \&c.$$

erit functionis propoſitae S differentiale completum:

$$dS = Adx + (B-A)dx + (C-B)dx + (D-C)dx + \&c.$$
$$- \mathfrak{B}dx - \mathfrak{C}(2xdx + dx^2) - \mathfrak{D}(3x^2dx + 3xdx^2 + dx^3)$$
$$- \mathfrak{E}(4x^3dx + 6x^2dx^2 + 4xdx^3 + dx^4) - \&c.$$

576. Hoc ergo modo functionis cuiusque inexplicabilis S differentiale aſſignari poteſt, ſi ſeriei $A + B + C + D + \&c.$ termini infiniteſimi vel euaneſcant vel inter ſe ſint aequales. Quodſi enim huius termini infiniteſimi non fuerint $= 0$, tum ſumma ſeriei $\mathfrak{B}$, quae ex termino generali $\frac{dX}{dx}$ formatur, fiet infinita; at vero cum ſerie $A + (B-A) + (C-B) + (D-C) + \&c.$ coniuncta ſummam finitam conſtituet. At fieri poteſt, vt termini ſeriei $A + B + C + D + \&c.$ ita in infinitum augeantur, vt non ſolum ſeriei $\mathfrak{B}$, ſed etiam ſeriei $\mathfrak{C}$ ſumma fiat infinite magna, quo caſu non ſufficit

ſeri-

ſeriem $A+(B-A)+(C-B)+$ &c. adieciſſe: ſed quoniam hoc caſu valores infiniteſimi §. 370. conſiderati, nempe $S^{|\infty|}$, $S^{|\infty+1|}$, $S^{|\infty+2|}$, non amplius in arithmetica ſunt progreſſione, vti aſſumſeramus, huius progreſſionis ratio erit habenda. Quemadmodum ergo aſſumſimus, horum terminorum differentias primas eſſe aequales; ita methodum amplius extendemus, ſi horum valorum differentias demum ſecundas, vel tertias, vel vlteriores conſtantes ſtatuamus.

377. Retento ergo eodem ratiocinio, quo §. 369. ſumus uſi, ponamus memoratorum valorum differentias demum ſecundas eſſe conſtantes:

$$S^{|\infty|},\ S^{|\infty+1|},\ S^{|\infty+2|};$$

DIFF. I. $\quad X^{|\infty+1|},\ X^{|\infty+2|}$

DIFF. II. $\quad X^{|\infty+2|}-X^{|\infty+1|}$

Hinc erit $\Sigma^{|\infty|}=S^{|\infty+\omega|}=S^{|\infty|}+\omega X^{|\infty+1|}+\frac{\omega(\omega-1)}{1.2}(X^{|\infty+2|}-X^{|\infty+1|})=S^{|\infty|}-\frac{\omega(\omega-3)}{1.2}X^{|\infty+1|}+\frac{\omega(\omega-1)}{1.2}X^{|\infty+2|}$.

Quamobrem habebimus hanc aequationem:

$$\Sigma+Z'+Z''+Z'''+\ \ldots\ +Z^{|\infty|}=$$
$$S+X'+X''+X'''+\ \ldots\ +X^{|\infty|}-$$
$$\frac{\omega(\omega-3)}{1.2}X^{|\infty+1|}+\frac{\omega(\omega-1)}{1.2}X^{|\infty+2|},$$

ex

ex qua elicitur:

$$\Sigma = S + X' + X'' + X''' + X'''' + \&c. \text{ in infinitum}$$
$$- Z' - Z'' - Z''' - Z'''' - \&c. \text{ in infinitum}$$
$$+ \omega X^{|\omega+1|} + \frac{\omega(\omega-1)}{1.2}(X^{|\omega+2|} - X^{|\omega+1|}).$$

Termini autem iſti infiniteſimi ita repraeſentari poterunt, vt ſit

$$\Sigma = S + X' + X'' + X''' + X'''' + \&c.$$
$$- Z' - Z'' - Z''' - Z'''' - \&c.$$
$$+ \omega X' + \omega \left\{ \begin{matrix} + X'' + X''' + X'''' + X''''' + \&c. \\ - X' - X'' - X''' - X'''' - \&c. \end{matrix} \right\}$$
$$\begin{matrix} + \dfrac{\omega(\omega-1)}{1.2} X'' \\ - \dfrac{\omega(\omega-1)}{1.2} X' \end{matrix} + \frac{\omega(\omega-1)}{1.2} \left\{ \begin{matrix} + X''' + X^{IV} + X^{V} + \&c. \\ - 2X'' - 2X''' - 2X^{IV} - \&c. \\ + X' + X'' + X''' + \&c. \end{matrix} \right\}$$

vnde ſimul lex patet, qua haec expreſſio erit comparata, ſi differentiae demum tertiae vel quartae vel vlteriores fuerint conſtantes.

378. Cum igitur ſit, vt ſupra demonſtrauimus:

$$Z = X + \frac{\omega dX}{1\, dx} + \frac{\omega^2 ddX}{1.2\, dx^2} + \frac{\omega^3 d^3X}{1.2.3\, dx^3} + \&c.$$

ſi loco Z', Z'', Z''', &c. valores hinc oriundos ſubſtituamus, erit valor ipſius S, ſi loco x ſcribatur $x + \omega$, ſequens:

$\Sigma = S$

$$\Sigma = S + \omega X' + \omega \begin{Bmatrix} + X'' + X''' + X^{IV} + X^{V} + \&c. \\ - X' - X'' - X''' - X^{IV} - \&c. \end{Bmatrix}$$

$$\begin{matrix} + \frac{\omega(\omega-1)}{1.\ 2} X'' \\ - \frac{\omega(\omega-1)}{1.\ 2} X' \end{matrix} + \frac{\omega(\omega-1)}{1.\ 2} \begin{Bmatrix} + X''' + X^{IV} + X^{V} + X^{VI} + \&c. \\ -2X'' - 2X''' - 2X^{IV} - 2X^{V} - \&c. \\ + X' + X'' + X''' + X^{IV} + \&c. \end{Bmatrix}$$

$$- \frac{\omega}{dx} d. [X' + X'' + X''' + X'''' + \&c.]$$

$$- \frac{\omega^2}{2\,dx^2} d^2. [X' + X'' + X''' + X'''' + \&c.]$$

$$- \frac{\omega^3}{6\,dx^3} d^3. [X' + X'' + X''' + X'''' + \&c.]$$

&c.

Si ergo loco ω ponatur dx, prodibit differentiale completum functionis inexplicabilis propoſitae S; ſcilicet

$$dS = X'dx + dx \begin{Bmatrix} + X'' + X''' + X^{IV} + X^{V} + \&c. \\ - X' - X'' - X''' - X^{IV} - \&c. \end{Bmatrix}$$

$$\begin{matrix} - X'' \frac{dx(1-dx)}{1.\ 2} \\ + X' \frac{dx(1-dx)}{1.\ 2} \end{matrix} - \frac{dx(1-dx)}{1.\ 2} \begin{Bmatrix} + X''' + X^{IV} + X^{V} + X^{VI} + \&c \\ -2X'' - 2X''' - 2X^{IV} - 2X^{V} - \&c. \\ + X' + X'' + X''' + X^{IV} + \&c. \end{Bmatrix}$$

$$\begin{matrix} + X''' \frac{dx(1-dx)(2-dx)}{1.\ 2.\ 3} \\ - 2X'' \frac{dx(1-dx)(2-dx)}{1.\ 2.\ 3} \\ + X' \frac{dx(1-dx)(2-dx)}{1.\ 2.\ 3} \end{matrix} + \frac{dx(1-dx)(2-dx)}{1.\ 2.\ 3} \begin{Bmatrix} + X^{IV} + X^{V} + \&c. \\ -3X''' - 3X^{IV} - \&c. \\ + 3X'' + 3X''' + \&c. \\ - X' - X'' - \&c. \end{Bmatrix}$$

&c.

$$\begin{aligned}&-d.\,[X' + X'' + X''' + X^{IV} + X^{V} + \&c.]\\ &-\tfrac{1}{2}dd.\,[X' + X'' + X''' + X^{IV} + X^{V} + \&c.]\\ &-\tfrac{1}{6}d^3.\,[X' + X'' + X''' + X^{IV} + X^{V} + \&c.]\\ &-\tfrac{1}{24}d^4.\,[X' + X'' + X''' + X^{IV} + X^{V} + \&c.]\\ &\qquad\&c.\end{aligned}$$

quae expreſſio latiſſime patet, & quotaecunque demum differentiae fuerint conſtantes, differentiale quaeſitum exhibet. Accommodata enim eſt haec formula ad differentias conſtantes, & ſimul lex patet, ſi forte vlterius progredi neceſſe ſit.

379. Quod ſi ſeries $A + B + C + D + \&c.$ ex qua formatur functio inexplicabilis

$$S = \overset{1}{A} + \overset{2}{B} + \overset{3}{C} + \overset{4}{D} + \ldots + \overset{x}{X}$$

ita fuerit comparata, vt eius termini infiniteſimi euaneſcant, tum vti iam notauimus erit:

$$\begin{aligned}dS = &-d.\,[X' + X'' + X''' + X^{IV} + \&c.]\\ &-\tfrac{1}{2}dd.\,[X' + X'' + X''' + X^{IV} + \&c.]\\ &-\tfrac{1}{6}d^3.\,[X' + X'' + X''' + X^{IV} + \&c.]\\ &-\tfrac{1}{24}d^4.\,[X' + X'' + X''' + X^{IV} + \&c.]\\ &\qquad\&c.\end{aligned}$$

Sin autem illius ſeriei termini infiniteſimi non ſint $= 0$, ſed tamen differentias habeant euaneſcentes, tum ad iſtam expreſſionem inſuper addi debet

$$dx\left\{X' \begin{matrix} + X'' + X''' + X^{IV} + X^{V} + \&c. \\ - X' - X'' - X''' - X^{IV} - \&c. \end{matrix}\right]$$

Ve-

Verum ſi terminorum infiniteſimorum huius ſeriei A + B + C + D + &c. differentiae demum ſecundae euaneſcant, tum praeterea adiici oportet:

$$\frac{dx(dx-1)}{1.\ 2.}\left\{\begin{array}{l} +X'' \\ \\ -X' \end{array}\right. \left.\begin{array}{l} +\ X''' + X^{IV} + X^{V} + \&c. \\ -2X'' - 2X''' - 2X^{IV} - \&c. \\ +\ X' + X'' + X''' + \&c. \end{array}\right]$$

Atque ſi memoratorum terminorum infiniteſimorum differentiae demum tertiae fuerint euaneſcentes, tum praeter has iam exhibitas expreſſiones inſuper addi debet.

$$\frac{dx(dx-1)(dx-2)}{1.\ 2.\ 3.}\left\{\begin{array}{l} X''' \\ -2X'' \\ +X' \end{array}\right. \left.\begin{array}{l} +X^{IV} + X^{V} + X^{VI} + \&c. \\ -3X''' - 3X^{IV} - 3X^{V} - \&c. \\ +3X'' + 3X''' + 3X^{IV} + \&c. \\ -X' - X'' - X''' - \&c. \end{array}\right]$$

Sicque porro expreſſiones inſuper addendae erunt comparatae, ſi vlteriores demum differentiae terminorum infiniteſimorum ſeriei A + B + C + D + &c. euaneſcant. Hincque adeo quaecunque ſeries aſſumatur, dummodo eius termini infiniteſimi tandem ad differentias euaneſcentes perducantur, functionis inexplicabilis ex ea formatae differentiale definiri poterit.

380. Si ponatur $x=0$, fiet $X'=A$, $X''=B$, $X'''=C$ &c. Quare vti A + B + C + D + &c. eſt ſeries, cuius terminus generalis eſt X, ſi ex terminis generalibus $\frac{dX}{dx}$; $\frac{ddX}{2dx^2}$; $\frac{d^3X}{6dx^2}$; $\frac{d^4X}{24dx^4}$; &c. ſimili modo formentur ſeries infinitae, earumque ſummae denotentur per litteras: $\mathfrak{B}$; $\mathfrak{C}$; $\mathfrak{D}$; $\mathfrak{E}$; &c. reſpectiue. Summa

 ω

ω terminorum ſeriei $A + B + C + D +$ &c. ita exprimetur, vt perinde ſit, ſiue ω ſit numerus integer ſiue ſecus. Scribamus ergo x pro ω, vt ſit,

$$S = \overset{1}{A} + \overset{2}{B} + \overset{3}{C} + \overset{4}{D} + \ldots + \overset{x}{X}$$

atque ſi huius ſeriei termini infiniteſimi euaneſcant, erit

$$S = -\mathfrak{B}x - \mathfrak{C}x^2 - \mathfrak{D}x^3 - \mathfrak{E}x^4 - \&c.$$

At ſi termini infiniteſimi differentias ſaltem primas habeant conſtantes, tum ad hunc valorem inſuper addi debet hic:

$$x\left\{A \begin{array}{l} + B + C + D + E + \&c. \\ - A - B - C - D - \&c. \end{array}\right]$$

ſin autem illorum terminorum infiniteſimorum differentiae demum ſecundae euaneſcant, tum praeterea addi debet:

$$\frac{x(x-1)}{1.\ 2}\left\{\begin{array}{l} + B \\ - A \end{array} \begin{array}{l} + C + D + E + F + \&c. \\ - 2B - 2C - 2D - 2E - \&c. \\ + A + B + C + D + \&c. \end{array}\right]$$

Si differentiae demum tertiae fuerint euaneſcentes, tum inſuper adiici debet haec ſeries infinita:

$$\frac{x(x-1)(x-2)}{1.\ 2.\ 3.}\left\{\begin{array}{l} + C \\ - 2B \\ + A \end{array} \begin{array}{l} + D + E + F + G + \&c. \\ - 3C - 3D - 3E - 3F - \&c. \\ + 3B + 3C + 3D + 3E + \&c. \\ - A - B - C - D - \&c. \end{array}\right]$$

&c.

381. Accommodemus haec quoque ad alterum functionum inexplicabilium genus, quae conſtant continuo producto terminorum aliquot ſeriei propoſitae $A + B + C + D +$ &c. ſitque

$$S = \overset{1}{A}\ \overset{2}{B}\ \overset{3}{C}\ \overset{4}{D} \ldots \overset{x}{X}$$

& quaeratur primo valor Σ, in quem S transmutatur, ſi loco x ſcribatur $x + \omega$; ponamus autem vt ante eſſe Z terminum ſeriei $A + B + C + D +$ &c. cuius index ſit $= x + \omega$, vti X reſpondet indici x. Quo ergo hunc caſum ad praecedentem reducamus ſumamus logarithmos, eritque

$$lS = lA + lB + lC + lD + \ldots + lX$$

Quod ſi iam huius ſeriei termini infiniteſimi euaneſcant, erit eandem methodum, qua ante vſi ſumus, adhibendo

$$l\Sigma = lS + lX' + lX'' + lX''' + \text{\&c.}$$
$$- lZ' - lZ'' - lZ''' - \text{\&c.}$$

hincque ad numeros regrediendo erit

$$\Sigma = S.\frac{X'}{Z'}\cdot\frac{X''}{Z''}\cdot\frac{X'''}{Z'''}\cdot\frac{X''''}{Z''''},\ \text{\&c.}$$

quae ergo expreſſio valet, ſi ſeriei A, B, C, D, &c. termini infiniteſimi vnitati aequentur. Sin autem logarithmi terminorum infiniteſimorum huius ſeriei non euaneſcant, at tamen differentias habeant euaneſcentes; tum ad illam ſeriem, quam pro $l\Sigma$ inuenimus, inſuper addi debet haec ſeries

$$\omega\, lX' + \omega\left(l\frac{X''}{X'} + l\frac{X'''}{X''} + l\frac{X''''}{X'''} + \text{\&c.}\right)$$

 ſicque

ſicque numeris ſumendis habebitur

$$\Sigma = SX^{I\omega} \cdot \frac{X^{II\omega} \cdot X^{I(1-\omega)}}{Z'} \cdot \frac{X^{III\omega} \cdot X^{II(1-\omega)}}{Z''} \cdot \frac{X^{IV\omega} \cdot X^{III(1-\omega)}}{Z'''} \cdot \text{ \&c.}$$

382. Quodſi ergo ponamus $x = 0$, quo caſu fit $S = 1$ & $X' = A$, $X'' = B$; $X''' = C$; &c. Σ denotabit productum ω terminorum huius ſeriei A, B, C, D &c. Si igitur pro ω ſcribamus x, vt Σ obtineat valorem, quem ante ipſi S tribueramus, ita vt ſit

$$S = \overset{1}{A}.\ \overset{2}{B}.\ \overset{3}{C}.\ \overset{4}{D}.\ \overset{\dots}{\dots}\ \overset{x}{X}$$

quia nunc Z', Z'', Z''', &c. abeunt in X', X'', X''' &c. ſi logarithmi terminorum infiniteſimorum iſtius ſeriei A, B, C, D, E, &c. euaneſcant, exprimetur S hoc modo

$$S = \frac{A}{X'} \cdot \frac{B}{X''} \cdot \frac{C}{X'''} \cdot \frac{D}{X^{IV}} \cdot \frac{E}{X^{V}}, \text{ \&c.}$$

Sin autem differentiae demum logarithmorum terminorum infiniteſimorum ſeriei A, B, C, D, &c. euaneſcant, tum iſta functio S ſequenti modo exprimetur, vt ſit:

$$S = A^x \cdot \frac{B^x A^{1-x}}{X'} \cdot \frac{C^x B^{1-x}}{X''} \cdot \frac{D^x C^{1-x}}{X'''} \cdot \frac{E^x D^{1-x}}{X^{IV}} \cdot \text{ \&c.}$$

ſi illorum logarithmorum differentiae ſecundae demum ſint euaneſcentes, ex praecedentibus facile colligitur, cuiusmodi factores inſuper addi debeant; quem caſum, cum vix occurrere ſoleat, hic praetermittamus. Ceterum

rum vſum harum expreſſionum in interpolationis negotio capite ſequente oſtendam.

383. Hic igitur cum differentiatio huiusmodi functionum inexplicabilium potiſſimum ſit propoſita: inueſtigemus differentiale huius functionis

$$S = A. B. C. D. \dots . X$$

Ad hoc reſumamus aequationem ante inuentam

$$l\Sigma = lS + lX' + lX'' + lX''' + \&c. \\ - lZ' - lZ'' - lZ''' - \&c.$$

& cum lZ oriatur ex lX, ſi loco x ponatur $x + \omega$, erit

$$lZ = lX + \frac{\omega}{dx} d. lX + \frac{\omega^2}{2dx^2} dd. lX + \frac{\omega^3}{6dx^3} d^3. lX + \&c.$$

quibus valoribus pro lZ', lZ'', lZ''', &c. ſubſtitutis habebitur

$$l\Sigma = lS - \frac{\omega}{dx} d \;[lX' + lX'' + lX''' + lX^{IV} + \&c.] \\ - \frac{\omega^2}{2dx^2} dd \;[lX' + lX'' + lX''' + lX^{IV} + \&c.] \\ - \frac{\omega^3}{6dx^2} d^3. [lX' + lX'' + lX''' + lX^{IV} + \&c.] \\ \&c.$$

Ponatur nunc $\omega = dx$, fietque $l\Sigma = lS + d. lS$, ideoque erit

$$\frac{dS}{S} = - d. \;[lX' + lX'' + lX''' + lX^{IV} + \&c.] \\ - \tfrac{1}{2} dd. [lX' + lX'' + lX''' + lX^{IV} + \&c.] \\ - \tfrac{1}{6} d^3. [lX' + lX'' + lX''' + lX^{IV} + \&c.] \\ \&c.$$

quae

quae formula valet, ſi logarithmi terminorum infiniteſimorum ſeriei A, B, C, D, &c. euaneſcant; ſin autem ipſi non euaneſcant, attamen differentias habeant euaneſcentes, tum ad praecedentem differentialis completi expreſſionem inſuper addi debet haec ſeries:

$$dx\,l\mathrm{X}' + dx\left(l\frac{\mathrm{X}''}{\mathrm{X}'} + l\frac{\mathrm{X}'''}{\mathrm{X}''} + l\frac{\mathrm{X}''''}{\mathrm{X}'''} + \&c.\right)$$

vt obtineatur differentiale completum.

384. Idem adhuc alio modo praeſtari poteſt. Ponatur $x = 0$, quo caſu abit $l\mathrm{S}$ in o. Tum formentur ſeries, quarum termini generales ſint:

$$l\mathrm{X};\quad \frac{d.l\mathrm{X}}{dx};\quad \frac{dd.l\mathrm{X}}{2dx^2};\quad \frac{d^3.l\mathrm{X}}{6dx^3};\quad \&c.$$

harumque ſerierum infinitarum ſummae ſint reſpectiue: $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. Scribatur x pro ω, vt ſit $\Sigma = \mathrm{S}$, eritque

$$l\mathrm{S} = -\mathfrak{B}x - \mathfrak{C}x^2 - \mathfrak{D}x^3 - \mathfrak{E}x^4 - \&c.$$

ſi quidem logarithmi terminorum infiniteſimorum ſeriei A, B, C, D, &c. cuius terminus generalis eſt X, euaneſcant: at ſi horum logarithmorum differentiae demum euaneſcant, erit:

$$l\mathrm{S} = x\,l\mathrm{A} + x\left(l\frac{\mathrm{B}}{\mathrm{A}} + l\frac{\mathrm{C}}{\mathrm{B}} + l\frac{\mathrm{D}}{\mathrm{C}} + l\frac{\mathrm{E}}{\mathrm{D}} + \&c.\right)$$
$$- \mathfrak{B}x - \mathfrak{C}x^2 - \mathfrak{D}x^3 - \mathfrak{E}x^4 - \&c.$$

Hincque adeo differentiale ipſius $l\mathrm{S}$ erit:

$$\frac{d\mathrm{S}}{\mathrm{S}} = dx\,l\mathrm{A} + dx\left(l\frac{\mathrm{B}}{\mathrm{A}} + l\frac{\mathrm{C}}{\mathrm{B}} + l\frac{\mathrm{D}}{\mathrm{C}} + l\frac{\mathrm{E}}{\mathrm{D}} + \&c.\right)$$
$$- \mathfrak{B}dx - 2\mathfrak{C}x\,dx - 3\mathfrak{D}x^2dx - 4\mathfrak{E}x^3dx - \&c.$$

At

At ſi differentiale completum deſideretur, erit id:

$$\frac{dS}{S} = dx\,l\,A + dx\left(l\frac{B}{A} + l\frac{C}{B} + l\frac{D}{C} + l\frac{E}{D} + \&c.\right)$$

$$-\mathfrak{B}dx - \mathfrak{C}(2xdx + dx^2) - \mathfrak{D}(3xxdx + 3xdx^2 + dx^3)$$
$$- \&c.$$

Ad quarum formularum vſum oſtendendum ſequentia exempla adiicimus, quae vtroque modo reſoluemus.

EXEMPLUM I.

Inuenire differentiale huius functionis inexplicabilis:

$$S = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot \;.\;.\;.\;.\; \frac{2X-1}{2X}.$$

Hic ante omnia notandum eſt, terminos infiniteſimos horum factorum abire in vnitates, ideoque eorum logarithmos euaneſcere. Cum igitur ſit $X = \frac{2x-1}{2x}$, erit $X' = \frac{2x+1}{2x+2}$; $X'' = \frac{2x+3}{2x+4}$; $X''' = \frac{2x+5}{2x+6}$; &c. & generaliter $\overset{[n]}{X} = \frac{2x+2n-1}{2x+2n}$;

vnde erit:

$$lX^{|n|} = l(2x+2n-1) - l(2x+2n)$$

$$d.\,lX^{|n|} = \frac{2\,dx}{2x+2n-1} - \frac{2\,dx}{2x+2n}$$

$$dd.\,lX^{|n|} = -\frac{4\,dx^2}{(2x+2n-1)^2} + \frac{4\,dx^2}{(2x+2n)^2}$$

$$d^3.\,lX^{|n|} = +\frac{2.2.\,4\,dx^3}{(2x+2n-1)^3} - \frac{2.2.\,4\,dx^3}{(2x+2n)^3}$$

$$d^4.\,lX^{|n|} = -\frac{2.2.4.\,6\,dx^4}{(2+2n-1)^4} + \frac{2.2.4.\,6\,dx^4}{(2x+2n)^4}$$

&c.

vnde erit differentiale completum:

$$\frac{dS}{S} = -2dx \left\{ \begin{array}{l} \frac{1}{2x+1} + \frac{1}{2x+3} + \frac{1}{2x+5} + \&c. \\ -\frac{1}{2x+2} - \frac{1}{2x+4} - \frac{1}{2x+6} - \&c. \end{array} \right\}$$

$$+\tfrac{4}{2}dx^2 \left\{ \begin{array}{l} \frac{1}{(2x+1)^2} + \frac{1}{(2x+3)^2} + \frac{1}{(2x+5)^2} + \&c. \\ -\frac{1}{(2x+2)^2} - \frac{1}{(2x+4)^2} - \frac{1}{(2x+6)^2} - \&c. \end{array} \right\}$$

$$-\tfrac{8}{3}dx^3 \left\{ \begin{array}{l} \frac{1}{(2x+1)^3} + \frac{1}{(2x+3)^3} + \frac{1}{(2x+5)^3} + \&c. \\ -\frac{1}{(2x+2)^3} - \frac{1}{(2x+4)^3} - \frac{1}{(2x+6)^3} - \&c. \end{array} \right\}$$

&c.

Quod

Quod ſi autem tantum differentiale primum quaeratur, erit id:

$$\frac{dS}{S} = - 2\,dx \text{ in}$$

$$\left(\frac{1}{(2x+1)(2x+2)} + \frac{1}{(2x+3)(2x+4)} + \frac{1}{(2x+5)(2x+6)} + \&c.\right)$$

quod idem altera methodo §. 394. tradita ita inueſtigatur. Cum ſit $lX = l\frac{2x-1}{2x}$, erit $\frac{d.\,lX}{dx} = \frac{2}{2x-1} - \frac{1}{x}$;

$$\frac{dd.\,lX}{2\,dx^2} = - \frac{2}{(2x-1)} + \frac{1}{2xx};\quad \frac{d^3.\,lX}{6\,dx^3} = + \frac{8}{3(2x-1)^3} - \frac{1}{3x^3}\&c.$$

ideoque fiet

$$\mathfrak{A} = l\frac{1}{2} + l\frac{3}{4} + l\frac{5}{6} + l\frac{7}{8} + \&c.$$

$$\mathfrak{B} = \left\{\begin{matrix} + \frac{2}{1} + \frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \frac{2}{9} + \&c. \\ - \frac{2}{2} - \frac{2}{4} - \frac{2}{6} - \frac{2}{8} - \frac{2}{10} - \&c. \end{matrix}\right\} = 2\,l2$$

$$\mathfrak{C} = - \frac{4}{2}\left\{\begin{matrix} \frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \&c. \\ - \frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{6^2} - \frac{1}{8^2} - \&c. \end{matrix}\right\}$$

$$\mathfrak{D} = \frac{8}{3}\left\{\begin{matrix} \frac{1}{1} + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \&c. \\ - \frac{1}{2^3} - \frac{1}{4^3} - \frac{1}{6^3} - \frac{1}{8^3} - \&c. \end{matrix}\right\}$$

&c.

ſiue erit:

$$\mathfrak{B} = +\frac{2}{1}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \&c.\right)$$

$$\mathfrak{C} = -\frac{4}{2}\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \&c.\right)$$

$$\mathfrak{D} = +\frac{8}{3}\left(1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} - \&c.\right)$$

$$\mathfrak{E} = -\frac{16}{4}\left(1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \&c.\right)$$

&c.

Quibus valoribus inuentis ſubſtitutis erit:

$$\frac{dS}{S} = -2\,dx\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \&c.\right)$$

$$+ 4x\,dx\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \&c.\right)$$

$$- 8x^2dx\left(1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} - \&c.\right)$$

$$+ 16x^3dx\left(1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \&c.\right)$$

&c.

Si igitur ſit $x = 0$, quo caſu ſit $lS = 0$ & $S = 1$, erit $dS = -2\,dx\,l2$.

EXEM-

EXEMPLUM II.

Inuenire differentiale huius functionis inexplicabilis:

$$S = 1 . 2 . 3 . 4 . . . . . x$$

Huius ſeriei 1, 2, 3, 4, &c. termini in infinitum ita creſcunt, vt logarithmorum differentiae euaneſcant: eſt enim $l(\infty + 1) - l\infty = l\left(1 + \frac{1}{\infty}\right) = \frac{1}{\infty} = 0$. Cum igitur ſit $X = x$ erit $X' = x + 1$; $X'' = x + 2$; $X''' = x + 3$; &c. porro autem ob $lX = lx$ fiet $d.lX = \frac{dx}{x}$; $dd.lX = -\frac{dx^2}{x^2}$; $d^3.lX = \frac{2dx^3}{x^3}$; $d^4.lX = -\frac{2.3dx^4}{x^4}$; &c. vnde ſi logarithmi vltimi euaneſcerent, foret

$$\frac{dS}{S} = -dx\left(\frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \frac{1}{x+4} + \&c.\right)$$
$$+ \frac{dx^2}{2}\left(\frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \frac{1}{(x+4)^2} + \&c.\right)$$
$$- \frac{dx^3}{3}\left(\frac{1}{(x+1)^3} + \frac{1}{(x+2)^3} + \frac{1}{(x+3)^3} + \frac{1}{(x+4)^3} + \&c.\right)$$
$$\&c.$$

At cum differentiae demum logarithmorum euaueſcant, inſuper addi debet haec expreſſio:

$$dx\, l(x+1) + dx\left(l\frac{x+2}{x+1} + l\frac{x+3}{x+2} + l\frac{x+4}{x+3} + l\frac{x+5}{x+4} + \&c.\right)$$

Quia vero eſt:

$$l\frac{x+2}{x+1} = \frac{1}{x+1} - \frac{1}{2(x+1)^2} + \frac{1}{3(x+1)^3} - \frac{1}{4(x+1)^4} + \&c.$$

$$l\frac{x+3}{x+2} = \frac{1}{x+2} - \frac{1}{2(x+2)^2} + \frac{1}{3(x+2)^3} - \frac{1}{4(x+2)^4} + \&c.$$

&c.

erit verum differentiale completum:

$$\frac{dS}{S} = dx\, l(x+1)$$

$$-\tfrac{1}{2}(dx - dx^2)\left(\frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \&c.\right)$$

$$+\tfrac{1}{3}(dx - dx^3)\left(\frac{1}{(x+1)^3} + \frac{1}{(x+2)^3} + \frac{1}{(x+3)^3} + \&c.\right)$$

$$-\tfrac{1}{4}(dx - dx^4)\left(\frac{1}{(x+1)^4} + \frac{1}{(x+2)^4} + \frac{1}{(x+3)^4} + \&c.\right)$$

$$+\tfrac{1}{5}(dx - dx^5)\left(\frac{1}{(x+1)^5} + \frac{1}{(x+2)^5} + \frac{1}{(x+3)^5} + \&c.\right)$$

&c.

Sin autem altero modo differentiale hoc exprimere velimus, quia eſt

$$lX = lx;\quad \frac{d.lX}{dx} = 1;\quad \frac{dd.lX}{2\,dx^2} = -\frac{1}{2x^3};$$

$$\frac{d^3.lX}{6\,dx^3} = \frac{1}{3x^3};\quad \frac{d^4.lX}{24\,dx^4} = -\frac{1}{4x^4};\quad \&c.$$

habe-

habebuntur ſequentes ſeries:

$$\mathfrak{A} = l1 + l2 + l3 + l4 + l5 + \&c.$$

$$\mathfrak{B} = 1\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \&c.\right)$$

$$\mathfrak{C} = -\tfrac{1}{2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \&c.\right)$$

$$\mathfrak{D} = \tfrac{1}{3}\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \&c.\right)$$

$$\mathfrak{E} = -\tfrac{1}{4}\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \&c.\right)$$

&c.

Hinc ob $lA = l1 = 0$, fiet ex §. 384:

$$lS = x\left(l\frac{2}{1} + l\frac{3}{2} + l\frac{4}{3} + l\frac{5}{4} + \&c.\right)$$

$$- x\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c.\right)$$

$$+ \tfrac{1}{2}x^2\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c.\right)$$

$$- \tfrac{1}{3}x^3\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \&c.\right)$$

$$+ \tfrac{1}{4}x^4\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \&c.\right)$$

&c.

Binae

Binae autem primae ſeries, per quas x eſt multiplicatum, etiamſi vtraque habeat ſummam infinitam, tamen ambae ſimul ſummam habent finitam. Si enim vtriusque n termini capiantur, prodibit:

$$l(n+1) - 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \ldots - \frac{1}{n}.$$

At ſupra §. 142. inuenimus eſſe

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} = \text{Conſt.} + ln$$
$$+ \frac{1}{2n} - \frac{\mathfrak{A}}{2n^2} + \frac{\mathfrak{B}}{4n^4} - \text{\&c.}$$

haecque conſtans prodit $= 0,5772156649015325$. Quodſi ergo ponatur $n = \infty$, erit:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{\infty} = \text{Conſt.} + l\infty,$$

vnde binarum illarum ſerierum in infinitum continuatarum valor erit $= l(\infty+1) - \text{Conſt.} - l\infty = -\text{Conſt.}$

Ex quo erit:

$$lS = -x \cdot 0,5772156649015325$$
$$+ \tfrac{1}{2}xx\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{\&c.}\right)$$
$$- \tfrac{1}{3}x^3\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \text{\&c.}\right)$$
$$+ \tfrac{1}{4}x^4\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{\&c.}\right)$$

&c.

vnde differentialia cuiusque ordinis facile reperiuntur.

Erit

Erit enim:

$$\frac{dS}{S} = - dx.\ 0,5772156649015325$$
$$+ xdx\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \&c.\right)$$
$$- x^2dx\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \&c.\right)$$
$$+ x^3dx\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \&c.\right)$$

&c.

At ſi hae ſeries in vnam colligantur erit:

$$\frac{dS}{S} = - dx.\ 0,5772156649015325$$
$$+ \frac{xdx}{1(1+x)} + \frac{xdx}{2(2+x)} + \frac{xdx}{3(3+x)} + \frac{xdx}{4(4+x)} + \&c.$$

Quare ſi ſit $x = 0$, fiet:

$$\frac{dS}{S} = - dx.\ 0,5772156649015325$$

Ex priori vero expreſſione hoc caſu erit:

$$\frac{dS}{S} = - \tfrac{1}{2}dx\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c.\right)$$
$$+ \tfrac{1}{3}dx\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \&c.\right)$$
$$- \tfrac{1}{4}dx\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \&c.\right)$$
$$+ \tfrac{1}{5}dx\left(1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \&c.\right)$$

&c.

385. Hinc ergo etiam huiusmodi functionum inexplicabilium differentialia quouis caſu ſpeciali exhiberi poſſunt, propterea quod hic differentialia completa eruimus. Quamobrem ſi tales functiones ingrediantur in expreſſiones, quae indeterminatae videntur, cuiusmodi capite praecedente tractauimus; valores eadem methodo definiri poterunt, vti ex adiunctis exemplis intelligetur.

EXEMPLUM I.

Determinare valorem huius expreſſionis:

$$\frac{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{x}}{x(x-1)}-\frac{1}{(x-1)(2x-1)}$$

eo caſu, quando ponitur $x=1$.

Ponamus $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{x}=S$, erit ex §. 372:

$$S=x\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\&c.\right)$$
$$-x^2\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\&c.\right)$$
$$+x^3\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\&c.\right)$$
$$\&c.$$

ſeu cum ſit quoque

$$S=+1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\&c.$$
$$-\frac{1}{1+x}-\frac{1}{2+x}-\frac{1}{3+x}-\frac{1}{4+x}-\frac{1}{5+x}-\&c.$$

ſi

ſi quiuis terminus ſuperioris ſeriei cum praecedente inferioris combinetur, prodibit:

$$S = 1 + \frac{x-1}{2(1+x)} + \frac{x-1}{3(2+x)} + \frac{x-1}{4(3+x)} + \&c.$$

quae expreſſio, quoniam poni debet $x = 1$ eſt commodior. Sit ergo $x = 1 + \omega$, fietque

$$S = 1 + \frac{\omega}{2(2+\omega)} + \frac{\omega}{3(3+\omega)} + \frac{\omega}{4(4+\omega)} + \&c.$$

ſiue

$$S = 1 + \omega\left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \&c.\right) = 1 + \mathfrak{B}\omega$$

$$- \omega^2\left(\frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \&c.\right) \quad - \mathfrak{C}\omega^2$$

$$+ \omega^3\left(\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \&c.\right) \quad + \mathfrak{D}\omega^3$$

&c. &c.

Tota ergo expreſſio poſito $x = 1 + \omega$ abibit in hanc:

$$\frac{1 + \mathfrak{B}\omega - \mathfrak{C}\omega^2 + \mathfrak{D}\omega^3 - \&c.}{\omega(1+\omega)} - \frac{1}{\omega(1+2\omega)} \text{ ſeu}$$

$$\frac{\omega + \mathfrak{B}\omega + 2\mathfrak{B}\omega^2 - \mathfrak{C}\omega^2}{\omega(1+\omega)(1+2\omega)} = \frac{1 + \mathfrak{B} + 2\mathfrak{B}\omega - \mathfrak{C}\omega - \&c.}{(1+\omega)(1+2\omega)}$$

Ponatur nunc $\omega = 0$, atque expreſſionis propoſitae valor caſu $x = 1$, erit:

$$= 1 + \mathfrak{B} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c.$$

quae ſeries cum ſit $= \frac{1}{6}\pi^2$, ſequitur valorem quaeſitum eſſe $= \frac{1}{6}\pi^2$.

EXEMPLUM II.

Inuenire valorem huius expreſſionis :

$$\frac{2x-xx}{(x-1)^2}+\frac{\pi\pi x}{6(x-1)}-\frac{(2x-1)(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{x})}{x(x-1)^2}$$

caſu quo ponitur $x=1$.

Ponatur $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{x}=S$, ſtatuaturque $x=1+\omega$, fiet vt in exemplo praecedente inuenimus :

$$S=1+\mathfrak{B}\omega-\mathfrak{C}\omega^2+\mathfrak{D}\omega^3-\&c. \quad \text{exiſtente}$$

$$\mathfrak{B}=\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\&c.=\tfrac{1}{6}\pi\pi-1$$

$$\mathfrak{C}=\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\&c.$$

$$\mathfrak{D}=\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\&c.$$

&c.

Poſito ergo $x=1+\omega$ expreſſio propoſita induet hanc formam :

$$\frac{1-\omega\omega}{\omega\omega}+\frac{(1+\mathfrak{B})(1+\omega)}{\omega}-\frac{(1+2\omega)(1+\mathfrak{B}\omega-\mathfrak{C}\omega^2+\mathfrak{D}\omega^3-\&c.)}{(1+\omega)\omega^2}$$

quae ad eandem denominationem $\omega^2(1+\omega)$ perducta fit :

$$\frac{1+\omega-\omega^2-\omega^3+\omega+2\omega^2+\omega^3+\mathfrak{B}\omega(1+2\omega+\omega\omega) -1-\mathfrak{B}\omega+\mathfrak{C}\omega^2-\mathfrak{D}\omega^3-2\omega-2\mathfrak{B}\omega^2+2\mathfrak{C}\omega^3\ \&c.}{\omega^2(1+\omega)}$$

quae reducitur ad hanc formam :

$$\frac{\omega^2+\mathfrak{C}\omega^2+\mathfrak{B}\omega^3+2\mathfrak{C}\omega^3-\mathfrak{D}\omega^3\ \&c.}{\omega^2(1+\omega)}$$

Fiat

Fiat nunc $\omega = 0$, atque prodibit $1 + \mathfrak{C}$. Quocirca expreſſionis propoſitae valor caſu $x = 1$, erit $= 1 + \mathfrak{C}$, ideoque per hanc ſeriem exprimetur:

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \&c.$$

cuius ſumma cum neque per logarithmos, neque per peripheriam circuli π exhiberi poſſit, valor quaeſitus etiamnum alio modo finite aſſignari non poteſt. Ex his ergo duobus exemplis vſus, quem differentiatio functionum inexplicabilium in doctrina ſerierum habere poteſt, ſatis luculenter perſpicitur.

386. In methodo hic tradita functiones inexplicabiles differentiandi aſſumſimus ſeriei A, B, C, D, E, &c. terminos infiniteſimos vel eſſe $= 0$, vel differentias tandem euaneſcentes habere; quorum ſi neutrum contingat, iſta methodo vti non licebit. Hancobrem aliam exponam methodum huic conditioni non adſtrictam, quam ſummatio generalis ſerierum ex termino generali petita & ſupra fuſius explicata ſuppeditat. Denotent igitur litterae $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, &c. numeros Bernullianos §. 122. exhibitos, ſitque functio inexplicabilis propoſita haec:

$$S = \overset{1}{A} + \overset{2}{B} + \overset{3}{C} + \overset{4}{D} + \ldots + \overset{x}{X}$$

& quia ſupra (130.) oſtendimus fore:

$$S = \int X dx + \tfrac{1}{2} X + \frac{\mathfrak{A}\, dX}{1.2\, dx} - \frac{\mathfrak{B}\, d^3X}{1.2.3.4\, dx^3} + \frac{\mathfrak{C}\, d^5X}{1.2.3.4.5.6\, dx^5} - \&c.$$

hinc facile erit iſtius functionis S differentiale exhibere erit enim:

$$dS = Xdx + \tfrac{1}{2}dX + \frac{\mathfrak{A}ddX}{1.2\,dx} - \frac{\mathfrak{B}d^4X}{1.2.3.4\,dx^3} + \frac{\mathfrak{C}d^6X}{1.2.3.4.5.6\,dx^5} - \&c.$$

387. Sin autem progreſſio propoſita coniuncta ſit cum geometrica, quo caſu termini eius infiniteſimi nunquam ad differentias conſtantes reducuntur, ac propterea methodus prior locum inuenit nullum; tum methodus §. 174. tradita medelam afferet. Si enim propoſita ſit haec functio:

$$S = Ap + Bp^2 + Cp^3 + Dp^4 + \ldots + Xp^x,$$

quaerantur valores litterarum α, β, γ, δ, &c. vt ſit

$$\frac{p-1}{p-e^u} = 1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \&c.$$

quibus inuentis, vti eos §. 170. exhibuimus, erit:

$$S = \frac{p}{p-1}\cdot p^x\left(X - \frac{\alpha dX}{dx} + \frac{\beta ddX}{dx^2} - \frac{\gamma d^3X}{dx^3} + \frac{\delta d^4X}{dx^4} - \&c.\right)$$

$\pm$ Conſtante, quae ſummam reddat $=0$, ſi ponatur $x=0$, ſeu quae cuiquam alii caſui ſatisfaciat. Sumto ergo differentiali haec conſtans ex computo abibit, eritque:

$$dS = \frac{p}{p-1}\cdot p^x dx\, lp\left(X - \frac{\alpha dX}{dx} + \frac{\beta ddX}{dx^2} - \frac{\gamma d^3X}{dx^3} + \&c.\right)$$
$$+ \frac{p}{p-1}\cdot p^x\left(dX - \frac{\alpha ddX}{dx} + \frac{\beta d^3X}{dx^2} - \frac{\gamma d^4X}{dx^3} + \&c.\right)$$

ſiue

$$dS = \frac{p^{x+1}}{p-1}\left(Xdx\,lp - (\alpha lp - 1)dX + (\beta lp - \alpha)\frac{ddX}{dx} - (\gamma lp - \beta)\frac{d^3X}{dx^2} + \&c.\right)$$

quod eſt differentiale quaeſitum functionis propoſitae S.

388.

388. Sin autem functio inexplicabilis propoſita ex factoribus conſtet, eorumque logarithmi infiniteſimi differentias habeant conſtantes ſiue minus ; tum hac quoque methodo differentiale functionis perpetuo exhiberi poterit. Sit enim

$$S = \overset{1}{A}.\ \overset{2}{B}.\ \overset{3}{C}.\ \overset{4}{D}.\ .\ .\ .\ .\ \overset{x}{X}.$$

Quia hinc fit

$$lS = lA + lB + lC + lD + \ .\ .\ .\ + lX$$

methodo ſuperiori, numeros Bernoullianos in ſubſidium vocando erit:

$$lS = \int dx\, lX + \tfrac{1}{2}lX + \frac{\mathfrak{A}d.lX}{1.2\,dx} - \frac{\mathfrak{B}d^3.lX}{1.2.3.4\,dx^3} + \&c.$$

qua expreſſione differentiata fit:

$$\frac{dS}{S} = dx\,lX + \tfrac{1}{2}d.lX + \frac{\mathfrak{A}dd.lX}{1.2\,dx} - \frac{\mathfrak{B}d^4.lX}{1.2.3.4\,dx^3} + \frac{\mathfrak{C}d^6.lX}{1.2.3.4.5.6.\,dx^5} - \frac{\mathfrak{D}d^8.lX}{1.2.3\ldots 8\,dx^7} + \&c.$$

Hinc ſi fuerit $X = x$, vt ſit:

$$S = 1.\ 2.\ 3.\ 4.\ .\ .\ .\ .\ .\ x$$

fiet applicatione facta

$$\frac{dS}{S} = dx\,lx + \frac{dx}{2x} - \frac{\mathfrak{A}dx}{2xx} + \frac{\mathfrak{B}dx}{4x^4} - \frac{\mathfrak{C}dx}{6x^6} + \&c.$$

quae forma, ſi x ſit numerus valde magnus, commodius vſurpatur, quam eae, quas ante inuenimus.

CAPUT

CAPUT XVII.

DE INTERPOLATIONE SERIERUM.

389.

Series interpolari dicitur, dum eius termini aſſignantur, qui reſpondent indicibus fractis vel etiam ſurdis. Si igitur ſeriei terminus generalis fuerit cognitus, interpolatio nullam habet difficultatem, cum quicunque numerus loco indicis x ſubſtituatur, iſta expreſſio praebeat terminum reſpondentem. Verum ſi ſeries ita fuerit comparata, vt eius terminus generalis nullo modo exhiberi queat; tum interpolatio huiusmodi ſerierum plerumque eſt maxime difficilis, neque maximam partem termini indicibus non integris reſpondentes aliter niſi per ſeries infinitas definiri poſſunt. Quoniam ergo in Capite praecedente huiusmodi expreſſionum, quae more conſueto finite exprimi non poſſunt, valores quibuscunque indicibus reſpondentes determinauimus; ea tractatio maximam afferet vtilitatem ad interpolationes perficiendas. Quam ob cauſam vſum, qui ex ſuperiori Capite in hoc negotium redundat, hic diligentius proſequemur.

390. Sit ergo propoſita ſeries quaecunque

$$\overset{1}{A} + \overset{2}{B} + \overset{3}{C} + \overset{4}{D} + \ldots \ldots \overset{x}{X}$$

cuius terminus generalis X ſit cognitus, ſummatorius autem S lateat. Hinc formetur alia ſeries, cuius terminus

gene-

generalis aequetur illius ſeriei termino ſummatorio, eritque iſta noua ſeries:

$$\overset{1}{A}; \overset{2}{(A+B)}; \overset{3}{(A+B+C)}; \overset{4}{(A+B+C+D)}; \overset{5}{(A+B+C+D+E)};$$
&c.

eiusque terminus generalis ſeu indici indefinito x reſpondens erit $= A+B+C+D+ \ldots +X = S$, qui cum explicite non ſit cognitus, interpolatio huius nouae ſeriei iisdem difficultatibus erit obnoxia, quas ante meminimus. Ad hanc ergo ſeriem interpolandam inueſtigari oportet valores ipſius S, quos recipit, ſi loco x numeri quicunque non integri ſubſtituantur. Si enim x eſſet numerus integer, tum conueniens ipſius S valor ſine difficultate reperiretur, additione ſcilicet tot terminorum ſeriei $A+B+C+D+$&c. quot x contineat vnitates.

391. Quo igitur ea, quae in Capite praecedente ſunt tradita, in vſum vocari poſſint, ponamus x eſſe numerum integrum, ita vt valor ei reſpondens $S = A+B+C+ \ldots +X$ ſit cognitus, & quaeramus valorem Σ, in quem S transmutetur, ſi loco x ſcribatur $x+\omega$, exiſtente ω fractione quacunque; eritque Σ terminus ſeriei propoſitae interpolandae, qui reſpondet indici $x+\omega$; quo ergo inuento, interpolatio huius ſeriei erit in promtu. Sit Z terminus ſeriei A, B, C, D, E, &c. qui reſpondet indici $x+\omega$, ſintque Z', Z'', Z''', &c. termini eius conſecutiui indices habentes $x+\omega+1$; $x+\omega+2$; $x+\omega+3$; &c.

Ac primo quidem ponamus ſeriei A, B, C, D, &c. terminos infiniteſimos euaneſcere. His ergo poſitis ſeries

$$\overset{1}{A}\,;\ \overset{2}{(A+B)}\,;\ \overset{3}{(A+B+C)}\,;\ \overset{4}{(A+B+C+D)}\,;\ \&c.$$

cuius terminus indici x reſpondens eſt

$$S = A + B + C + \dots\dots + X$$

interpolabitur quaerendo eius terminum Σ, qui indici fracto $x + \omega$ reſpondeat, erit autem vti inuenimus:

$$\Sigma = S \begin{array}{l} +X' + X'' + X''' + X'''' + \&c. \\ -Z' - Z'' - Z''' - Z'''' - \&c. \end{array}$$

ſicque habebitur ſeries infinita iſti termino quaeſito Σ aequalis, quae ob

$$Z = X + \frac{\omega dX}{dx} + \frac{\omega^2 ddX}{1.2\,dx^2} + \frac{\omega^3 d^3X}{1.2.3\,dx^3} + \&c.$$

in hanc formam transmutatur, vt ſit:

$$\begin{aligned} \Sigma = S &- \frac{\omega}{dx} d.\,[X' + X'' + X''' + X'''' + \&c.] \\ &- \frac{\omega^2}{2\,dx^2} dd.\,[X' + X'' + X''' + X'''' + \&c.] \\ &- \frac{\omega^3}{6\,dx^3} d^3.\,[X' + X'' + X''' + X'''' + \&c.] \\ &\qquad \&c. \end{aligned}$$

quarum formularum ea, quae quouis caſu commodior videatur, adhiberi poterit.

392. Sumamus pro A, B, C, D, &c. ſeriem harmonicam quamcunque $\frac{1}{a} + \frac{1}{a+b} + \frac{1}{a+2b} + \frac{1}{a+3b} + \&c.$ cuius

cuius terminus generalis ſeu indici x reſpondens eſt $= \frac{1}{a+(x-1)b} = X$. Hinc formata ſit iſta ſeries:

$$\overset{1}{\frac{1}{a}}; \quad \overset{2}{\left(\frac{1}{a}+\frac{1}{a+b}\right)}; \quad \overset{3}{\left(\frac{1}{a}+\frac{1}{a+b}+\frac{1}{a+2b}\right)}; \quad \overset{4}{\left(\frac{1}{a}+\frac{1}{a+b}+\frac{1}{a+2b}+\frac{1}{a+3b}\right)} \text{ \&c.}$$

cuius propterea terminus indici x reſpondens erit:

$$S = \frac{1}{a} + \frac{1}{a+b} + \frac{1}{a+2b} + \frac{1}{a+3b} + \cdots + \frac{1}{a+(x-1)b}.$$

Si iam Σ denotet terminum iſtius ſeriei indici $x+\omega$ reſpondentem, ob $Z = \frac{1}{a+(x+\omega-1)b}$, erit

$$X' = \frac{1}{a+bx}; \quad Z' = \frac{1}{a+bx+b\omega}$$

$$X'' = \frac{1}{a+b+bx}; \quad Z'' = \frac{1}{a+b+bx+b\omega}$$

$$X''' = \frac{1}{a+2b+bx}; \quad Z''' = \frac{1}{a+2b+bx+b\omega}$$

&c. &c.

hincque orietur:

$$\Sigma = S + \frac{1}{a+bx} + \frac{1}{a+b+bx} + \frac{1}{a+2b+bx} + \text{\&c.}$$

$$- \frac{1}{a+bx+b\omega} - \frac{1}{a+b+bx+b\omega} - \frac{1}{a+2b+bx+b\omega} - \text{\&c.}$$

altera expreſſio autem erit huiusmodi:

$$\Sigma = S + b\omega\left(\frac{1}{(a+bx)^2} + \frac{1}{(a+b+bx)^2} + \frac{1}{(a+2b+bx)^2} + \&c.\right)$$
$$- b^2\omega^2\left(\frac{1}{(a+bx)^3} + \frac{1}{(a+b+bx)^3} + \frac{1}{(a+2b+bx)^3} + \&c.\right)$$
$$+ b^3\omega^3\left(\frac{1}{(a+bx)^4} + \frac{1}{(a+b+bx)^4} + \frac{1}{(a+2b+bx)^4} + \&c.\right)$$
$$\&c.$$

EXEMPLUM I.

Propoſita ſit iſta ſeries:

$$\overset{1}{1};\ \overset{2}{(1+\tfrac{1}{2})};\ \overset{3}{(1+\tfrac{1}{2}+\tfrac{1}{3})};\ \overset{4}{(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4})};\ \&c.$$

cuius terminos, qui indicibus fractis reſpondent, inueniri oporteat.

Erit ergo $a = 1$ & $b = 1$; vnde ſi terminus indici integro x reſpondens ponatur

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{x},$$

terminusque indici fracto $x + \omega$ reſpondens vocetur $= \Sigma$, erit:

$$\Sigma = S + \frac{1}{1+x} + \frac{1}{2+x} + \frac{1}{3+x} + \frac{1}{4+x} + \frac{1}{5+x} + \&c.$$
$$- \frac{1}{1+x+\omega} - \frac{1}{2+x+\omega} - \frac{1}{3+x+\omega} - \frac{1}{4+x+\omega} - \frac{1}{5+x+\omega} - \&c.$$

Notandum autem eſt, ſi inuentus fuerit terminus reſpondens indici fracto ω, quem ponamus $= T$, ex eo termi-

minum indicis $x+\omega$ facile inueniri poſſe; erit enim, ſi T', T'', T''', &c. denotent terminos indicibus $1+\omega$, $2+\omega$, $3+\omega$, &c. reſpondentes:

$$T' = T + \frac{1}{1+\omega}$$

$$T'' = T + \frac{1}{1+\omega} + \frac{1}{2+\omega}$$

$$T''' = T + \frac{1}{1+\omega} + \frac{1}{2+\omega} + \frac{1}{3+\omega} \quad \&c.$$

vnde ſufficit eos tantum terminos, qui reſpondent indicibus ω vnitate minoribus, inueſtigaſſe. Quem in finem ponamus $x=0$, erit quoque $S=0$, atque terminus ſeriei T indici fracto ω reſpondens ita exprimetur:*

$$T = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c.$$
$$- \frac{1}{1+\omega} - \frac{1}{2+\omega} - \frac{1}{3+\omega} - \frac{1}{4+\omega} - \&c.$$

vel his fractionibus in ſeries infinitas conuerſis prodibit altera expreſſio:

$$T = +\omega\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \&c.\right)$$
$$-\omega^2\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \&c.\right)$$
$$+\omega^3\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \&c.\right)$$
$$-\omega^4\left(1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \frac{1}{5^5} + \&c.\right)$$
$$\&c.$$

quae ad valorem ipſius T proxime inueniendum perquam eſt apta.

Quaeratur ergo propoſitae ſeriei terminus reſpondens indici, $\frac{1}{2}$ qui ſi ponatur $= T$, erit:

$$T = 1 - \frac{2}{3} + \frac{1}{2} - \frac{2}{5} + \frac{1}{3} - \frac{2}{7} + \frac{1}{4} - \frac{2}{9} + \&c.$$

ſeu $T = 2\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \&c.\right)$

cuius ſeries valor eſt $= 2 - 2l2$, ſicque terminus indicis $= \frac{1}{2}$ finite exprimi poteſt. Erunt ergo termini ſequentes, quorum indices ſunt $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$, &c. ita expreſſi:

Ind.	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	
Term.	$2-2l2$;	$2+\frac{2}{3}-2l2$;	$2+\frac{2}{3}+\frac{2}{5}-2l2$;	$2+\frac{2}{3}+\frac{2}{5}+\frac{2}{7}-2l2$;	&c.

EXEMPLUM II.

Propoſita ſit iſta ſeries:

$$\overset{1}{1}\;;\;\overset{2}{\left(1+\tfrac{1}{3}\right)}\;;\;\overset{3}{\left(1+\tfrac{1}{3}+\tfrac{1}{5}\right)}\;;\;\overset{4}{\left(1+\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}\right)}\;;\;\&c.$$

cuius terminos indicibus fractis reſpondentes exprimere oporteat.

Erit ergo $a = 1$, $b = 2$, vnde ſi terminus indici integro x reſpondens ponatur

$$S = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2x-1}$$

terminusque indici fracto $x + \omega$ vocetur $= \Sigma$, erit

$$\Sigma = S + \frac{1}{1+2x} + \frac{1}{3+2x} + \frac{1}{5+2x} + \frac{1}{7+2x} + \&c.$$
$$- \frac{1}{1+2(x+\omega)} - \frac{1}{3+2(x+\omega)} - \frac{1}{5+2(x+\omega)} - \frac{1}{7+2(x+\omega)} - \&c.$$

Cum

Cum igitur ſufficiat terminos indicibus vnitate minoribus aſſignaſſe, ſit $x = 0$, & $S = 0$: quocirca ſi terminus indici ω conueniens ponatur $= T$, erit:

$$T = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \&c.$$

$$- \frac{1}{1+2\omega} - \frac{1}{3+2\omega} - \frac{1}{5+2\omega} - \frac{1}{7+2\omega} - \frac{1}{9+2\omega} - \&c.$$

& ſi ω numerum quemcunque denotare ponatur, quoniam T eſt terminus indici ω reſpondens, erit T terminus generalis ſeriei propoſitae, qui etiam hoc modo exprimetur:

$$T = \frac{2\omega}{1(1+2\omega)} + \frac{2\omega}{3(3+2\omega)} + \frac{2\omega}{5(5+2\omega)} + \frac{2\omega}{7(7+2\omega)} + \&c.$$

vel ita:

$$T = 2\omega \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c.\right)$$

$$- 4\omega^2 \left(1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{9^3} + \&c.\right)$$

$$+ 8\omega^3 \left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \&c.\right)$$

$$- 16\omega^4 \left(1 + \frac{1}{3^5} + \frac{1}{5^5} + \frac{1}{7^5} + \frac{1}{9^5} + \&c.\right)$$

&c.

Pona-

Ponamus eſſe $\omega = \frac{1}{2}$, erit terminus huic indici reſpondens $\quad T = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \&c. = l2,$

eruntque

Ind.	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$
Term.	$l2;$	$\frac{1}{2} + l2;$	$\frac{1}{2} + \frac{1}{4} + l2;$	$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + l2;$

&c.

Si ſit $\omega = \frac{1}{4}$; erit

$$T = +1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \&c. \quad \text{ſiue}$$
$$- \frac{2}{3} - \frac{2}{7} - \frac{2}{11} - \frac{2}{15} - \&c.$$
$$T = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \&c. - \frac{1}{2} l2 = \frac{\pi}{4} - \frac{1}{2} l2.$$

393. Quod ſi ergo huius ſeriei generalis:

$$\frac{1}{a}\,;\ \left(\frac{1}{a} + \frac{1}{a+b}\right);\ \left(\frac{1}{a} + \frac{1}{a+b} + \frac{1}{a+2b}\right);\ \&c.$$

quaeratur terminus reſpondens indici $= \frac{1}{2}$, ponatur in expreſſionibus §. praeced. $x = 0$, & $\omega = \frac{1}{2}$; fietque $S = 0$, & terminus indici $\frac{1}{2}$ reſpondens quaeſitus erit

$$\Sigma = \frac{1}{a} - \frac{2}{2a+b} + \frac{1}{a+b} - \frac{2}{2a+3b} + \frac{1}{a+2b} - \frac{2}{2a+5b} + \&c.$$

ſiue terminis ad maiorem vniformitatem perductis erit

$$\tfrac{1}{2}\Sigma = \frac{1}{2a} - \frac{1}{2a+b} + \frac{1}{2a+2b} - \frac{1}{2a+3b} + \frac{1}{2a+4b} - \&c.$$

in qua ſerie cum ſigna + & — alternentur, ſumendis continuis differentiis per methodum ſupra expoſitam valor ipſius $\frac{1}{2}\Sigma$ per ſeriem magis conuergentem exprimetur.

Erunt

Erunt autem differentiarum ſeries:

$$\frac{b}{2a(2a+b)};\ \frac{b}{(2a+b)(2a+2b)};\ \frac{b}{(2a+2b)(2a+3b)};\ \&c.$$

$$\frac{2bb}{2a(2a+b)(2a+2b)};\ \frac{2bb}{(2a+b)(a+2b)(2a+3b)};\ \&c.$$

$$\frac{6b^3}{2a(2a+b)(2a+2b)(2a+3b)};\ \&c.$$

$$\&c.$$

Ex quibus concluditur fore:

$$\tfrac{1}{2}\Sigma = \frac{1}{4a} + \frac{1b}{8a(2a+b)} + \frac{1.2bb}{16a(2a+b)(2a+2b)} + \frac{1.2.3b^3}{32a(2a+b)(2a+2b)(2a+3b)} + \&c.$$

Hincque ergo habebitur:

$$\Sigma = \frac{1}{2a} + \frac{\frac{1}{2}\cdot b}{2a(2a+b)} + \frac{\frac{1}{2}\cdot\frac{2}{2}bb}{2a(2a+b)(2a+2b)} + \frac{\frac{1}{2}\cdot\frac{2}{2}\cdot\frac{3}{2}b^3}{2a(2a+b)(2a+2b)(2a+3b)} + \&c.$$

quae ſeries maxime conuergit, atque valorem termini Σ facili labore proxime exhibet.

394. Quod ſi autem in genere ſeriei A, B, C, D, E, &c. termini infiniteſimi euaneſcant, terminusque indici ω reſpondens fuerit $= Z$, eiusque ſequentes, qui indicibus $\omega+1$, $\omega+2$, $\omega+3$, &c. reſpondeant, ſint Z', Z'', Z''', Z^{IV}, &c. Si in ſuperioribus (391) ponatur

 $x=0$,

$x = 0$, vt ſit $S = 0$ & $X' = A$, $X'' = B$, $X''' = C$, &c. ſequetur, ſi formetur huiusmodi ſeries:

$$\overset{1}{A},\ (\overset{2}{A + B}),\ (\overset{3}{A + B + C}),\ (\overset{4}{A + B + C + D}),\ \&c.$$

eiusque terminus indici ω reſpondens ponatur $= \Sigma$, fore

$$\Sigma = (A - Z') + (B - Z'') + (C - Z''') + (D - Z^{IV}) + \&c.$$

ex qua expreſſione termini quicunque intermedii definiri poterunt. Sufficiet autem ad interpolationem perficiendam eos terminos inueſtigaſſe, qui reſpondeant indicibus ω vnitate minoribus. Si enim terminus Σ indici huiusmodi cuicunque ω reſpondens fuerit repertus, iique qui conueniant indicibus $\omega + 1$, $\omega + 2$, $\omega + 3$, &c. ponantur Σ', Σ'', Σ''', Σ^{IV}, &c. erit

$$\Sigma' = \Sigma + Z'$$
$$\Sigma'' = \Sigma + Z' + Z''$$
$$\Sigma''' = \Sigma + Z' + Z'' + Z'''$$
$$\&c.$$

EXEMPLUM I.

Interpolare hanc ſeriem:

$$\overset{1}{1};\ (\overset{2}{1 + \tfrac{1}{4}});\ (\overset{3}{1 + \tfrac{1}{4} + \tfrac{1}{9}});\ (\overset{4}{1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16}});$$

&c.

Sit Σ huius ſeriei terminus reſpondens indici ω, & cum haec ſeries formata ſit ex ſummatione huius:

$$1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \tfrac{1}{25} + \&c.$$

cuius terminus indici ω reſpondens eſt $= \frac{1}{\omega^2}$ erit

$Z =$

$$Z = +1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \&c.$$
$$- \frac{1}{(1+\omega)^2} - \frac{1}{(2+\omega)^2} - \frac{1}{(3+\omega)^2} - \frac{1}{(4+\omega)^2} - \&c.$$

Quod ſi ergo ſeriei propoſitae quaeratur terminus indici $\frac{1}{2}$ reſpondens, poni debit $\omega = \frac{1}{2}$, fietque:

$\Sigma = 1 - \frac{4}{9} + \frac{1}{4} - \frac{4}{25} + \frac{1}{9} - \frac{4}{49} + \&c.$ ſiue

$\Sigma = 4(\frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \frac{1}{36} - \frac{1}{49} + \&c.)$

Cum igitur ſit $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{25} + \&c. = \frac{\pi^2}{12}$, erit

$\Sigma = 4\left(1 - \frac{\pi\pi}{12}\right) = 4 - \frac{1}{3}\pi^2$, qui eſt terminus indici $\frac{1}{2}$ reſpondens. Hinc ergo reſpondebunt

Indicibus $\frac{1}{2}$ $\frac{3}{2}$ $\frac{5}{3}$ &c.

Termini $4 - \frac{1}{3}\pi^2$; $\frac{4}{1} + \frac{4}{9} - \frac{1}{3}\pi^2$; $\frac{4}{1} + \frac{4}{9} + \frac{4}{25} - \frac{1}{3}\pi^1$; &c.

EXEMPLUM II.

Interpolare hanc ſeriem:

$$\overset{1}{1};\ \overset{2}{(1 + \tfrac{1}{9})};\ \overset{3}{(1 + \tfrac{1}{9} + \tfrac{1}{25})};\ \overset{4}{(1 + \tfrac{1}{9} + \tfrac{1}{25} + \tfrac{1}{49})}$$
&c.

Sit Σ terminus reſpondens indici cuicunque ω, & cum haec ſeries formata ſit ex ſummatione huius:

$$1 + \tfrac{1}{9} + \tfrac{1}{25} + \tfrac{1}{49} + \tfrac{1}{81} + \&c.$$

ex qua fit terminus indici ω reſpondens $Z = \frac{1}{(2\omega - 1)^2}$

erit $Z' = \frac{1}{(2\omega + 1)^2}$; $Z'' = \frac{1}{(2\omega + 3)^2}$; $Z''' = \frac{1}{(2\omega + 5)^2}$
&c.

 Quam-

Quamobrem habebitur:

$$\Sigma = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \&c.$$

$$- \frac{1}{(1+2\omega)^2} - \frac{1}{(3+2\omega)^2} - \frac{1}{(5+2\omega)^2} - \frac{1}{(7+2\omega)^2} - \&c.$$

Ponamus $\omega = \frac{1}{2}$, vt inueniamus terminum ſeriei propoſitae reſpondentem indici $= \frac{1}{2}$, qui erit:

$$\Sigma = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \&c. = \frac{\pi\pi}{12},$$

ex quo termini, qui medium interiacent inter binos quosvis datos, ſequenti modo exprimentur. Reſpondebunt

Ind. $\frac{1}{2}$; $\frac{3}{2}$; $\frac{5}{2}$; $\frac{7}{2}$; &c.

Term. $\frac{\pi\pi}{12}$; $\frac{1}{4} + \frac{\pi\pi}{12}$; $\frac{1}{4} + \frac{1}{16} + \frac{\pi\pi}{12}$; $\frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \frac{\pi\pi}{12}$; &c.

EXEMPLUM III.

Interpolare hanc ſeriem:

$$\overset{1}{1};\ \overset{2}{\left(1 + \frac{1}{2^n}\right)};\ \overset{3}{\left(1 + \frac{1}{2^n} + \frac{1}{3^n}\right)};\ \overset{4}{\left(1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n}\right)};\ \&c.$$

Sit vt ante Σ terminus indici ω reſpondens, erit

$Z = \frac{1}{\omega^3}$; & $Z' = \frac{1}{(1+\omega)^n}$; $Z'' = \frac{1}{(2+\omega)^n}$; $Z''' = \frac{1}{(3+\omega)^n}$ &c. hincque habebitur:

$$\Sigma = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \&c.$$

$$- \frac{1}{(1+\omega)^n} - \frac{1}{(2+\omega)^n} - \frac{1}{(3+\omega)^n} - \frac{1}{(4+\omega)^n} - \&c.$$

Si

Si igitur deſideretur terminus indici $\frac{1}{2}$ reſpondens, erit

$$\text{is} = 1 - \frac{2^n}{3^n} + \frac{1}{2^n} - \frac{2^n}{5^n} + \frac{1}{3^n} - \frac{2^n}{7^n} + \&c.$$

$$\text{ſeu} = 2^n\left(\frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} + \&c.\right)$$

Quare ſi ponatur:

$$\mathfrak{N} = 1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} + \&c.$$

erit ſeriei propoſitae terminus qui indici $\frac{1}{2}$ reſpondet $= 2^n(1 - \mathfrak{N})$; hincque reſpondebunt

Indic. $\frac{1}{2}$; $\frac{3}{2}$; $\frac{5}{2}$

Term. $2^n - 2^n\mathfrak{N}$; $2^n + \frac{2^n}{3^n} - 2^n\mathfrak{N}$; $2^n + \frac{2^n}{3^n} + \frac{2^n}{5^n} - 2^n\mathfrak{N}$; &c.

EXEMPLUM IV.

Interpolare hanc ſeriem:

$$\overset{1}{1};\ \overset{2}{\left(1+\frac{1}{3^n}\right)};\ \overset{3}{\left(1+\frac{1}{3^n}+\frac{1}{5^n}\right)};\ \overset{4}{\left(1+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}\right)};\ \&c.$$

Sit Σ terminus qui indici cuicunque ω reſpondeat, & cum ſit $Z = \frac{1}{(2\omega-1)^n}$, erit:

$$Z' = \frac{1}{(2\omega+1)^n};\ Z'' = \frac{1}{(2\omega+3)^n};\ Z''' = \frac{1}{(2\omega+5)^n};\ \&c.$$

atque

$$\Sigma = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \&c.$$

$$- \frac{1}{(1+2\omega)^n} - \frac{1}{(3+2\omega)^n} - \frac{1}{(5+2\omega)^n} - \frac{1}{(7+2\omega)^n} - \&c.$$

Ponatur $\omega = \frac{1}{2}$, & prodibit terminus indici $\frac{1}{2}$ reſpondens

$$= 1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} + \&c. = \mathfrak{N},$$

ex quo porro erunt reliqui termini inter binos datos medii

$$\begin{array}{llll} \text{Indices:} & \frac{1}{2}; & \frac{3}{2}; & \frac{5}{2}; \quad \&c. \\ \text{Termini:} & \mathfrak{N}; & \frac{1}{2^n} + \mathfrak{N}; & \frac{1}{2^n} + \frac{1}{4^n} + \mathfrak{N}; \quad \&c. \end{array}$$

395. Ponamus nunc ſeriei A, B, C, D, E, &c. ex cuius ſummatione ſeries interpolanda formatur, terminos infiniteſimos non euaneſcere, ſed ita eſſe comparatos, vt eorum differentiae euaneſcant; ſitque X huius ſeriei terminus reſpondens indici x, & Z terminus reſpondens exponenti $x+\omega$, tum vero ſint X', X'', X''', X'''', &c. termini ipſum X ſequentes, & Z', Z'', Z''', &c. termini ipſum Z ſequentes. Quibus poſitis proponatur haec ſeries interpolanda:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \\ A; & (A+B); & (A+B+C); & (A+B+C+D); & \&c. \end{array}$$

cuius terminus indici x reſpondens ſit $= S$, at terminus indici $x+\omega$ reſpondens ſit $= \Sigma$; eritque ex iis, quae Capite praecedente ſunt tradita:

$$\begin{aligned} \Sigma = S &+ X' + X'' + X''' + \&c. \\ &- Z' - Z'' - Z''' - \&c. \\ &+ \omega X' + \omega \left\{ \begin{array}{l} X'' + X''' + X'''' + \&c. \\ -X' - X'' - X''' - \&c. \end{array} \right\} \end{aligned}$$

Quia autem vt ante ſufficit terminos indicibus vnitate minoribus reſpondentes inueſtigaſſe, ponamus $x = 0$, vt ſit $S = 0$, $X' = A$, $X'' = B$, &c. eritque terminus indici ω reſpondens:

$$\Sigma = (A - Z') + (B - Z'') + (C - Z''') + (D - Z'''') \text{ \&c.}$$
$$+ \omega A + \omega [(B - A) + (C - B) + (D - C) + (E - D) + \text{\&c.}]$$

Vel ſi differentias has more ſupra recepto exprimere velimus quo eſt $\Delta A = B - A$; $\Delta B = C - B$; &c. habebitur:

$$\Sigma = (A - Z') + (B - Z'') + (C - Z''') + (D - Z'''') + \text{\&c.}$$
$$+ \omega (A + \Delta A + \Delta B + \Delta C + \Delta D + \text{\&c.})$$

395. Sin autem ſeriei A, B, C, D, E, &c. ex cuius ſummatione ſeries interpolanda formatur, termini infiniteſimi neque ipſi euaneſcant, neque differentias primas habeant euaneſcentes; tum plures ſeries ad valorem ipſius Σ exprimendum adiici debebunt, quoad ſcilicet ad differentias terminorum infiniteſimorum euaneſcentes perueniatur. Sit enim vt ante ſeriei A, B, C, D, E, &c. terminus indici x reſpondens $= X$, eumque ſequentes X', X'', X''', &c. indici autem $x + \omega$ reſpondeat terminus Z, quem ſequantur Z', Z'', &c. atque proponatur haec ſeries:

$$\overset{1}{A};\ \overset{2}{(A+B)};\ \overset{3}{(A+B+C)};\ \overset{4}{(A+B+C+D)};\ \text{\&c.}$$

cuius terminus indici x reſpondens ſit

$$S = A + B + C + D + \ldots\ldots + X$$

in-

indici vero $x+\omega$ reſpondeat terminus Σ; ita vt

indicibus	reſpondeant termini
$x+\omega+1$	$\Sigma' = \Sigma + Z'$
$x+\omega+2$	$\Sigma'' = \Sigma + Z' + Z''$
$x+\omega+3$	$\Sigma''' = \Sigma + Z' + Z'' + Z'''$
&c.	&c.

Si iam differentiae terminorum ita exprimantur, vt ſit

$$\Delta X' = X'' - X';\ \Delta X'' = X''' - X'';\ \Delta X''' = X'''' - X''';\ \&c.$$
$$\Delta^2 X' = \Delta X'' - \Delta X';\ \Delta^2 X'' = \Delta X''' - \Delta X'';\ \Delta^2 X''' = \Delta X'''' - \Delta X''';\ \&c.$$
$$\Delta^3 X' = \Delta^2 X'' - \Delta^2 X';\ \Delta^3 X'' = \Delta^2 X''' - \Delta^2 X'';\ \&c.$$

ex §. 377. terminus Σ ſequenti modo exprimetur:

$$\begin{aligned}\Sigma = S\ &+ X' + X'' + X''' + X'''' + \&c.\\ &- Z' - Z'' - Z''' - Z'''' - \&c.\\ &+ \omega[X' + \Delta X' + \Delta X'' + \Delta X''' + \Delta X'''' + \&c.]\\ &+ \frac{\omega(\omega-1)}{1.\,2}[\Delta X' + \Delta^2 X' + \Delta^2 X'' + \Delta^2 X''' + \Delta^2 X'''' + \&c.]\\ &+ \frac{\omega(\omega-1)(\omega-2)}{1.\,2.\,3}[\Delta^2 X' + \Delta^3 X' + \Delta^3 X'' + \Delta^3 X''' + \Delta^3 X'''' + \&c.]\\ &\&c.\end{aligned}$$

397. Sufficit, vti iam notauimus, tot huiusmodi ſeries adieciſſe, donec ad terminorum infiniteſimorum differentias euaneſcentes perueniatur: ſi enim has ipſas ſeries quoque in infinitum continuare velimus, vel eo vsque ſaltem, donec terminorum finitorum differentiae euaneſcant; tum ob

$Z' =$

$$Z' = X' + \omega\Delta X' + \frac{\omega(\omega-1)}{1.\,2.}\Delta^2 X' + \frac{\omega(\omega-1)(\omega-2)}{1.\,2.\,3}\Delta^3 X' + \&c.$$

tota expreſſio inuenta contrahetur in hanc:

$$\Sigma = S + \omega X' + \frac{\omega(\omega-1)}{1.\,2}\Delta X' + \frac{\omega(\omega-1)(\omega-2)}{1.\,2.\,3}\Delta^2 X' + \&c.$$

quae terminum ſummatorium ſeriei A + B + C + D + &c. inuoluit; qui autem ſi eſſet cognitus, interpolatio nullam haberet difficultatem. Interim tamen & hac formula vti licebit, quippe quae, quoties abrumpitur, quemvis terminum interpolandum finite & algebraicae expreſſum exhibet: ſin autem in infinitum progrediatur, plerumque praeſtat priorem formulam adhibere, in qua ratio terminorum infiniteſimorum habetur. Haec vero, ſi ponatur $x = 0$, vt Σ denotet terminum indici ω reſpondentem, ob $S = 0$ hanc formam induet:

$$\begin{aligned}
\Sigma = \quad & + A + B + C + D + \&c.\\
& - Z' - Z'' - Z''' - Z'''' - \&c.\\
& + \omega[A + \Delta A + \Delta B + \Delta C + \Delta D + \&c.]\\
& + \frac{\omega(\omega-1)}{1.\,2}[\Delta A + \Delta^2 A + \Delta^2 B + \Delta^2 C + \Delta^2 D + \&c.]\\
& + \frac{\omega(\omega-1)(\omega-2)}{1.\,2.\,3}[\Delta^2 A + \Delta^3 A + \Delta^3 B + \Delta^3 C + \Delta^3 D + \&c.]\\
& \&c.
\end{aligned}$$

Vel ſi ponatur breuitatis gratia:

$$\omega = \alpha;\quad \frac{\omega(\omega-1)}{1.\,2} = \beta;\quad \frac{\omega(\omega-1)(\omega-2)}{1.\,2.\,3} = \gamma;\quad \&c.$$

erit:

$$\begin{aligned}\Sigma = \quad & \alpha A + \beta\Delta A + \gamma\Delta^2 A + \delta\Delta^3 A + \&c.\\ & + A + \alpha\Delta A + \beta\Delta^2 A + \gamma\Delta^3 A + \&c. - Z'\\ & + B + \alpha\Delta B + \beta\Delta^2 B + \gamma\Delta^3 B + \&c. - Z''\\ & + C + \alpha\Delta C + \beta\Delta^2 C + \gamma\Delta^3 C + \&c. - Z'''\\ & \&c.\end{aligned}$$

quarum ſerierum horizontalium numerus in infinitum quidem progreditur, at quaelibet finito terminorum numero conſtat.

EXEMPLUM.

Interpolare hanc ſeriem:

$$\overset{1}{\tfrac{1}{2}};\ \overset{2}{\tfrac{1}{2}+\tfrac{2}{3}};\ \overset{3}{\tfrac{1}{2}+\tfrac{2}{3}+\tfrac{3}{4}};\ \overset{4}{\tfrac{1}{2}+\tfrac{2}{3}+\tfrac{3}{4}+\tfrac{4}{5}};\ \&c.$$

Sit huius ſeriei terminus indici ω reſpondens $= \Sigma$, & cum ea oriatur ex ſummatione huius ſeriei:

$$\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5}\ \&c.\quad \text{erit}\quad Z = \frac{\omega}{\omega+1};$$

& quia termini infiniteſimi differentias ſuas primas iam habent euaneſcentes, differentiae tantum primae ſunt accipiendae, quae erunt:

$$\text{ob}\quad A = \frac{1}{2};\quad B = \frac{2}{3};\quad C = \frac{3}{4};\quad D = \frac{4}{5};\quad \&c.$$

$$\Delta A = \frac{1}{2.3};\quad \Delta B = \frac{1}{3.4};\quad \Delta C = \frac{1}{4.5};\quad \&c.$$

Hinc

Hinc ergo habebitur:

$$\Sigma = \frac{\omega}{2} + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \&c.$$
$$+ \frac{\omega}{2.3} + \frac{\omega}{3.4} + \frac{\omega}{4.5} + \frac{\omega}{5.6} + \&c.$$
$$- \frac{(\omega+1)}{\omega+2} - \frac{(\omega+2)}{\omega+3} - \frac{(\omega+3)}{\omega+4} - \frac{(\omega+4)}{\omega+5} - \&c.$$

ſeu ob $\frac{\omega}{2} + \frac{\omega}{2.3} + \frac{\omega}{3.4} + \frac{\omega}{4.5} + \&c. = \omega$; erit

$$\Sigma = \omega + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \&c.$$
$$- \frac{(\omega+1)}{\omega+2} - \frac{(\omega+2)}{\omega+3} - \frac{(\omega+3)}{\omega+4} - \frac{(\omega+4)}{\omega+5} - \&c.$$

Si ergo quaeratur terminus indici $\frac{1}{2}$ reſpondens, erit is

$$\Sigma = \frac{1}{2} + \frac{1}{2} - \frac{3}{5} + \frac{2}{3} - \frac{5}{7} + \frac{3}{4} - \frac{7}{9} + \frac{4}{5} - \frac{9}{11} + \&c.$$

ſeu $\Sigma = \frac{1}{2} - \frac{1}{2.5} - \frac{1}{3.7} - \frac{1}{4.9} - \frac{1}{5.11} - \frac{1}{6.13} - \&c.$

ideoque $\frac{1}{2}\Sigma = \frac{1}{4} - \frac{1}{4.5} - \frac{1}{6.7} - \frac{1}{8.9} - \frac{1}{10.11} - \frac{1}{12.13} - \&c.$

ſeu $\frac{1}{2}\Sigma = \frac{1}{4} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \frac{1}{10} - \frac{1}{13} - \&c.$

$$+ \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \&c.$$

 Qua-

Quare cum ſit $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \&c. = l2$

erit $\frac{1}{2}\Sigma = l2 - 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \&c. = l2 - \frac{7}{12}$,

ideoque $\Sigma = 2\,l\,2 - \frac{7}{6}$.

398. Pergamus nunc ad ſeries interpolandas, quarum termini ex factoribus ſunt conflati, ſitque propoſita haec ſeries generaliſſima:

$$\overset{1}{A}\,;\ \overset{2}{AB}\,;\ \overset{3}{ABC}\,;\ \overset{4}{ABCD}\,;\ \overset{5}{ABCDE}\,;\ \&c.$$

cuius terminus indici ω reſpondens ſit $= \Sigma$. Erit ergo $l\Sigma$ terminus reſpondens indici ω in hac ſerie:

$$\overset{1}{lA}\,;\ \overset{2}{(lA + lB)}\,;\ \overset{3}{(lA + lB + lC)}\,;\ \overset{4}{(lA + lB + lC + lD)}\ \&c.$$

Quodſi ergo ponamus huius ſeriei terminos infiniteſimos euaneſcere; atque ſeriei A, B, C, D, E, &c. terminum indici ω reſpondentem eſſe Z, eiusque ſequentes indicibus $\omega + 1$, $\omega + 2$, $\omega + 3$, &c. reſpondentes eſſe Z', Z'', Z''', Z'''', &c. erit ex ſupra demonſtratis:

$$l\Sigma = \begin{matrix} + lA + lB + lC + lD + \&c. \\ - lZ' - lZ'' - lZ''' - lZ'''' - \&c. \end{matrix}$$

Hinc igitur ad numeros progrediendo habebitur:

$$\Sigma = \frac{A}{Z'} \cdot \frac{B}{Z''} \cdot \frac{C}{Z'''} \cdot \frac{D}{Z''''} \cdot \&c.$$

399.

399. Quodſi autem terminorum infiniteſimorum ſeriei A, B, C, D, &c. logarithmi non euaneſcant, ſed habeant differentias euaneſcentes, erit vti vidimus:

$$l\Sigma = + lA + lB + lC + \&c.$$
$$- lZ' - lZ'' - lZ''' - \&c.$$
$$+ \omega lA + \omega\left(l\frac{B}{A} + l\frac{C}{B} + l\frac{D}{C} + \&c.\right)$$

hincque ad numeros a logarithmis procedendo fiet

$$\Sigma = A^{\omega}\cdot\frac{A^{1-\omega}B^{\omega}}{Z'}\cdot\frac{B^{1-\omega}C^{\omega}}{Z''}\cdot\frac{C^{1-\omega}D^{\omega}}{Z'''}\cdot\frac{D^{1-\omega}E^{\omega}}{Z''''}\cdot\&c.$$

At ſi illorum logarithmorum infiniteſimorum differentiae demum ſecundae euaneſcant, erit:

$$l\Sigma = lA + lB + lC + lD + \&c.$$
$$+ lZ' - lZ'' - lZ''' - lZ'''' - \&c.$$
$$+ \omega\left(lA + l\frac{B}{A} + l\frac{C}{B} + l\frac{D}{C} + l\frac{E}{D} + \&c.\right)$$
$$+ \frac{\omega(\omega-1)}{1.2}\left(l\frac{B}{A} + l\frac{AC}{B^2} + l\frac{BD}{C^2} + l\frac{CE}{D^2} + l\frac{DE}{E^2} + \&c.\right)$$

Ex his itaque obtinebitur:

$$\Sigma = A^{\frac{\omega(3-\omega)}{2}}\cdot B^{\frac{\omega(\omega-1)}{1.2}}\cdot$$

$$\frac{A^{\frac{(\omega-1)(\omega-2)}{1.2}}B^{\omega(2-\omega)}C^{\frac{\omega(\omega-1)}{1.2}}}{Z'}\cdot\frac{B^{\frac{(\omega-1)(\omega-2)}{1.2}}C^{\omega(2-\omega)}D^{\frac{\omega(\omega-1)}{1.2}}}{Z''}\ \&c.$$

quae ſi $\omega < 1$ commodius ita exprimetur :

$$\Sigma = \frac{A^{\frac{\omega(3-\omega)}{1.\,2}}}{B^{\frac{\omega(1-\omega)}{1.\,2}}} \cdot \frac{A^{\frac{(1-\omega)(2-\omega)}{1.\,2}}\, B^{\omega(2-\omega)}}{C^{\frac{\omega(1-\omega)}{1.\,2}}\, Z'} \cdot \frac{B^{\frac{(1-\omega)(2-\omega)}{1.\,2}}\, C^{\omega(2-\omega)}}{D^{\frac{\omega(1-\omega)}{1.\,2}}\, Z''} . \&c.$$

400. Accommodemus hanc interpolationem ad iſtam ſeriem :

$$\overset{1}{\frac{a}{b}} ; \overset{2}{\frac{a(a+c)}{b(b+c)}} ; \overset{3}{\frac{a(a+c)(a+2c)}{b(b+c)(b+2c)}} ; \overset{4}{\frac{a(a+c)(a+2c)(a+3c)}{b(b+c)(b+2c)(b+3c)}} ; \&c.$$

cuius factores deſumti ſunt ex hac ſerie :

$$\overset{1}{\frac{a}{b}} ; \overset{2}{\frac{a+c}{b+c}} ; \overset{3}{\frac{a+2c}{b+2c}} ; \overset{4}{\frac{a+3c}{b+3c}} ; \&c.$$

cuius terminorum infiniteſimorum logarithmi ſunt $= 0$. Erit ergo $Z = \frac{a-c+c\omega}{b-c+c\omega}$; $Z' = \frac{a+c\omega}{b+c\omega}$; &c. Hinc ſi illius ſeriei terminus indici ω reſpondens ponatur $= \Sigma$, erit ex §. 398 :

$$\Sigma = \frac{a(b+c\omega)}{b(a+c\omega)} \cdot \frac{(a+c)(b+c+c\omega)}{(b+c)(a+c+c\omega)} \cdot \frac{(a+2c)(b+2c+c\omega)}{(b+2c)(b+2c+c\omega)} . \&c.$$

Quare ſi deſideretur terminus indici $\frac{1}{2}$ reſpondens, facto $\omega = \frac{1}{2}$, erit :

$$\Sigma = \frac{a(2b+c)}{b(2a+c)} \cdot \frac{(a+c)(2b+3c)}{(b+c)(2a+3c)} \cdot \frac{(a+2c)(2b+5c)}{(b+2c)(2a+5c)} . \&c.$$

EXEM-

EXEMPLUM.

Interpolare hanc ſeriem:

$$\overset{1}{\frac{1}{2}};\ \overset{2}{\frac{1.3}{2.4}};\ \overset{3}{\frac{1.3.5}{2.4.6}};\ \overset{4}{\frac{1.3.5.7}{2.4.6.8}};\ \overset{5}{\frac{1.3.5.7.9}{2.4.6.8.10}};\ \&c.$$

Cum hic ſit $a = 1$, $b = 2$, & $c = 2$; ſi terminus indici cuicunque ω reſpondens $= \Sigma$, erit

$$\Sigma = \frac{1(2+2\omega)}{2(1+2\omega)} \cdot \frac{3(4+2\omega)}{4(3+2\omega)} \cdot \frac{5(6+2\omega)}{6(5+2\omega)} \cdot \frac{7(8+2\omega)}{8(7+2\omega)} \cdot \&c.$$

Hinc ſi termini, qui indicibus $\omega+1$, $\omega+2$, $\omega+3$, &c. reſpondent, ponantur Σ', Σ'', Σ''', &c. erit:

$$\Sigma' = \frac{1+2\omega}{2+2\omega} \cdot \Sigma$$

$$\Sigma'' = \frac{1+2\omega}{2+2\omega} \cdot \frac{3+2\omega}{4+2\omega} \cdot \Sigma$$

$$\Sigma''' = \frac{1+2\omega}{2+2\omega} \cdot \frac{3+2\omega}{4+2\omega} \cdot \frac{5+2\omega}{6+2\omega} \cdot \Sigma$$

&c.

Si itaque deſideretur terminus indici $\frac{1}{2}$ reſpondens, facto $\omega = \frac{1}{2}$, erit:

$$\Sigma = \frac{1.3}{2.2} \cdot \frac{3.5}{4.4} \cdot \frac{5.7}{6.6} \cdot \frac{7.9}{8.8} \cdot \frac{9.11}{10.10} \cdot \&c.$$

Verum poſito $\pi =$ ſemicircumferentiae circuli, cuius radius eſt $= 1$, ſupra oſtendimus eſſe:

$$\pi = 2 \cdot \frac{2.2}{1.3} \cdot \frac{4.4}{3.5} \cdot \frac{6.6}{5.7} \cdot \frac{8.8}{7.9} \cdot \&c.$$

Hanc-

Hancobrem termini intermedii indicibus $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, &c. per peripheriam circuli exprimi poterunt, hoc modo:

Indices: $\frac{1}{2}$ $\frac{3}{2}$ $\frac{5}{2}$ $\frac{7}{2}$

Termini: $\frac{2}{\pi}$; $\frac{2}{3}\cdot\frac{2}{\pi}$; $\frac{2.4}{3.5}\cdot\frac{2}{\pi}$; $\frac{2.4.6}{3.5.7}\cdot\frac{2}{\pi}$ &c.

Quam eandem interpolationem *Wallisius* in arithmetica infinitorum inuenit.

401. Conſideremus nunc iſtam ſeriem:

$$\overset{1}{a};\ \overset{2}{a(a+b)};\ \overset{3}{a(a+b)(a+2b)};\ \overset{4}{a(a+b)(a+2b)(a+3b)};\ \&c.$$

cuius factores hanc progreſſionem arithmeticam conſtituunt: a, $(a+b)$, $(a+2b)$, $(a+3b)$, $(a+4b)$, &c. huiusque termini infiniteſimi ita ſunt comparati, vt eorum logarithmorum differentiae euaneſcant. Cum igitur ſit $Z = a - b + b\omega$, & $Z' = a + b\omega$; $Z'' = a + b + b\omega$; $Z''' = a + 2b + b\omega$; &c. ſi Σ denotet terminum ſeriei propoſitae, cuius index eſt $= \omega$, erit:

$$\Sigma = a^{\omega}\cdot\frac{a^{1-\omega}(a+b)^{\omega}}{a+b\omega}\cdot\frac{(a+b)^{1-\omega}(a+2b)^{\omega}}{a+b+b\omega}\cdot\frac{(a+2b)^{1-\omega}(a+3b)^{\omega}}{a+2b+b\omega}\cdot \&c.$$

Hocque valore inuento, ſi ω denotet numerum quemvis fractum vnitate minorem, termini ſequentes indicibus $1+\omega$, $2+\omega$, $3+\omega$, &c. reſpondentes ita determinabuntur, vt ſit

$\Sigma' =$

$$\Sigma' = (a+b\omega)\,\Sigma$$

$$\Sigma'' = (a+b\omega)(a+b+b\omega)\,\Sigma$$

$$\Sigma''' = (a+b\omega)(a+b+b\omega)(a+2b+b\omega)\,\Sigma$$

&c.

Quare ſi deſideretur terminus indici $\frac{1}{2}$ reſpondens, facto $\omega = \frac{1}{2}$, erit:

$$\Sigma = a^{\frac{1}{2}}.\frac{a^{\frac{1}{2}}(a+b)^{\frac{1}{2}}}{a+\frac{1}{2}b}.\frac{(a+b)^{\frac{1}{2}}(a+2b)^{\frac{1}{2}}}{a+\frac{3}{2}b}.\frac{(a+2b)^{\frac{1}{2}}(a+3b)^{\frac{1}{2}}}{a+\frac{5}{2}b}.\ \&c.$$

ideoque ſumtis quadratis:

$$\Sigma^2 = a.\frac{a(a+b)}{(a+\frac{1}{2}b)(a+\frac{1}{2}b)}.\frac{(a+b)(a+2b)}{(a+\frac{3}{2}b)(a+\frac{3}{2}b)}.\frac{(a+2b)(a+3b)}{(a+\frac{5}{2}b)(a+\frac{5}{2}b)}.\ \&c.$$

402. Ponatur in ſerie quam ſupra tractauimus:

$$\overset{1}{\frac{f}{g}};\ \overset{2}{\frac{f(f+h)}{g(g+h)}};\ \overset{3}{\frac{f(f+h)(f+2h)}{g(g+h)(g+2h)}};\ \overset{4}{\frac{f(f+h)(f+2h)(f+3h)}{g(g+h)(g+2h)(g+3h)}};\ \&c.$$

terminus indici $\frac{1}{2}$ reſpondens $= \Theta$, erit:

$$\Theta = \frac{f(g+\frac{1}{2}h)}{g(f+\frac{1}{2}h)}.\frac{(f+h)(g+\frac{3}{2}h)}{(g+h)(f+\frac{3}{2}h)}.\frac{(f+2h)(g+\frac{5}{2}h)}{(g+2h)(f+\frac{5}{2}h)}.\ \&c.$$

ſtatuatur nunc: $f = a$; $g = a+\frac{1}{2}b$; & $h = b$; erit:

$$\Theta = \frac{a(a+b)}{(a+\frac{1}{2}b)(a+\frac{1}{2}b)}.\frac{(a+b)(a+2b)}{(a+\frac{3}{2}b)(a+\frac{3}{2}b)}.\ \&c.$$

ideoque fiet $\Sigma^2 = a\Theta$, & $\Sigma = \sqrt{a\Theta}$. Quocirca ſi huius ſeriei:

$$\overset{1}{a};\ \overset{2}{a(a+b)};\ \overset{3}{a(a+b)(a+2b)};\ \overset{4}{a(a+b)(a+2b)(a+3b)};\ \&c.$$

 ter-

terminus indici $\frac{1}{2}$ reſpondens ſtatuatur $= \Sigma$; atque huius ſeriei :

$$\overset{1}{\frac{a}{a+\frac{1}{2}b}};\ \overset{2}{\frac{a(a+b)}{(a+\frac{1}{2}b)(a+\frac{3}{2}b)}};\ \overset{3}{\frac{a(a+b)(a+2b)}{(a+\frac{1}{2}b)(a+\frac{3}{2}b)(a+\frac{5}{2}b)}};\ \&c.$$

terminus indici $\frac{1}{2}$ reſpondens ponatur $= \Theta$; erit $\Sigma = V a \Theta$.

Cum igitur hic ſeriei ſolorum numeratorum terminus indici $\frac{1}{2}$ reſpondens ſit $= \Sigma$, ſi in ſerie denominatorum terminus indici $\frac{1}{2}$ reſpondens ponatur $= \Lambda$; erit $\Theta = \frac{\Sigma}{\Lambda}$; at eſt $\Theta = \frac{\Sigma^2}{a}$, vnde fiet $\Sigma = \frac{a}{\Lambda}$, ſeu $\Sigma\Lambda = a$, quibus theorematibus interpolatio huiusmodi ſerierum non mediocriter illuſtratur.

EXEMPLUM I.

Sit propoſita haec ſeries interpolanda :

1, 1.2, 1.2.3, 1.2.3.4, &c.

Quia hic eſt $a = 1$, & $b = 1$, ſi terminus indici ω reſpondens ponatur $= \Sigma$, erit :

$$\Sigma = \frac{1^{1-\omega} . 2^{\omega}}{1+\omega} . \frac{2^{1-\omega} . 3^{\omega}}{2+\omega} . \frac{3^{1-\omega} . 4^{\omega}}{3+\omega} . \frac{4^{1-\omega} . 5^{\omega}}{4+\omega}\ \&c.$$

Hic pro ω ſemper fractio vnitate minor accipi poteſt nihilominus enim interpolatio per totam ſeriem extendetur. Nam ſi termini indicibus $1+\omega$, $2+\omega$, $3+\omega$, &c. reſpondentes ponantur Σ', Σ'', Σ''', &c.

erit :

erit:

$$\Sigma' = (1+\omega)\,\Sigma$$
$$\Sigma'' = (1+\omega)(2+\omega)\,\Sigma$$
$$\Sigma''' = (1+\omega)(2+\omega)(3+\omega)\,\Sigma$$
&c.

Seriei ergo propoſitae terminus indici $\frac{1}{2}$ reſpondens erit:

$$\Sigma = \frac{1^{\frac{1}{2}}\,.\,2^{\frac{1}{2}}}{\frac{3}{2}}\,;\ \frac{2^{\frac{1}{2}}\,.\,3^{\frac{1}{2}}}{2\frac{1}{2}}\,;\ \frac{3^{\frac{1}{2}}\,.\,4^{\frac{1}{2}}}{3\frac{1}{2}}\,;\ \&c. \quad \text{ſiue}$$

$$\Sigma^2 = \frac{2.4}{3.3}\cdot\frac{4.6}{5.5}\cdot\frac{6.8}{7.7}\cdot\frac{8.10}{9.9}\cdot\ \&c.$$

Vnde cum ſit $\pi = 2\cdot\frac{2.2}{1.3}\cdot\frac{4.4}{3.5}\cdot\frac{6.6}{5.7}\cdot\ \&c.$

erit $\Sigma^2 = \frac{\pi}{4}$ & $\Sigma = \frac{\sqrt{\pi}}{2}$: hincque reſpondebunt

Indicibus:	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$
Termini:	$\frac{\sqrt{\pi}}{2}$;	$\frac{3}{2}\cdot\frac{\sqrt{\pi}}{2}$;	$\frac{3.5}{2.2}\cdot\frac{\sqrt{\pi}}{2}$;	$\frac{3.5.7}{2.2.2}\cdot\frac{\sqrt{\pi}}{2}$; &c.

EXEMPLUM II.

Sit propoſita haec ſeries interpolanda:

$$\overset{1}{1}\,;\ \overset{2}{1.3}\,;\ \overset{3}{1.3.5}\,;\ \overset{4}{1.3.5.7}\,;\ \&c.$$

Quia hic eſt $a = 1$; $b = 2$; ſi terminus indici ω reſpondens ponatur $= \Sigma$, erit:

$$\Sigma = \frac{1^{1-\omega}\,.\,3^{\omega}}{1+2\omega}\cdot\frac{3^{1-\omega}\,.\,5^{\omega}}{3+2\omega}\cdot\frac{5^{1-\omega}\,.\,7^{\omega}}{5+2\omega}\cdot\ \&c.$$

 ter-

terminique ordine ſequentes ita erunt comparati:

$$\Sigma' = (1+2\omega)\,\Sigma$$
$$\Sigma'' = (1+2\omega)(3+2\omega)\,\Sigma$$
$$\Sigma''' = (1+2\omega)(3+2\omega)(5+2\omega)\,\Sigma$$
&c.

Si igitur ſeriei propoſitae deſideretur terminus indici $\frac{1}{2}$ reſpondens, isque vocetur $= \Sigma$, erit:

$$\Sigma = \frac{\sqrt{1.3}}{2}\cdot\frac{\sqrt{3.5}}{4}\cdot\frac{\sqrt{5.7}}{6}\cdot\frac{\sqrt{7.9}}{8}\cdot\&c. \quad \text{ergo}$$

$$\Sigma^2 = \frac{1.3}{2.2}\cdot\frac{3.5}{4.4}\cdot\frac{5.7}{6.6}\cdot\frac{7.9}{8.8}\cdot\&c. = \frac{2}{\pi},$$

ideoque habebitur $\Sigma = \sqrt{\frac{2}{\pi}}$. At reſpondebunt

Indicibus:	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	&c.
Termini:	$\sqrt{\frac{2}{\pi}}$;	$2.\sqrt{\frac{2}{\pi}}$;	$2.4\sqrt{\frac{2}{\pi}}$;	$2.4.6\sqrt{\frac{2}{\pi}}$;	&c.

Quodſi ergo prior ſeries & haec inuicem multiplicentur vt habeatur haec ſeries:

$$\overset{1}{1^2};\ \overset{2}{1^2.2.3};\ \overset{3}{1^2.2.3^2.5};\ \overset{4}{1^2.2.3^2.4.5.7};\ \overset{5}{1^2.2.3^2.4.5^2.7.9};$$
&c.

cuius terminus indici $\frac{1}{2}$ reſpondens erit $=\frac{\sqrt{\pi}}{2}.\sqrt{\frac{2}{\pi}}=\frac{1}{\sqrt{2}}$; quod facile perſpicitur, ſi iſti ſeriei haec forma tribuatur:

$$\overset{1}{\frac{1.2}{2}};\ \overset{2}{\frac{1.2.3.4}{2^2}};\ \overset{3}{\frac{1.2.3.4.5.6}{2^3}};\ \overset{4}{\frac{1.2.3.4.5.6.7.8}{2^4}};\ \&c.$$

cuius terminus indici $\frac{1}{2}$ reſpondens manifeſto eſt $=\frac{1}{\sqrt{2}}$.

EXEM-

EXEMPLUM III.

Sit iſta ſeries propoſita interpolanda :

$$\overset{1}{\frac{n}{1}}\ ;\ \overset{2}{\frac{n(n-1)}{1.\ 2}}\ ;\ \overset{3}{\frac{n(n-1)(n-2)}{1.\ 2.\ 3}}\ ;\ \overset{4}{\frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4}}\ ;\ \&c.$$

Conſiderentur huius ſeriei numeratores ac denominatores ſeorſim, & cum numeratores ſint:

$$\overset{1}{n}\ ;\ \overset{2}{n(n-1)}\ ;\ \overset{3}{n(n-1)(n-2)}\ ;\ \overset{4}{n(n-1)(n-2)(n-3)}\ ;\ \&c.$$

fiet applicatione facta, $a = n$, & $b = -1$, vnde huius ſeriei terminus indici ω reſpondens erit $=$

$$n^{\omega}\cdot\frac{n^{1-\omega}(n-1)^{\omega}}{n-\omega}\cdot\frac{(n-1)^{1-\omega}(n-2)^{\omega}}{n-1-\omega}\cdot\frac{(n-2)^{1-\omega}(n-3)^{\omega}}{n-2-\omega}\cdot\&c.$$

quae autem expreſſio ob factores in negatiuos abeuntes nihil certi monſtrat. Transformetur ergo ſeries propoſita, ponendo breuitatis gratia $1.\ 2.\ 3\ .\ .\ .\ n = N$, in hanc:

$$N\left(\overset{1}{\frac{1}{1.1.2.3\ldots(n-1)}}\ ;\ \overset{2}{\frac{1}{1.2.\ 1.2.3\ldots(n-2)}}\ ;\ \overset{3}{\frac{1}{1.2.3.1.2.3\ldots(n-3)}}\ ;\ \&c.\right)$$

cuius denominatores cum conſtent duobus factoribus, alteri conſtituent hanc ſeriem:

$$\overset{1}{1.2.3\ldots.(n-1)}\ ;\ \overset{2}{1.2.3\ldots.(n-2)}\ ;\ \overset{3}{1.2.3\ldots.(n-3)}\ ;\ \&c.$$

cuius terminus indici ω reſpondens, conuenit cum termino huius ſeriei:

N n n n n 3 1 ;

$$\overset{1}{1}\ ;\ \overset{2}{1.2}\ ;\ \overset{3}{1.2.3}\ ;\ \overset{4}{1.2.3.4}\ ;\ \overset{5}{1.2.3.4.5}\ ;\ \&c.$$

indici $n-\omega$ reſpondente: qui eſt

$$\frac{1^{1-n+\omega}\,.\,2^{n-\omega}}{1+n-\omega}\,.\,\frac{2^{1-n+\omega}\,.\,3^{n-\omega}}{2+n-\omega}\,.\,\frac{3^{1-n+\omega}\,.\,4^{n-\omega}}{3+n-\omega}\,.\ \&c.$$

Sit autem huius ſeriei terminus indici $1-\omega$ reſpondens $=\Theta$; erit:

$$\Theta=\frac{1^{\omega}\,.\,2^{1-\omega}}{2-\omega}\,.\,\frac{2^{\omega}\,.\,3^{1-\omega}}{3-\omega}\,.\,\frac{3^{\omega}\,.\,4^{1-\omega}}{4-\omega}\,.\ \&c.$$

atque cum reſpondeant:

Indicibus: $1-\omega$; $2-\omega$; $3-\omega$

Termini: Θ ; $(2-\omega)\Theta$; $(2-\omega)(3-\omega)\Theta$; &c.

indici $n-\omega$ reſpondebit hic terminus:

$$(2-\omega)(3-\omega)(4-\omega)\ \ldots\ (n-\omega)\Theta.$$

Deinde illorum denominatorum alteri factores conſtituent hanc ſeriem:

$$\overset{1}{1}\ ;\ \overset{2}{1.2}\ ;\ \overset{3}{1.2.3}\ ;\ \overset{4}{1.2.3.4}\ ;\ \overset{5}{1.2.3.4.5}\ ;\ \&c.$$

ſi terminus indici ω reſpondens ponatur $=\Lambda$, erit:

$$\Lambda=\frac{1^{1-\omega}\,.\,2^{\omega}}{1+\omega}\,.\,\frac{2^{1-\omega}\,.\,3^{\omega}}{2+\omega}\,.\,\frac{3^{1-\omega}\,.\,4^{\omega}}{3+\omega}\,.\ \&c.$$

Quibus inuentis ſi ipſius ſeriei propoſitae:

$$\overset{1}{\frac{n}{1}}\ ;\ \overset{2}{\frac{n(n-1)}{1.\ 2}}\ ;\ \overset{3}{\frac{n(n-1)(n-2)}{1.\ 2.\ 3}}\ ;\ \overset{4}{\frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4}}\ ;\ \&c.$$

ter-

terminus indici ω reſpondens ponatur Σ, erit:

$$\Sigma = \frac{N}{\Lambda.(2-\omega)(3-\omega)(4-\omega)\ .\ .\ .\ .\ (n-\omega)\Theta}.$$

At vero eſt:

$$\frac{N}{(2-\omega)(3-\omega)(4-\omega)\ .\ .\ .\ .\ (n-\omega)} =$$

$$\frac{2}{2-\omega}.\frac{3}{3-\omega}.\frac{4}{4-\omega}.\ .\ .\ .\ .\ .\frac{n}{n-\omega},$$

atque

$$\Lambda\Theta = \frac{1.2}{(1+\omega)(2-\omega)}.\frac{2.3}{(2+\omega)(3-\omega)}.\frac{3.4}{(3+\omega)(4-\omega)}.\ \&c.$$

Ex quibus terminus indici ω reſpondens quaeſitus erit:

$$\Sigma = \frac{2}{2-\omega}.\frac{3}{3-\omega}.\frac{4}{4-\omega}.\frac{5}{5-\omega}.\ .\ .\ .\ .\ .\ .\frac{n}{n-\omega}.$$

$$\frac{(1+\omega)(2-\omega)}{1.\ 2}.\frac{(2+\omega)(3-\omega)}{2.\ 3}.\frac{(3+\omega)(4-\omega)}{3.\ 4}.\ \&c.\ \text{in infinitum.}$$

Indici ergo $\frac{1}{2}$ reſpondebit iſte terminus:

$$\frac{4}{3}.\frac{6}{5}.\frac{8}{7}.\frac{10}{9}.\frac{12}{11}.\ .\ .\ .\ .\ .\ .\ .\frac{2n}{2n-1}.$$

$$\frac{3.3}{2.4}.\frac{5.5}{4.6}.\frac{7.7}{6.8}.\frac{9.9}{8.10}.\ \&c.$$

qui reducitur ad $\frac{4.6.8.10\ .\ .\ .\ .\ .\ 2n}{3.5.7.9\ .\ .\ .\ (2n-1)}.\frac{4}{\pi}$, ſeu

erit $= \frac{2}{\pi}.\frac{2.4.6.8.10\ .\ .\ .\ .\ .\ 2n}{1.3.5.7.9\ .\ .\ .\ (2n-1)}.$

Si

Si fuerit $n = 2$, prodibit iſta ſeries interpolanda:

$$\begin{array}{cccccccc} 0, & 1, & 2, & 3, & 4, & 5, & 6, & \&c. \\ 1, & 2, & 1, & 0, & 0, & 0, & 0, & \&c. \end{array}$$

cuius propterea terminus indici $\frac{1}{2}$ reſpondens eſt $= \frac{16}{3\pi}$.

EXEMPLUM IV.

Quaeratur terminus reſpondens indici $= \frac{1}{2}$ *in hac ſerie:*

$$\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ 1 + & \frac{1}{2} - & \frac{1.1}{2.4} + & \frac{1.1.3}{2.4.6} - & \frac{1.1.3.5}{2.4.6.8} + \&c. \end{array}$$

Oritur haec ſeries ex praecedente ſi ponatur $n = \frac{1}{2}$, eritque propterea terminus quaeſitus, qui ſit $= \Sigma$:

$$\Sigma = \frac{2}{\pi} \cdot \frac{2.4.6.8 \ldots \ldots 2n}{1.3.5.7 \ldots (2n-1)} \text{ poſito } n = \tfrac{1}{2}.$$

Ponatur $\frac{2.4.6.8 \ldots \ldots 2n}{1.3.5.7 \ldots (2n-1)} = \Theta$ ſi ſit $n = \frac{1}{2}$, eritque Θ terminus reſpondens indici $\frac{1}{2}$ in hac ſerie:

$$\frac{2}{1};\ \frac{2.4}{1.3};\ \frac{2.4.6}{1.3.5};\ \frac{2.4.6.8}{1.3.5.7};\ \&c.$$

qui ex ſuperioribus prodit $= \frac{\pi}{2}$. Quocirca ſeriei propoſitae terminus indici $\frac{1}{2}$ reſpondens, qui quaeritur, erit $= 1$. Quoniam autem in iſta ſerie, ſi terminus indici cuicunque ω reſpondens ponatur $= \Sigma$, ſequens eum erit

$\Sigma' =$

$\Sigma' = \frac{1-2\omega}{2+2\omega}\Sigma$; ſeries propoſita ita mediis terminis interiiciendis interpolabitur:

Indices: $0 \quad \frac{1}{2} \quad 1 \quad \frac{3}{2} \quad 2 \quad \frac{5}{2} \quad 3 \quad \frac{7}{2}$

Termini: $1;\ 1;\ \frac{1}{2};\ 0;\ \frac{-1.1}{2.4};\ 0;\ \frac{1.1.3}{2.4.6};\ 0;$ &c.

EXEMPLUM. V.

Si n *fuerit numerus quicunque fractus, inuenire terminum indici* ω *reſpondentem in ſerie:*

$$\overset{0}{1};\ \overset{1}{\frac{n}{1}};\ \overset{2}{\frac{n(n-1)}{1.\,2}};\ \overset{3}{\frac{n(n-1)(n-2)}{1.\ 2.\ 3}};\ \overset{4}{\frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4}}$$

&c.

Si expreſſionem $\frac{2}{2-\omega}\cdot\frac{3}{3-\omega}\cdot\frac{4}{4-\omega}\cdot \ldots\ldots \frac{n}{n-\omega}$ cum §. 400. comparemus, fiat $a=1$, $c=1$, $b=1-\omega$, ibique loco ω poſito n, erit:

$$\frac{1}{1-\omega}\cdot\frac{2}{2-\omega}\cdot\frac{3}{3-\omega}\cdot\ldots\ldots\frac{n}{n-\omega}=\frac{1(1-\omega+n)}{(1-\omega)(1+n)}\cdot\frac{2(2-\omega+n)}{(2-\omega)(2+n)}.\text{\&c.}$$

vnde terminus quaeſitus indici ω reſpondens ſi ponatur $=\Sigma$, erit:

$$\Sigma=\frac{(1-\omega+n).2}{(1+n)(2-\omega)}\cdot\frac{(2-\omega+n)3}{(2+n)(3-\omega)}.\text{\&c.}\frac{(1+\omega)(2-\omega)}{1.\ \ 2}\cdot\frac{(2+\omega)(3-\omega)}{2.\ \ 3}.\text{\&c.}$$

ideoque

$$\Sigma=\frac{(1+\omega)(1+n-\omega)}{1\ (1+n)}\cdot\frac{(2+\omega)(2+n-\omega)}{2\ (2+n)}\cdot\frac{(3+\omega)(3+n-\omega)}{3\ (3+n)}.\text{\&c.}$$

 quo-

quoties ergo $n-\omega$ fuerit numerus integer valor ipſius Σ rationaliter exprimi poteſt.

Sic ſi ſit $n = \omega$ erit $\Sigma = 1$

ſi $n = 1 + \omega$ erit $\Sigma = n$

ſi $n = 2 + \omega$ erit $\Sigma = \frac{n(n-1)}{1.\ 2}$

ſi $n = 3 + \omega$ erit $\Sigma = \frac{n(n-1)(n-2)}{1.\ 2.\ 3}$

&c.

At ſi fuerit $\omega - n$ numerus integer affirmatiuus, erit ſemper $\Sigma = 0$.

CAPUT

CAPUT XVIII.

DE VSU CALCULI DIFFERENTIALIS IN RESOLUTIONE FRACTIONUM.

403.

Methodus fractionem quamuis propoſitam in fractiones ſimplices reſoluendi, quam in Introductione expoſuimus etſi, per ſe ſatis eſt facilis; tamen ope calculi differentialis ita perfici poteſt, vt ſaepenumero multo minori negotio in vſum vocari poſſit. Praecipue vero ſi denominator fractionis reſoluendae fuerit indefiniti gradus, methodus ante expoſita plerumque non mediocriter impeditur, dum loco quantitatis incognitae ſubſtitutio valoris, quem ex quopiam factore induit, fieri debet. Imprimis autem his caſibus diuiſio denominatoris per factorem iam inuentum nimis fit moleſta. Quae operatio, ſi calculus differentialis in ſubſidium vocetur, euitari poterit, ita vt non opus ſit alterum denominatoris factorem, qui oritur ſi denominator per factorem iam cognitum diuidatur, noſſe. Hunc autem vſum praeſtat methodus determinandi valorem fractionis, cuius numerator ac denominator certo caſu ambo euaneſcunt, cuius beneficio, quemadmodum reſolutio fractionum iam ſupra tradita commodior & tractabilior reddi queat, hoc Capite doceamus, ſimulque finem huic libro, in quo vſum calculi differentialis in Analyſi expoſuimus, imponamus.

404. Si igitur propoſita fuerit fractio quaecunque $\frac{P}{Q}$, cuius numerator ac denominator ſint functiones variabilis quantitatis x, rationales & integrae; primum videndum eſt, vtrum x in numeratore P tot pluresue dimenſiones habeat, quam in denominatore Q. Quodſi eueniat, complectetur fractio $\frac{P}{Q}$ in ſe partem integram huius formae $A + Bx + Cx^2 +$ &c. quae diuiſionis ope inde erui poterit: pars reliqua erit fractio eundem denominatorem Q habens, ſed cuius numerator erit functio puta R pauciores ipſius x dimenſiones continens, quam denominator Q, ita vt vlterior reſolutio inſtituenda ſit in fractione $\frac{R}{Q}$. Interim tamen non opus eſt noſſe hunc nouum numeratorem R, ſed eaedem fractiones ſimplices, quas fractio $\frac{R}{Q}$ ſuppeditatura eſſet, elici poſſunt immediate ex fractione propoſita $\frac{P}{Q}$; prouti iam ſupra notauimus.

405. Praeter partem integram igitur, ſi quam continet fractio $\frac{P}{Q}$, erui debent fractiones ſimplices, quarum denominatores ſint vel binomiales huius formae $f + gx$, vel trinomiales huiusmodi $f + 2x\cos\phi . \sqrt{fg} + gxx$, vel eiusmodi formularum quadrata, cubiue ſeu altiores poteſtates. Hique denominatores omnes erunt factores denominatoris Q, ita vt quilibet denominatoris ipſius Q factor praebeat fractionem ſimplicem. Scilicet ſi denomi-

minator Q factorem habeat $f+gx$, ex eo naſcetur fractio ſimplex huiusmodi $\frac{\mathfrak{A}}{f+gx}$; ſin autem factor fuerit $(f+gx)^2$, binae fractiones $\frac{\mathfrak{A}}{(f+gx)^2}+\frac{\mathfrak{B}}{f+gx}$. Atque ex denominatoris Q factore cubico $(f+gx)^3$ orientur tres fractiones ſimplices huius formae: $\frac{\mathfrak{A}}{(f+gx)^3}+\frac{\mathfrak{B}}{(f+gx)^2}+\frac{\mathfrak{C}}{f+gx}$; & ita porro. Quodſi autem denominator Q factorem habuerit trinomialem huiusmodi $ff-2fgx\cos\varphi+ggxx$, ex eo orietur fractio ſimplex talis formae $\frac{\mathfrak{A}+\mathfrak{a}x}{ff-2fgx\cos\varphi+ggxx}$; &, ſi duo huiusmodi factores fuerint aequales vti $(ff-2fgx\cos\varphi+ggxx)^2$, hinc prodibunt duae fractiones $\frac{\mathfrak{A}+\mathfrak{a}x}{(ff-2fgx\cos\varphi+ggxx)^2}+\frac{\mathfrak{B}+\mathfrak{b}x}{ff-2fgx\cos\varphi+ggxx}$. Huiusmodi autem factor cubicus $(ff-2fgx\cos\varphi+ggxx)^3$ dabit tres fractiones ſimplices, biquadratus quatuor, & ita porro.

406. Reſolutio ergo fractionis cuiuscunque $\frac{P}{Q}$ ita inſtituatur. Quaerantur primo omnes factores tam ſimplices ſeu binomiales, quam trinomiales denominatoris Q, &, ſi qui fuerint inter ſe aequales, ii probe notentur, & inſtar vnius habeantur. Tum ex ſingulis his denominatoris factoribus eliciantur fractiones ſimplices, vel modo iam ſupra oſtenſo, vel eo, quem hic ſumus

tradituri, & qui pro lubitu in locum prioris ſubſtitui poterit. Quo facto aggregatum omnium iſtarum fractionum ſimplicium vna cum parte integra, ſi quam continet fractio propoſita $\frac{P}{Q}$, huius valorem exhaurient. Inventionem quidem factorum denominatoris Q hic tanquam cognitam aſſumimus, cum pendeat a reſolutione aequationis $Q = 0$; methodumque hic trademus per calculum differentialem pro dato quouis denominatoris factore fractionem ſimplicem inde ortam definiendi. Quod, cum iſtarum fractionum ſimplicium denominatores iam habeantur, praeſtabitur, ſi numeratorem cuiusque fractionis inueſtigare doceamus.

407. Ponamus ergo fractionis $\frac{P}{Q}$ denominatorem Q factorem habere $f + gx$, ita vt ſit $Q = (f + gx)S$ neque vero hic alter factor S inſuper eundem factorem $f + gx$ contineat. Sit fractio ſimplex ex iſto factore orta $= \frac{\mathfrak{A}}{f + gx}$; & complementum huiusmodi formam habebit $\frac{V}{S}$, ita vt ſit $\frac{\mathfrak{A}}{f + gx} + \frac{V}{S} = \frac{P}{Q}$. Erit ergo $\frac{V}{S} = \frac{P}{Q} - \frac{\mathfrak{A}}{f + gx} = \frac{P - \mathfrak{A}S}{(f + gx)S}$; ideoque $V = \frac{P - \mathfrak{A}S}{f + gx}$. Cum igitur V ſit functio integra ipſius x, neceſſe eſt vt $P - \mathfrak{A}S$ ſit diuiſibile per $f + gx$; ac propterea ſi ponatur $f + gx = 0$ ſeu $x = \frac{-f}{g}$, expreſ-

ſio $P - \mathfrak{A}S$ euaneſcet. Fiat ergo $x = \frac{-f}{g}$, & cum ſit $P - \mathfrak{A}S = 0$, erit $\mathfrak{A} = \frac{P}{S}$, vti iam ſupra inuenimus. Cum autem ſit $S = \frac{Q}{f + gx}$, fiet $\mathfrak{A} = \frac{(f + gx)P}{Q}$, ſi vbique ponatur $f + gx = 0$, ſeu $x = \frac{-f}{g}$. Quoniam vero hoc caſu tam numerator $(f + gx)P$, quam denominator Q euaneſcit; per ea, quae de valore huiusmodi fractionum inueſtigando expoſuimus, erit $\mathfrak{A} = \frac{(f + gx)\,dP + Pg\,dx}{dQ}$, ſi quidem ponatur $x = \frac{-f}{g}$. Hoc autem caſu ob $(f + gx)dP = 0$, erit $\mathfrak{A} = \frac{gP\,dx}{dQ}$; ſicque per differentiationem valor numeratoris $\mathfrak{A}$ expedite reperitur.

408. Si igitur fractionis propoſitae $\frac{P}{Q}$ denominator Q factorem habeat ſimplicem $f + gx$, ex eo orietur fractio ſimplex $\frac{\mathfrak{A}}{f + gx}$, exiſtente $\mathfrak{A} = \frac{gP\,dx}{dQ}$, poſtquam hic vbique loco x valor $\frac{-f}{g}$ ex aequatione $f + gx = 0$ oriundus fuerit ſubſtitutus. Hoc ergo modo non neceſſe eſt, vt ante quaeratur alter denominatoris Q factor S, qui oritur, ſi Q per $f + gx$ diuidatur. Hinc ſi Q non in factoribus exprimatur, hanc diuiſionem ſaepe non parum moleſtam, praecipue ſi x in deno-

nominatore Q habeat exponentes indefinitos, omittere poterimus, cum valor ipſius $\mathfrak{A}$ ex formula $\frac{gP\,dx}{dQ}$ obtineatur. Sin autem denominator Q iam in factoribus fuerit expreſſus, ita vt inde valor ipſius S ſponte pateat, tum praeferenda erit altera expreſſio, qua inuenimus $\mathfrak{A} = \frac{P}{S}$, ponendo pariter vbique $x = \frac{-f}{g}$. Sicque pro inueniendo valore ipſius $\mathfrak{A}$ quouis caſu ea formula adhiberi poterit, quae commodior & expeditior videatur. Vſum autem nouae formulae aliquot exemplis illuſtrabimus.

EXEMPLUM I.

Sit propoſita iſta fractio $\frac{x^9}{1+x^{17}}$, *cuius fractionem ſimplicem ex denominatoris factore* $1+x$ *oriundam definiri oporteat.*

Quoniam hic eſt $Q = 1 + x^{17}$, cuius etſi factor $1 + x$ conſtat, tamen ſi, vti prima methodus poſtulat per eum diuidere velimus, prodiret
$S = 1 - x + xx - x^3 + \ldots + x^{16}$.
Commodius igitur vtemur noua formula $\mathfrak{A} = \frac{gP\,dx}{dQ}$; quia itaque eſt $f = 1, g = 1$, & $P = x^9$, ob $dQ = 17x^{16}dx$, fiet $\mathfrak{A} = \frac{x^9}{17x^{16}} = \frac{1}{17x^7}$, poſito $x = -1$, vnde fit $\mathfrak{A} = -\frac{1}{17}$, & fractio ſimplex ex denominatoris factore $1 + x$ oriunda erit $\frac{-1}{17(1+x)}$.

EX-

EXEMPLUM II.

Propoſita fractione $\frac{x^m}{1-x^{2n}}$, *fractionem ſimplicem ex denominatoris factore* $1-x$ *oriundam inueſtigare.*

Ob factorem propoſitum $1-x$, erit $f=1$, & $g=-1$. Tum vero denominator $Q=1-x^{2n}$ dat $dQ=-2nx^{2n-1}dx$; vnde propter $P=x^m$ obtinebitur $\mathfrak{A}=\frac{-x^m}{-2nx^{2n-1}}$. Poſitoque ex aequatione $1-x=0$, $x=1$, fiet $\mathfrak{A}=\frac{1}{2n}$; ita vt fractio ſimplex futura ſit haec $\frac{}{2n(1-x)}$.

EXEMPLUM III.

Propoſita fractione $\frac{x^m}{1-4x^k+3x^n}$, *eius fractionem ſimplicem ex denominatoris factore* $1-x$ *oriundam determinare.*

Hic ergo fit $f=1$; $g=-1$; $P=x^m$; $Q=1-4x^k+3x^n$ & $\frac{dQ}{dx}=-4kx^{k-1}+3nx^{n-1}$; vnde fit $\mathfrak{A}=\frac{-x^m}{-4kx^{k-1}+3nx^{n-1}}$ & poſito $x=1$, erit $\mathfrak{A}=\frac{1}{4k-3n}$. Fractio ergo ſimplex ex iſto denominatoris factore ſimplici $1-x$ oriunda erit $=\frac{1}{(4k-3n)(1-x)}$.

409. Ponamus nunc fractionis $\frac{P}{Q}$ denominatorem Q factorem habere quadratum $(f+gx)^2$, & fractiones ſimplices hinc oriundas eſſe $= \frac{\mathfrak{A}}{(f+gx)^2} + \frac{\mathfrak{B}}{f+gx}$. Sit $Q = (f+gx)^2 S$ & complementum $= \frac{V}{S}$; ita vt ſit $\frac{V}{S} = \frac{P}{Q} - \frac{\mathfrak{A}}{(f+gx)^2} - \frac{\mathfrak{B}}{f+gx}$; & $V = \frac{P - \mathfrak{A}S - \mathfrak{B}(f+gx)S}{(f+gx)^2}$. Quia nunc V eſt functio integra, neceſſe eſt vt ſit $P - \mathfrak{A}S - \mathfrak{B}S(f+gx)$ diuiſibile per $(f+gx)^2$; & cum S factorem $f+gx$ amplius non contineat, quoque haec expreſſio $\frac{P}{S} - \mathfrak{A} - \mathfrak{B}(f+gx)$ diuiſibilis erit per $(f+gx)^2$; ideoque facto $f+gx=0$, ſeu $x = \frac{-f}{g}$ non ſolum ipſa, ſed etiam eius differentiale $d.\frac{P}{S} - \mathfrak{B}g\,dx$ euaneſcet. Fiat ergo $x = \frac{-f}{g}$, eritque ex priori aequatione $\mathfrak{A} = \frac{P}{S}$; ex poſteriori vero erit $\mathfrak{B} = \frac{1}{g\,dx} d.\frac{P}{S}$; quibus valoribus inuentis habebuntur fractiones quaeſitae: $\frac{\mathfrak{A}}{(f+gx)^2} + \frac{\mathfrak{B}}{f+gx}$.

EXEM-

EXEMPLUM.

Propoſita fractione $\frac{x^m}{1-4x^3+3x^4}$, *cuius denominator factorem habet* $(1-x)^2$, *inuenire fractiones ſimplices hinc oriundas.*

Cum hic ſit $f=1$, $g=-1$, $P=x^m$ & $Q=1-4x^3+3x^4$, erit $S=1+2x+3xx$; $\frac{P}{S}=\frac{x^m}{1+2x+3xx}$, &

$$d.\frac{P}{S}=\frac{mx^{m-1}dx+2(m-1)x^m dx+3(m-2)x^{m+1}dx}{(1+2x+3xx)^2}.$$

Hinc poſito $x=1$, erit:

$$\mathfrak{A}=\frac{1}{6} \ \& \ \mathfrak{B}=-1.\frac{6m-8}{36}=\frac{4-3m}{18};$$

vnde fractiones quaeſitae erunt: $\frac{1}{6(1-x)^2}+\frac{4-3m}{18(1-x)}$.

410. Habeat fractionis $\frac{P}{Q}$ denominator Q tres factores ſimplices aequales, ſeu ſit $Q=(f+gx)^3$, ſintque fractiones ſimplices ex hoc factore cubico $(f+gx)^3$ oriundae hae: $\frac{\mathfrak{A}}{(f+gx)^3}+\frac{\mathfrak{B}}{(f+gx)^2}+\frac{\mathfrak{C}}{f+gx}$; complementum vero harum fractionum ad fractionem propoſitam $\frac{P}{Q}$ conſtituendam ſit $\frac{V}{S}$, eritque $V=\frac{P-\mathfrak{A}S-\mathfrak{B}S(f+gx)-\mathfrak{C}S(f+gx)^2}{(f+gx)^3}$. Quare haec expreſſio $\frac{P}{S}-\mathfrak{A}-\mathfrak{B}(f+gx)-\mathfrak{C}(f+gx)^2$

 diui-

diuiſibilis erit per $(f+gx)^3$; vnde poſito $f+gx=0$ ſeu $x=\frac{-f}{g}$, non ſolum ipſa haec expreſſio, ſed etiam eius differentiale primum & ſecundum euadet $=0$. Erit ſcilicet ponendo $x=\frac{-f}{g}$:

$$\frac{P}{S}-\mathfrak{A}-\mathfrak{B}(f+gx)-\mathfrak{C}(f+gx)^2=0$$

$$d.\frac{P}{S}-\mathfrak{B}g\,dx-2\mathfrak{C}g\,dx(f+gx)=0$$

$$dd.\frac{P}{S}-\mathfrak{C}g^2dx^2=0.$$

Ex prima aequatione ergo erit $\mathfrak{A}=\frac{P}{S}$

Ex ſecunda vero erit $\mathfrak{B}=\frac{1}{gdx}\,d.\frac{P}{S}$

Ex tertia denique definitur $\mathfrak{C}=\frac{1}{2g^2dx^2}\,dd.\frac{P}{S}$.

411. Generaliter ergo ſi fractionis $\frac{P}{S}$ denominator Q factorem habeat $(f+gx)^n$, ita vt ſit $Q=(f+gx)^nS$; poſitis fractionibus ſimplicibus ex hoc factore $(f+gx)^n$ oriundis his:

$$\frac{\mathfrak{A}}{(f+gx)^n}+\frac{\mathfrak{B}}{(f+gx)^{n-1}}+\frac{\mathfrak{C}}{(f+gx)^{n-2}}+\frac{\mathfrak{D}}{(f+gx)^{n-3}}+\frac{\mathfrak{E}}{(f+gx)^{n-4}}+\&c.$$

quoad ad vltimam, cuius denominator eſt $f+gx$, perveniatur, ſi ratiocinium vt ante inſtituatur, reperietur haec expreſſio:

$$\frac{P}{S}-\mathfrak{A}-\mathfrak{B}(f+gx)-\mathfrak{C}(f+gx)^2-\mathfrak{D}(f+gx)^3-\mathfrak{E}(f+gx)^4-\&c.$$

diui-

diuiſibilis eſſe debere per $(f+gx)^n$, hinc tam ipſa, quam ſingula eius differentialia vsque ad gradum $n-1$, caſu $x=\frac{-f}{g}$ euaneſcere debebunt. Ex quibus aequationibus concludetur fore ponendo vbique $x=\frac{-f}{g}$:

$$\mathfrak{A}=\frac{P}{S}$$

$$\mathfrak{B}=\frac{1}{1\,gdx}\,d.\frac{P}{S}$$

$$\mathfrak{C}=\frac{1}{1.2\,g^2dx^2}\,dd.\frac{P}{S}$$

$$\mathfrak{D}=\frac{1}{1.2.3\,g^3dx^3}\,d^3.\frac{P}{S}$$

$$\mathfrak{E}=\frac{1}{1.2.3.4\,g^4dx^4}\,d^4.\frac{P}{S}\quad \&c.$$

Vbi quidem notandum eſt, differentialia iſta ipſius $\frac{P}{S}$ ante capi oportere, quam loco x ponatur $\frac{-f}{g}$, alias enim variabilitas ipſius x tolleretur.

412. Facilius ergo hoc modo iſti numeratores $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. exprimuntur, quam eo modo, qui in Introductione eſt traditus, & ſaepenumero quoque hac noua ratione eorum valores expeditius reperiuntur. Quae comparatio quo facilius inſtitui queat, valores litterarum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. priori modo definiamus:

Poſito $x = \frac{-f}{g}$	Statuatur relicto x variabili.
$\mathfrak{A} = \frac{P}{S}$	$\frac{P - \mathfrak{A}S}{f + gx} = \mathfrak{P}$ erit
$\mathfrak{B} = \frac{\mathfrak{P}}{S}$	$\frac{\mathfrak{P} - \mathfrak{B}S}{f + gx} = \mathfrak{Q}$ erit
$\mathfrak{C} = \frac{\mathfrak{Q}}{S}$	$\frac{\mathfrak{Q} - \mathfrak{C}S}{f + gx} = \mathfrak{R}$ erit
$\mathfrak{D} = \frac{\mathfrak{R}}{S}$	$\frac{\mathfrak{R} - \mathfrak{D}S}{f + gx} = \mathfrak{S}$ erit
$\mathfrak{E} = \frac{\mathfrak{S}}{S}$	& ita porro.

413. Quodſi autem fractionis $\frac{P}{Q}$ denominator Q non omnes factores ſimplices habeat reales, tum bini imaginariorum iunctim ſumantur, quorum productum erit reale. Sit ergo denominatoris Q factor $ff - 2fgx\,\text{coſ}\,\varphi + ggxx$, qui poſitus $= 0$ dat hunc duplicem valorem imaginarium:

$$x = \frac{f}{g}\,\text{coſ}\,\varphi \pm \frac{f}{g\sqrt{-1}}\,\text{ſin}\,\varphi\,; \qquad \text{ex quo erit}$$

$$x^n = \frac{f^n}{g^n}\,\text{cſ}\,n\varphi \pm \frac{f^n}{g^n\sqrt{-1}}\,\text{ſin}\,.\,n\varphi\,.$$

Ponamus eſſe $Q = (ff - 2fgx\,\text{coſ}\,\varphi + ggxx)\,S$, atque S praeterea per $ff - 2fgx\,\text{coſ}\,\varphi + ggxx$ non eſſe diuiſibile. Sit fractio ex iſto factore denominatoris oriunda:

$$\frac{\mathfrak{A} + \mathfrak{a}x}{ff - 2fgx\,\text{coſ}\,\varphi + ggxx}$$

&

& complementum ad propoſitam $\frac{P}{Q}$ ſit $= \frac{V}{S}$, erit $V = \frac{P - (\mathfrak{A} + \mathfrak{a}x)S}{ff - 2fgx\operatorname{cof}\phi + ggxx}$; vnde $P - (\mathfrak{A} + \mathfrak{a}x)S$ ac propterea quoque $\frac{P}{S} - \mathfrak{A} - \mathfrak{a}x$ diuiſibile erit per $ff - 2fgx\operatorname{cof}\phi + ggxx$. Euaneſcet ergo $\frac{P}{S} - \mathfrak{A} - \mathfrak{a}x$ ſi ponatur $ff - 2fgx\operatorname{cof}\phi + ggxx = 0$, hoc eſt ſi ponatur

vel $\quad x = \frac{f}{g}\operatorname{cof}\phi + \frac{f}{g\sqrt{-1}}\operatorname{ſin}\phi$

vel $\quad x = \frac{f}{g}\operatorname{cof}\phi - \frac{f}{g\sqrt{-1}}\operatorname{ſin}\phi$.

414. Quoniam P & S ſunt functiones integrae ipſius x, fiat in vtroque ſeorſim vtraque ſubſtitutio; & quia pro quauis poteſtate ipſius x, puta x^n binomium hoc $x^n = \frac{f^n}{g^n}\operatorname{cof} n\phi \pm \frac{f^n}{g^n\sqrt{-1}}\operatorname{ſin}.n\phi$ ſubſtitui debet. Ponamus primo vbique $\frac{f^n}{g^n}\operatorname{cof}\phi$ pro x^n, hocque facto abeat P in $\mathfrak{P}$, & S in $\mathfrak{S}$. Deinde ponatur vbique $\frac{f^n}{g^n}\operatorname{ſin} n\phi$ pro x^n, hocque facto abeat P in $\mathfrak{p}$ & S in $\mathfrak{s}$; vbi notandum eſt ante has ſubſtitutiones vtramque functionem P & S penitus debere euolui, ita vt, ſi forte factoribus ſint implicatae, ii per actualem multiplicationem tollantur. His valoribus $\mathfrak{P}$, $\mathfrak{p}$, $\mathfrak{S}$, $\mathfrak{s}$, inuentis, manifeſtum erit, ſi ponatur:

$$x = \frac{f}{g}\operatorname{cof}\phi \pm \frac{f}{g\sqrt{-1}}\operatorname{ſin}\phi$$

func-

functionem P abituram eſſe in $\mathfrak{P} \pm \frac{\mathfrak{p}}{\sqrt{-1}}$, & functionem S abituram eſſe in $\mathfrak{S} \pm \frac{\mathfrak{s}}{\sqrt{-1}}$. Hinc cum $\frac{P}{S} - \mathfrak{A} - \mathfrak{a}x$ ſeu $P - (\mathfrak{A} + \mathfrak{a}x)S$ vtroque caſu euaneſcere debeat, erit:

$$\mathfrak{P} \pm \frac{\mathfrak{p}}{\sqrt{-1}} = \left(\mathfrak{A} + \frac{\mathfrak{a}f}{g}\operatorname{cof}\phi \pm \frac{\mathfrak{a}f}{g\sqrt{-1}}\operatorname{fin}\phi\right)\left(\mathfrak{S} \pm \frac{\mathfrak{s}}{\sqrt{-1}}\right)$$

vnde ob ſigna ambigua hae duae aequationes orientur:

$$\mathfrak{P} = \mathfrak{A}\mathfrak{S} + \frac{\mathfrak{a}f\mathfrak{S}}{g}\operatorname{cof}\phi - \frac{\mathfrak{a}f\mathfrak{s}}{g}\operatorname{fin}\phi$$

$$\mathfrak{p} = \mathfrak{A}\mathfrak{s} + \frac{\mathfrak{a}f\mathfrak{s}}{g}\operatorname{cof}\phi + \frac{\mathfrak{a}f\mathfrak{S}}{g}\operatorname{fin}\phi$$

ex quibus eliminando $\mathfrak{A}$ eruitur:

$$\mathfrak{S}\mathfrak{p} - \mathfrak{s}\mathfrak{P} = \frac{\mathfrak{a}f(\mathfrak{S}^2 + \mathfrak{s}^2)}{g}\operatorname{fin}\phi; \text{ ideoque erit}$$

$$\mathfrak{a} = \frac{g(\mathfrak{S}\mathfrak{p} - \mathfrak{s}\mathfrak{P})}{f(\mathfrak{S}^2 + \mathfrak{s}^2)\operatorname{fin}\phi}.$$

Deinde eliminando ſin ϕ erit:

$$\mathfrak{S}\mathfrak{P} + \mathfrak{s}\mathfrak{p} = (\mathfrak{S}^2 + \mathfrak{s}^2)(\mathfrak{A} + \frac{\mathfrak{a}f}{g}\operatorname{cof}\phi). \quad \text{Ergo}$$

$$\mathfrak{A} = \frac{\mathfrak{S}\mathfrak{P} + \mathfrak{s}\mathfrak{p}}{\mathfrak{S}^2 + \mathfrak{s}^2} - \frac{(\mathfrak{S}\mathfrak{p} - \mathfrak{s}\mathfrak{P})\operatorname{cof}\phi}{(\mathfrak{S}^2 + \mathfrak{s}^2)\operatorname{fin}\phi}.$$

415. Cum iam ſit $S = \frac{Q}{ff - 2fgx\operatorname{cof}\phi + ggxx}$; quia poſito $ff - 2fgx\operatorname{cof}\phi + ggxx = 0$ tam numerator quam denominator euaneſcent, erit hoc caſu:

$$S = \frac{dQ : dx}{2ggx - 2fg\operatorname{cof}\phi}.$$

Po-

Ponamus nunc, ſi vbique ſubſtituatur $x^n = \frac{f^n}{g^n}\,\text{coſ}\, n\phi$, functionem $\frac{dQ}{dx}$ abire in $\mathfrak{Q}$; ſin autem ſtatuatur $x^n = \frac{f^n}{g^n}\,\text{ſin}\, n\phi$, eam abire in $\mathfrak{q}$; atque manifeſtum eſt ſi ponatur $x = \frac{f}{g}\,\text{coſ}\,\phi \pm \frac{f}{g\sqrt{-1}}\,\text{ſin}\,\phi$ functionem $\frac{dQ}{dx}$ abire in $\mathfrak{Q} \pm \frac{\mathfrak{q}}{\sqrt{-1}}$. Ex quo functio S abibit in $\frac{\mathfrak{Q} \pm \mathfrak{q} : \sqrt{-1}}{\pm 2fg\,\text{ſin}\,\phi : \sqrt{-1}}$. Cum ergo ſit $S = \mathfrak{S} \pm \frac{\mathfrak{s}}{\sqrt{-1}}$ eodem valore pro x poſito, habebitur:

$$\mathfrak{Q} \pm \frac{\mathfrak{q}}{\sqrt{-1}} = \pm \frac{2fg\mathfrak{S}}{\sqrt{-1}}\,\text{ſin}\,\phi - 2fg\mathfrak{s}\,\text{ſin}\,\phi.$$

Erit ergo $\mathfrak{s} = \frac{-\mathfrak{Q}}{2fg\,\text{ſin}\,\phi}$ & $\mathfrak{S} = \frac{\mathfrak{q}}{2fg\,\text{ſin}\,\phi}$.

Hisque valoribus ſubſtitutis, fiet $\mathfrak{a} = \frac{2gg(\mathfrak{p}\mathfrak{q} + \mathfrak{P}\mathfrak{Q})}{\mathfrak{Q}^2 + \mathfrak{q}^2}$

& $\mathfrak{A} = \frac{2fg(\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q})\,\text{ſin}\,\phi}{\mathfrak{Q}^2 + \mathfrak{q}^2} - \frac{2fg(\mathfrak{p}\mathfrak{q} + \mathfrak{P}\mathfrak{Q})\,\text{coſ}\,\phi}{\mathfrak{Q}^2 + \mathfrak{q}^2}$.

416. Hinc ergo idonea obtinetur ratio ex quouis factore ſecundae poteſtatis fractionem ſimplicem formandi, hicque cum ipſe fractionis propoſitae denominator in computo retineatur, diuiſionem, qua valor litterae S definiri deberet, & quae ſaepe non parum eſt moleſta,

euitamus. Si igitur fractionis $\frac{P}{Q}$ denominator Q factorem habeat talem $ff - 2fgx\cos\phi + ggxx$, ſequenti modo fractio ſimplex ex hoc factore oriunda, quam fingamus $= \frac{\mathfrak{A} + \mathfrak{a}x}{ff - 2fgx\cos\phi + ggxx}$, definietur. Ponatur $x = \frac{f}{g}\cos\phi$, & pro quauis ipſius x poteſtate x^n ſcribatur $\frac{f^n}{g^n}\cos n\phi$; quo facto abeat P in $\mathfrak{P}$, & functio $\frac{dQ}{dx}$ in $\mathfrak{Q}$. Deinde ibidem ponatur $x = \frac{f}{g}\sin\phi$, & poteſtas eius quaeuis $x^n = \frac{f^n}{g^n}\sin n\phi$; abeatque P in $\mathfrak{p}$, & $\frac{dQ}{dx}$ in $\mathfrak{q}$. Inuentisque hoc modo valoribus litterarum $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{p}$ & $\mathfrak{q}$ quantitates $\mathfrak{A}$ & $\mathfrak{a}$ ita definientur, vt ſit

$$\mathfrak{A} = \frac{2fg(\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q})\sin\phi}{\mathfrak{Q}^2 + \mathfrak{q}^2} - \frac{2fg(\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q})\cos\phi}{\mathfrak{Q}^2 + \mathfrak{q}^2}$$

$$\mathfrak{a} = \frac{2gg(\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q})}{\mathfrak{Q}^2 + \mathfrak{q}^2}.$$

Fractio ergo ex denominatoris Q factore $ff - 2fgx\cos\phi + ggxx$ oriunda erit:

$$\frac{2fg(\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q})\sin\phi + 2g(\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q})(gx - f\cos\phi)}{(\mathfrak{Q}^2 + \mathfrak{q}^2)(ff - 2fgx\cos\phi + ggxx)}.$$

EXEM-

EXEMPLUM I.

Si propoſita fuerit haec fractio $\frac{x^m}{a+bx^n}$, *cuius denominator* $a+bx^n$ *factorem habeat hunc:* $ff-2fgx\cos\phi+ggxx$ *inuenire fractionem ſimplicem huic factori conuenientem.*

Quoniam hic eſt $P = x^m$ & $Q = a + bx^n$, erit $\frac{dQ}{dx} = nbx^{n-1}$, vnde fiet:

$$\mathfrak{P} = \frac{f^m}{g^m}\cos m\phi \quad ; \quad \mathfrak{p} = \frac{f^m}{g^m}\sin m\phi$$

$$\mathfrak{Q} = \frac{nbf^{n-1}}{g^{n-1}}\cos(n-1)\phi \quad ; \quad \mathfrak{q} = \frac{nbf^{n-1}}{g^{n-1}}\sin(n-1)\phi.$$

Ex his erit: $\mathfrak{Q}^2 + \mathfrak{q}^2 = \frac{n^2b^2f^{2(n-1)}}{g^{2(n-1)}}$;

$$\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q} = \frac{nbf^{m+n-1}}{g^{m+n-1}}\sin(n-m-1)\phi;$$

atque $\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q} = \frac{nbf^{m+n-1}}{g^{m+n-1}}\cos(n-m-1)\phi$.

Quamobrem erit fractio ſimplex quaeſita:

$$\frac{2g^{n-m}[f\sin\phi.\sin(n-m-1)\phi+gx\cos(n-m-1)\phi-f\cos\phi.\cos(n-m-1)\phi]}{nbf^{n-m-1}(ff-2fgx\cos\phi+ggxx)}$$

ſeu

$$\frac{2g^{n-m}[gx\cos(n-m-1)\phi - f\cos(n-m)\phi]}{nbf^{n-m-1}(ff-2fgx\cos\phi+ggxx)}.$$

EXEMPLUM II.

Sit propoſita haec fractio $\frac{1}{x^m(a+bx^n)}$, *cuius denominator factorem habeat* ff$-$2fgx*coſ*ϕ+ggxx, *inuenire fractionem ſimplicem inde oriundam.*

Cum ſit $P = 1$, & $Q = ax^m + bx^{m+n}$, erit $\frac{dQ}{dx} = max^{m-1} + (m+n)bx^{m+n-1}$, ideoque poſito $x^n = \frac{f^n}{g^n}\operatorname{coſ} n\phi$ ob $P = x^0$ & $\mathfrak{P} = 1$.

$$\mathfrak{Q} = \frac{maf^{m-1}}{g^{m-1}}\operatorname{coſ}(m-1)\phi + \frac{(m+n)bf^{m+n-1}}{g^{m+n-1}}\operatorname{coſ}(m+n-1)\phi$$

& $\mathfrak{p} = 0$; atque

$$\mathfrak{q} = \frac{maf^{m-1}}{g^{m-1}}\operatorname{ſin}(m-1)\phi + \frac{(m+n)bf^{m+n-1}}{g^{m+n-1}}\operatorname{ſin}(m+n-1)\phi$$

Ergo

$$\mathfrak{Q}^2 + \mathfrak{q}^2 = \frac{m^2a^2f^{2(m-1)}}{g^{2(m-1)}} + \frac{2m(m+n)abf^{2m+n-2}}{g^{2m+n-2}}\operatorname{coſ} n\phi + \frac{(m+n)^2b^2f^{2(m+n-1)}}{g^{2(m+n-1)}}.$$

Quodſi vero eſt $ff - 2fgx\operatorname{coſ}\phi + ggxx$ diuiſor ipſius $a+bx^n$, erit $a + \frac{bf^n}{g^n}\operatorname{coſ} n\phi = 0$ & $\frac{bf^n}{g^n}\operatorname{ſin} n\phi = 0$,

vnde $aa = \frac{bbf^{2n}}{g^{2n}}$.

Erit ergo:

$$\mathfrak{Q}^2 + \mathfrak{q}^2 = \frac{(m+n)^2 bbf^{2(m+n-1)}}{g^{2(m+n-1)}} - \frac{m(2n+m)aaf^{2(m-1)}}{g^{2(m-1)}}$$

$$= \frac{nnaaf^{2(m-1)}}{g^{2(m-1)}} = \frac{nnbbf^{2(m+n-1)}}{g^{2(m+n-1)}}.$$

Deinde vero erit:

$$\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q} = \frac{maf^{m-1}}{g^{m-1}} \operatorname{ſin}(m-1)\phi + \frac{(m+n)bf^{m+n-1}}{g^{m+n-1}} \operatorname{cſ}(m+n-1)\phi$$

$$= \frac{bf^{m+n-1}}{g^{m+n-1}} [(m+n)\operatorname{ſin}(m+n-1)\phi - m\operatorname{coſ}n\phi . \operatorname{ſin}(m-1)\phi]$$

$$= \frac{bf^{m+n-1}}{g^{m+n-1}} [n\operatorname{coſ}n\phi \operatorname{ſin}(m-1)\phi + (m+n)\operatorname{ſin}n\phi \operatorname{coſ}(m-1)\phi]$$

$$\& \quad \mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q} =$$

$$= \frac{bf^{m+n-1}}{g^{m+n-1}} [(m+n)\operatorname{coſ}(m+n-1)\phi - m\operatorname{coſ}n\phi . \operatorname{coſ}(m-1)\phi]$$

Vel cum $ff - 2fg\operatorname{coſ}\phi + ggxx$ ſit quoque diuiſor ipſius $ax^{m-1} + bx^{m+n-1}$, erit:

$$\frac{af^{m-1}}{g^{m-1}} \operatorname{coſ}(m-1)\phi + \frac{bf^{m+n-1}}{g^{m+n-1}} \operatorname{coſ}(m+n-1)\phi = 0$$

$$\& \quad \frac{af^{m-1}}{g^{m-1}} \operatorname{ſin}(m-1)\phi + \frac{bf^{m+n-1}}{g^{m+n-1}} \operatorname{ſin}(m+n-1)\phi = 0,$$

vnde erit:

$$\mathfrak{Q} = \frac{nbf^{m+n-1}}{g^{m+n-1}} \operatorname{cſ}(m+n-1)\phi \quad \& \quad \mathfrak{q} = \frac{nbf^{m+n-1}}{g^{m+n-1}} \operatorname{ſin}(m+n-1)\phi$$

ſeu

$$\mathfrak{Q} = \frac{-naf^{m-1}}{g^{m-1}} \operatorname{coſ}(m-1)\phi \quad \& \quad \mathfrak{q} = \frac{-naf^{m-1}}{g^{m-1}} \operatorname{ſin}(m-1)\phi.$$

Ex quibus reſultabit fractio quaeſita:

$$+\frac{2g^m[f\,\text{coſ}\,m\phi - gx\,\text{coſ}\,(m-1)\phi]}{naf^{m-1}(ff - 2fgx\,\text{coſ}\,\phi + ggxx)}.$$

Quae formula ex priori exemplo ſequitur, ſi ponatur m negatiuum, vnde non opus fuiſſet hunc caſum peculiarem conſtituiſſe.

EXEMPLUM III.

Si huius fractionis $\frac{x^m}{a+bx^n+cx^{2n}}$ *denominator habuerit factorem* $ff - 2fgx\,\text{coſ}\,\phi + ggxx$, *fractionem ſimplicem inueſtigare ex hoc factore oriundam.*

Si $ff - 2fgx\,\text{coſ}\,\phi + ggxx$ eſt factor denominatoris $a + bx^n + cx^{2n}$, erit vt ſupra oſtendimus:

$$a + \frac{bf^n}{g^n}\,\text{coſ}\,n\phi + \frac{cf^{2n}}{g^{2n}}\,\text{coſ}\,2n\phi = 0$$

$$\& \quad \frac{bf^n}{g^n}\,\text{ſin}\,n\phi + \frac{cf^{2n}}{g^{2n}}\,\text{ſin}\,2n\phi = 0.$$

Cum igitur ſit $P = x^m$ & $Q = a + bx^n + cx^n$, erit $\frac{dQ}{dx} = nbx^{n-1} + 2ncx^{2n-1}$; vnde efficitur:

$$\mathfrak{P} = \frac{f^m}{g^m}\,\text{coſ}\,m\phi \quad \& \quad \mathfrak{p} = \frac{f^m}{g^m}\,\text{ſin}\,m\phi:$$

$$\mathfrak{Q} = \frac{nbf^{n-1}}{g^{n-1}}\,\text{coſ}\,(n-1)\phi + \frac{2ncf^{2n-1}}{g^{2n-1}}\,\text{coſ}\,(2n-1)\phi$$

$$\mathfrak{q} = \frac{nbf^{n-1}}{g^{n-1}}\,\text{ſin}\,(n-1)\phi + \frac{2ncf^{2n-1}}{g^{2n-1}}\,\text{ſin}\,(2n-1)\phi.$$

Quam-

Quamobrem habebimus:

$$\mathfrak{Q}^2 + \mathfrak{q}^2 = \frac{n^2 f^{2(n-1)}}{g^{2(n-1)}}\left(bb + \frac{4bcf^n}{g^n}\operatorname{coſ} n\phi + \frac{4ccf^{2n}}{g^{2n}}\right).$$

At ex duabus prioribus aequationibus eſt:

$$\frac{f^{2n}}{g^{2n}}\left(bb + \frac{2bcf^n}{g^n}\operatorname{coſ} n\phi + \frac{ccf^{2n}}{g^{2n}}\right) = aa;$$

ideoque

$$\frac{4bcf^n}{g^n}\operatorname{coſ} n\phi = \frac{2g^{2n}aa}{f^{2n}} - 2bb - \frac{2ccf^{2n}}{g^{2n}}$$

quo valore ibi ſubſtituto erit:

$$\mathfrak{Q}^2 + \mathfrak{q}^2 = \frac{n^2 f^{2n-2}}{g^{2n-2}}\left(\frac{2aag^{2n}}{f^{2n}} - bb + \frac{2ccf^{2n}}{g^{2n}}\right)$$

ſeu

$$\mathfrak{Q}^2 + \mathfrak{q}^2 = \frac{n^2(2aag^{4n} - bbf^{2n}g^{2n} + 2ccf^{4n})}{ff\,g^{4n-2}}.$$

Deinde erit: $\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q} =$

$$\frac{nbf^{m+n-1}}{g^{m+n-1}}\operatorname{ſin}(n-m-1)\phi + \frac{2ncf^{m+2n-1}}{g^{m+2n-1}}\operatorname{ſin}(2n-m-1)\phi$$

$$\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q} =$$

$$\frac{nbf^{m+n-1}}{g^{m+n-1}}\operatorname{coſ}(n-m-1)\phi + \frac{2ncf^{m+2n-1}}{g^{m+2n-1}}\operatorname{ſin}(2n-m-1)\phi.$$

Quibus valoribus inuentis erit fractio ſimplex quaeſita:

$$\frac{2fg(\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q})\operatorname{ſin}\phi + 2g(\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q})(gx - f\operatorname{coſ}\phi)}{(\mathfrak{Q}^2 + \mathfrak{q}^2)(ff - 2fgx\operatorname{coſ}\phi + ggxx)}.$$

417.

417. Hae autem fractiones facilius exprimentur, ſi ipſos denominatorum factores determinemus. Sit igitur denominator fractionis propoſitae:

$$a + bx^n$$

cuius factor trinomialis ſi ponatur:

$$ff - 2fgx \operatorname{coſ}\phi + ggxx$$

erit vti in Introductione oſtendimus:

$$a + \frac{bf^n}{g^n}\operatorname{coſ} n\phi = 0 \quad \& \quad \frac{bf^n}{g^n}\operatorname{ſin} n\phi = 0,$$

cum igitur ſit $\operatorname{ſin} n\phi = 0$, erit vel $n\phi = (2k-1)\pi$, vel $n\phi = 2k\pi$, priori caſu erit $\operatorname{coſ} n\phi = -1$, poſteriori $\operatorname{coſ} n\phi = +1$. Si ergo a & b ſint quantitates affirmatiuae, prior caſus ſolus locum habebit, quo ſit $a = \frac{bf^n}{g^n}$; ac propterea:

$$f = a^{\frac{1}{n}} \quad \& \quad g = b^{\frac{1}{n}}$$

retineamus autem loco harum quantitatum irrationalium litteras f & g, ſeu ponamus potius $a = f^n$ & $b = g^n$, ita vt factores inueſtigari debeant huius functionis:

$$f^n + g^n x^n$$

Cum igitur ſit $\phi = \frac{(2k-1)\pi}{n}$, vbi k numerum quemcunque affirmatiuum integrum deſignare poteſt; at vero maiores numeri pro k non ſunt ſumendi, quam qui reddant $\frac{2k-1}{n}$ vnitate minorem; hinc fractionis propoſitae $f^n + g^n x^n$ factores erunt ſequentes:

ff—

$$ff - 2fgx\,\text{coſ}\,\frac{\pi}{n} + ggxx$$

$$ff - 2fgx\,\text{coſ}\,\frac{3\pi}{n} + ggxx$$

$$ff - 2fgx\,\text{coſ}\,\frac{5\pi}{n} + ggxx$$

&c.

vbi notandum eſt ſi n ſit numerus impar, vnum factorem haberi binomium hunc:

$$f + gx$$

ſin autem n ſit numerus par, nullus factor aderit binomius.

EXEMPLUM I.

Reſoluere hanc fractionem $\frac{x^m}{f^n + g^n x^n}$ *in ſuas fractiones ſimplices.*

Cum denominatoris vnusquisque factor trinomialis contineatur in hac forma:

$$ff - 2fgx\,\text{coſ}\,\frac{(2k-1)\pi}{n} + ggxx$$

erit in §. praecedente Exempl. 1. $a = f^n$, $b = g^n$, & $\varphi = \frac{(2k-1)\pi}{n}$, vnde erit:

$$\text{ſin}\,(n-m-1)\varphi = \text{ſin}\,(m+1)\varphi = \text{ſin}\,\frac{(m+1)(2k-1)\pi}{n} \quad \&$$

$$\text{coſ}(n-m-1)\varphi = -\text{coſ}(m+1)\varphi = -\text{coſ}\,\frac{(m+1)(2k-1)\pi}{n}.$$

 Hinc

Hinc ex iſto factore oritur fractio ſimplex haec:

$$\frac{2f\,\text{ſ}\text{n}\frac{(2k-1)\pi}{n}\,\text{ſ}\text{n}\frac{(m+1)(2k-1)\pi}{n} - 2\,\text{cſ}\frac{(m+1)(2k-1)\pi}{n}\left(gx - f\,\text{cſ}\frac{(2k-1)}{n}\pi\right)}{nf^{n-m-1}g^{m}\left(ff - 2fgx\,\text{coſ}\frac{(2k-1)}{n}\pi + ggxx\right)}.$$

Quamobrem fractio propoſita reſoluetur in has ſimplices:

$$\frac{2f\,\text{ſin}\frac{\pi}{n}.\,\text{ſin}\frac{(m+1)\pi}{n} - 2\,\text{coſ}\frac{(m+1)\pi}{n}\left(gx - f\,\text{coſ}\frac{\pi}{n}\right)}{nf^{n-m-1}g^{m}\left(ff - 2fgx\,\text{coſ}\frac{\pi}{n} - ggxx\right)}$$

$$\frac{2f\,\text{ſin}\frac{3\pi}{n}.\,\text{ſin}\frac{3(m+1)\pi}{n} - 2\,\text{coſ}\frac{3(m+1)\pi}{n}\left(gx - f\,\text{coſ}\frac{3\pi}{n}\right)}{nf^{n-m-1}g^{m}\left(ff - 2fgx\,\text{coſ}\frac{3\pi}{n} + ggxx\right)}$$

$$\frac{2f\,\text{ſin}\frac{5\pi}{n}.\,\text{ſin}\frac{5(m+1)\pi}{n} - 2\,\text{coſ}\frac{5(m+1)\pi}{n}\left(gx - f\,\text{coſ}\frac{5\pi}{n}\right)}{nf^{n-m-1}g^{m}\left(ff - 2fgx\,\text{coſ}\frac{5\pi}{n} + ggxx\right)}$$

&c.

Si ergo n fuerit numerus par, hoc modo omnes oriuntur fractiones ſimplices; ſin autem m ſit numerus impar, ob factorem binomium $f + gx$, ad fractiones hoc modo reſultantes inſuper addi debet haec:

$$\frac{\pm 1}{ng^{n-m-1}g^{m}(f+gx)}$$

vbi ſignum $+$ valet, ſi m fuerit numerus par, contra ſignum $-$. Si m fuerit numerus maior quam n, tum ad

has fractiones accedent inſuper partes integrae huiusmodi

$$Ax^{m-n} + Bx^{m-2n} + Cx^{m-3n} + Dx^{m-4n} + \&c.$$

quamdiu exponentes manent affirmatiui, eritque:

$$Ag^n = 1 \;;\quad \text{Ergo} \quad A = \frac{1}{g^n}$$

$$Af^n + Bg^n = 0 \quad . \; . \quad B = -\frac{f^n}{g^{2n}}$$

$$Bf^n + Cg^n = 0 \quad . \; . \quad C = +\frac{f^{2n}}{g^{3n}}$$

$$Cf^n + Dg^n = 0 \quad . \; . \quad D = -\frac{f^{3n}}{g^{4n}}$$

$$\&c. \qquad \&c.$$

EXEMPLUM II.

Reſoluere hanc fractionem $\frac{1}{x^m(f^n + g^n x^n)}$ *in ſuas fractiones ſimplices.*

Quod ad factores ipſius $f^n + g^n x^n$ attinet, ex iis oriuntur eaedem fractiones, quas exemplo praecedente eruimus, dummodo ibi ſumatur m negatiue: ſuper eſt igitur tantum, vt fractiones ſimplices ex denominatoris altero factore x^m definiamus, quod hoc modo commodiſſime fit: ſtatuatur fractio propoſita $= \frac{\mathfrak{A}}{x^m} + \frac{\mathfrak{N} x^{n-m}}{f^n + g^n x^n}$, eritque

$$\mathfrak{A} f^n = 1 \;;\quad \text{Ergo} \quad \mathfrak{A} = \frac{1}{f^n}$$

$$\mathfrak{A} g^n + \mathfrak{N} = 0 \quad . \; . \quad \mathfrak{N} = -\frac{g^n}{f^n}.$$

 Si

Si $n-m$ adhuc fuerit numerus negatiuus, ſimili modo erit operandum, ita vt, ſi m fuerit numerus quantumuis magnus, reſultent huiusmodi fractiones ſimplices

$$\frac{\mathfrak{A}}{x^m} + \frac{\mathfrak{B}}{x^{m-n}} + \frac{\mathfrak{C}}{x^{m-2n}} + \frac{\mathfrak{D}}{x^{m-3n}} + \&c.$$

cuius ſeriei tot termini ſunt ſumendi, quot habentur ipſius x exponentes affirmatiui in denominatore. Eritque

$$\mathfrak{A}f^n = 1 \; ; \quad \text{Ergo} \quad \mathfrak{A} = \frac{1}{f^n}$$

$$\mathfrak{A}g^n + \mathfrak{B}f^n = 0 \;\; . \;\; . \quad \mathfrak{B} = -\frac{g^n}{f^{2n}}$$

$$\mathfrak{B}g^n + \mathfrak{C}f^n = 0 \;\; . \;\; . \quad \mathfrak{C} = +\frac{g^{2n}}{f^{3n}}$$

$$\mathfrak{C}g^n + \mathfrak{D}f^n = 0 \;\; . \;\; . \quad \mathfrak{D} = -\frac{g^{3n}}{f^{4n}}$$

$$\&c. \qquad\qquad \&c.$$

Fractio ergo propoſita omnino in has fractiones ſimplices reſoluetur:

$$\frac{1}{f^n x^m} - \frac{g^n}{f^{2n}x^{m-n}} + \frac{g^{2n}}{f^{3n}x^{m-2n}} + \frac{g^{3n}}{f^{4n}x^{m-3n}} + \&c.$$

$$\frac{-2fg^m \operatorname{ſin}\frac{\pi}{n} . \operatorname{ſin}\frac{(m-1)\pi}{n} - 2g^m \operatorname{coſ}\frac{(m-1)\pi}{n}\left(gx - f\operatorname{coſ}\frac{\pi}{n}\right)}{nf^{n+m-1}\left(ff - 2fgx\operatorname{coſ}\frac{\pi}{n} + ggxx\right)}$$

$$\frac{-2fg^m \operatorname{ſin}\frac{3\pi}{n} . \operatorname{ſin}\frac{3(m-1)\pi}{n} - 2g^m \operatorname{coſ}\frac{3(m-1)\pi}{n}\left(gx - f\operatorname{coſ}\frac{3\pi}{n}\right)}{nf^{n+m-1}\left(ff - 2fgx\operatorname{coſ}\frac{3\pi}{n} + ggxx\right)}$$

$$\frac{-2fg^m \text{ſin}\frac{5\pi}{n}.\text{ſin}\frac{5(m-1)\pi}{n} - 2g^m \text{coſ}\frac{5(m-1)\pi}{n}\left(gx - f\text{coſ}\frac{5\pi}{n}\right)}{nf^{n+m-1}\left(ff - 2fgx\,\text{coſ}\frac{5\pi}{n} + ggxx\right)}$$

&c.

Quibus formulis ſi n fuerit numerus impar, ob $f+gx$ factorem denominatoris, inſuper adiici debet:

$$\frac{\pm\; g^m}{nf^{n+m-1}(f+gx)}$$

vbi ſignorum ambiguorum $\pm$ ſuperius valet, ſi m fuerit numerus par, inferius vero ſi m impar.

418. Conſideremus nunc quoque formulam $a+bx^n$, ſi b ſit numerus negatiuus, ſitque propoſita haec functio:

$$f^n - g^n x^n$$

cuius primo ſemper erit factor $f-gx$; atque ſi n ſit numerus par, quoque $f+gx$ eius erit factor. Reliqui vero erunt trinomiales, quorum forma generalis ſi ponatur

$$ff - 2fgx\,\text{coſ}\,\phi + ggxx$$

erit $f^n - f^n\text{coſ}n\phi = 0$ & $f^n\text{ſin}n\phi = 0$ ſiue $\text{ſin}n\phi = 0$ & $\text{coſ}n\phi = 1$. Quibus vt ſatisfiat, oportet eſſe $n\phi = 2k\pi$ exiſtente k numero quocunque integro, atque propterea erit $\phi = \frac{2k\pi}{n}$. Factor ergo generalis erit:

$$ff - 2fgx\,\text{coſ}\frac{2k\pi}{n} + ggxx$$

ſumendo ergo pro $2k$ omnes numeros pares exponente n minores, prodibunt factores trinomiales omnes:

$$ff - 2fgx \operatorname{coſ} \frac{2\pi}{n} + ggxx$$

$$ff - 2fgx \operatorname{coſ} \frac{4\pi}{n} + ggxx$$

$$ff - 2fgx \operatorname{coſ} \frac{6\pi}{n} + ggxx$$

&c.

EXEMPLUM I.

Reſoluere hanc fractionem $\frac{x^m}{f^n - g^n x^n}$ *in ſuas fractiones ſimplices.*

Quoniam denominatoris factor eſt $f-gx$, inde orietur fractio huiusmodi $\frac{\mathfrak{A}}{f-gx}$, ad cuius numeratorem inueniendum, ponatur $x^m = P$ & $f^n - g^n x^n = Q$, erit $dQ = -ng^n x^{n-1}$, fietque $\mathfrak{A} = \frac{-gx^m}{-ng^n x^{n-1}} = \frac{x^m}{ng^{n-1}x^{n-1}}$, poſito $x = \frac{f}{g}$. Ergo erit $\mathfrak{A} = \frac{1}{nf^{n-m-1}g^m}$, hincque fractio ſimplex ex factore $f-gx$ orta erit:

$$\frac{1}{nf^{n-m-1}g^m(f-gx)}$$

Si n ſit numerus par, quia tum denominatoris factor quoque eſt $f+gx$, ponatur fractio ſimplex inde orivnda $= \frac{\mathfrak{A}}{f+gx}$, erit $\mathfrak{A} = \frac{-gx^m}{ng^n x^{n-1}} = \frac{-x^m}{ng^{n-1}x^{n-1}}$, poſito

ſito $x = \frac{-f}{g}$. Fiet ergo ob $n-1$ numerum imparem $g^{n-1}x^{n-1} = -f^{n-1}$: at erit $x^m = \frac{\pm f^m}{g^m}$, vbi ſignum ſuperius valet, ſi m fuerit numerus par, inferius ſi m ſit numerus impar. Quare cum ſit $\mathfrak{A} = \frac{\mp 1}{nf^{n-m-1}g^m}$, erit fractio ſimplex ex factore $f+gx$ oriunda haec:

$$\frac{\mp 1}{nf^{n-m-1}g^m(f+gx)}.$$

Deinde cum factorum trinomialium forma generalis ſit:

$$ff - 2fgx\,\mathrm{coſ}\,\frac{2k\pi}{n} + ggxx,$$

ſi comparatio cum Exemplo 1. §. 416. inſtituatur, erit $a = f^n$, $b = -g^n$ & $\phi = \frac{2k\pi}{n}$; vnde $\mathrm{ſin}\,\phi = 0$ & $\mathrm{coſ}\,n\phi = 1$; atque $\mathrm{ſin}(n-m-1)\phi = -\mathrm{ſin}(m+1)\phi = -\mathrm{ſin}\,\frac{2k(m+1)\pi}{n}$, & $\mathrm{coſ}(n-m-1)\phi = \mathrm{coſ}(m+1)\phi = \mathrm{coſ}\frac{2k(m+1)\pi}{n}$.

Ex quibus erit fractio ſimplex hinc oriunda:

$$\frac{2f\,\mathrm{ſin}\,\frac{2k\pi}{n}.\mathrm{ſin}\,\frac{2k(m+1)\pi}{n} - 2\,\mathrm{coſ}\frac{2k(m+1)\pi}{n}\left(gx - f\,\mathrm{coſ}\frac{2k\pi}{n}\right)}{nf^{n-m-1}g^m\left(ff - 2fgx\,\mathrm{coſ}\frac{2k\pi}{n} + ggxx\right)}$$

Hancobrem fractiones ſimplices quaeſitae erunt:

$$\frac{1}{nf^{n-m-1}g^m(f-gx)}$$

+

$$+\frac{2f\,\text{ſin}\frac{2\pi}{n}.\text{ſin}\frac{2(m+1)\pi}{n} - 2\,\text{coſ}\frac{2(m+1)\pi}{n}\left(gx - f\,\text{coſ}\frac{2\pi}{n}\right)}{nf^{n-m-1}g^{m}\left(ff - 2fgx\,\text{coſ}\frac{2\pi}{n} + ggxx\right)}$$

$$+\frac{2f\,\text{ſin}\frac{4\pi}{n}.\text{ſin}\frac{4(m+1)\pi}{n} - 2\,\text{coſ}\frac{4(m+1)\pi}{n}\left(gx - f\,\text{coſ}\frac{4\pi}{n}\right)}{nf^{n-m-1}g^{m}\left(ff - 2fgx\,\text{coſ}\frac{4\pi}{n} + ggxx\right)}$$

$$+\frac{2f\,\text{ſin}\frac{6\pi}{n}.\text{ſin}\frac{6(m+1)\pi}{n} - 2\,\text{coſ}\frac{6(m+1)\pi}{n}\left(gx - f\,\text{coſ}\frac{6\pi}{n}\right)}{nf^{n-m-1}g^{m}\left(ff - 2fgx\,\text{coſ}\frac{6\pi}{n} + ggxx\right)}\ \&c.$$

quibus ſi n fuerit numerus par, inſuper addi debet haec fractio:

$$\frac{\mp 1}{nf^{n-m-1}\,g^{m}\,(f+gx)}$$

cuius ſignum ſuperius − eſt ſumendum, ſi m fuerit numerus par, inferius ſi impar. Praeterea vero ſi m ſit numerus non minor quam n, adiiciendae ſunt partes integrae:

$$Ax^{m-n} + Bx^{m-2n} + Cx^{m-3n} + Dx^{m-4n} + \&c.$$

quamdiu exponentes non fuerint negatiui, eritque:

$$-Ag^{n} = 1 \quad \text{ſeu} \quad A = -\frac{1}{g^{n}}$$

$$Af^{n} - Bg^{n} = 0 \quad . \quad . \quad B = -\frac{f^{n}}{g^{2n}}$$

$$Bf^{n} - Cg^{n} = 0 \quad . \quad . \quad C = -\frac{f^{2n}}{g^{3n}}$$

$$Cf^{n} - Dg^{n} = 0 \quad . \quad . \quad D = -\frac{f^{3n}}{g^{4n}}$$

&c. &c.

EX-

EXEMPLUM II.

Reſoluere hanc fractionem $\frac{1}{x^m(f^n - g^n x^n)}$ *in ſuas fractiones ſimplices.*

Fractiones quae ex denominatoris factore $f^n - g^n x^n$ oriuntur, eaedem erunt quae ante, dummodo in illis formulis m negatiue accipiatur. Quare ad alterum factorem x^m eſt reſpiciendum, ex quo ſi ponamus has fractiones reſultare:

$$\frac{\mathfrak{A}}{x^m} + \frac{\mathfrak{B}}{x^{m-n}} + \frac{\mathfrak{C}}{x^{m-2n}} + \frac{\mathfrak{D}}{x^{m-3n}} + \&c.$$

quae ſeries eousque eſt continuanda, donec exponentes ipſius x fiant negatiui. Erit vero

$$\mathfrak{A} f^n = 1 \; ; \quad \text{Ergo} \quad \mathfrak{A} = \frac{1}{f^n}$$

$$\mathfrak{B} f^n - \mathfrak{A} g^n = 0 \; . \; . \quad \mathfrak{B} = \frac{g^n}{f^{2n}}$$

$$\mathfrak{C} f^n - \mathfrak{B} g^n = 0 \; . \; . \quad \mathfrak{C} = \frac{g^{2n}}{f^{3n}}$$

$$\mathfrak{D} f^n - \mathfrak{C} g^n = 0 \; . \; . \quad \mathfrak{D} = \frac{g^{3n}}{f^{4n}}$$

&c. &c.

Fractio ergo propoſita reſoluetur in has fractiones ſimplices:

$$\frac{1}{f^n x^m} + \frac{g^n}{f^{2n} x^{m-n}} + \frac{g^{2n}}{f^{2n} x^{m-2n}} + \frac{g^{3n}}{f^{4n} x^{m-3n}} + \&c.$$

$$+ \frac{g^m}{n f^{n+m-1}(f - gx)}$$

$$+\frac{2fg^m \text{ſin}\frac{2\pi}{n}\text{ſin}\frac{2(m-1)\pi}{n} - 2g^m \text{coſ}\frac{2(m-1)\pi}{n}\left(gx - f\text{coſ}\frac{2\pi}{n}\right)}{nf^{n+m-1}\left(ff - 2fgx\,\text{coſ}\frac{2\pi}{n} + ggxx\right)}$$

$$+\frac{2fg^m \text{ſin}\frac{4\pi}{n}\text{ſin}\frac{4(m-1)\pi}{n} - 2g^m \text{coſ}\frac{4(m-1)\pi}{n}\left(gx - f\text{coſ}\frac{4\pi}{n}\right)}{nf^{n+m-1}\left(ff - 2fgx\,\text{coſ}\frac{4x}{n} + ggxx\right)}$$

$$+\frac{2fg^m \text{ſin}\frac{6\pi}{n}\text{ſin}\frac{6(m-1)\pi}{n} - 2g^m \text{coſ}\frac{6(m-1)\pi}{n}\left(gx - f\text{coſ}\frac{6\pi}{n}\right)}{nf^{n+m-1}\left(ff - 2fgx\,\text{coſ}\frac{6\pi}{n} + ggxx\right)}$$

&c.

quibus ſi n fuerit numerus par, inſuper addi debet haec fractio : $\frac{\mp g^m}{nf^{n+m-1}(f+gx)}$

quae autem praetermittitur, ſi n fuerit numerus impar. Signorum ambiguorum vero ſuperius — valet, ſi m ſit numerus par, inferius vero $+$, ſi m ſit numerus impar.

419. Hoc ergo modo omnes fractiones, quarum denominator ex duobus conſtat membris huiusmodi $a+bx^n$, in fractiones ſimplices reſoluuntur. At ſi denominator conſtet tribus huiusmodi membris $a+bx^n+cx^{2n}$, tum primum videndum eſt, vtrum is in duos factores reales prioris formae reſolui poſſit. Hoc enim ſi eueniat, reſolutio in fractiones ſimplices modo ante expoſito inſtitui poterit. Si enim proponatur huiusmodi fractio

$$\frac{x^m}{(f^n+g^nx^n)(f^n+h^nx^n)}$$

ea

ea primum in duas fractiones transformabitur huiusmodi:

$$\frac{\alpha x^m}{f^n + g^n x^n} + \frac{\beta x^m}{f^n + h^n x^n}$$

eritque $\alpha f^n + \beta f^n = 1$ & $\alpha h^n + \beta g^n = 0$, vnde fit

$$\alpha = \frac{1}{f^n} - \beta = -\frac{\beta g^n}{h^n}; \text{ ideoque habebitur}$$

$$\beta = \frac{h^n}{f^n(h^n - g^n)} \ \& \ \alpha = \frac{g^n}{f^n(g^n - h^n)}.$$

Si exponens m fuerit maior quam n, transmutatio in ſequentes fractiones erit commodior:

$$\frac{\alpha x^{m-n}}{f^n + g^n x^n} + \frac{\beta x^{m-n}}{f^n + h^n x^n}$$

qua fit $\alpha + \beta = 0$ & $\alpha h^n + \beta g^n = 1$, ideoque $\alpha = \frac{1}{h^n - g^n}$ & $\beta = \frac{1}{g^n - h^n}$. Vtra autem transformatio adhibeatur, vtraque fractio hoc modo oriunda methodo ante expoſita reſoluetur in ſuas fractiones ſimplices, quae iunctim ſumtae fractioni propoſitae erunt aequales.

420. Simili modo methodus hactenus tradita ſufficiet, ſi denominator ex pluribus membris conſtet huiusmodi $a + bx^n + cx^{2n} + dx^{3n} + ex^{4n} + \&c.$ dummodo is in factores formae $f^n \pm g^n x^n$ reſolui queat. Ponamus enim occurrere hanc fractionem in ſuas fractiones ſimplices reſoluendam:

$$\frac{x^m}{(a - x^n)(b - x^n)(c - x^n)(d - x^n)}.$$

Haec primum reſoluetur in has:

$$\frac{Ax^m}{a-x^n}+\frac{Bx^m}{b-x^n}+\frac{Cx^m}{c-x^n}+\frac{Dx^m}{d-x^n}+\&c.$$

quarum numeratores ſequenti modo determinabuntur, vt ſit

$$A=\frac{1}{(b-a)(c-a)(d-a)}$$

$$B=\frac{1}{(a-b)(c-b)(d-b)}$$

$$C=\frac{1}{(a-c)(b-c)(d-c)}\ \&c.$$

Hac ergo praeparatione facta, ſingulae iſtae fractiones methodo ante expoſita in ſuas fractiones ſimplices reſoluentur; quae cunctae in vnam ſummam erunt colligendae.

421. Quodſi vero huiusmodi denominator $a+bx^n+cx^{2n}+dx^{3n}+\&c.$ non omnes factores formae $f^n+g^nx^n$ habeant reales, bini imaginarii erunt coniungendi. Ponamus ergo huiusmodi binorum factorum productum eſſe:

$$f^{2n}-2f^ng^nx^n\operatorname{coſ}\omega+g^{2n}x^{2n}$$

& cum haec expreſſio nullos habeat factores ſimplices reales, ponamus factores trinomiales in hac forma generali contineri:

$$ff-2fgx\operatorname{coſ}\phi+ggxx$$

quorum numerus erit $=n$. Poſito ergo $x^n=\frac{f^n}{g^n}\operatorname{coſ}n\phi$, orietur haec aequatio:

$$1-2\operatorname{coſ}\omega.\operatorname{coſ}n\phi+\operatorname{coſ}2n\phi=0.$$

De-

Deinde poſito $x^n = \frac{f^n}{g^n}$ ſin $n\phi$; erit quoque:

$$-2\,\mathrm{coſ}\,\omega\,.\,\mathrm{ſin}\,n\phi + \mathrm{ſin}\,2\,n\phi = 0$$

quae diuiſa per ſin $n\phi$ dat coſ $n\phi =$ coſ ω, ſicque ſimul priori aequationi ſatisfit. Erit ergo $n\phi = 2k\pi \pm \omega$ denotante k numerum quemuis integrum, ideoque erit $\phi = \frac{2k\pi \pm \omega}{n}$, & factores omnes continebuntur in hac forma: $ff - 2fgx\,\mathrm{coſ}\frac{2k\pi \pm \omega}{n} + ggxx$

vnde ſequentes habebuntur factores:

$$ff - 2fgx\,\mathrm{coſ}\frac{\omega}{n} + ggxx$$

$$ff - 2fgx\,\mathrm{coſ}\frac{2\pi - \omega}{n} + ggxx$$

$$ff - 2fgx\,\mathrm{coſ}\frac{2\pi + \omega}{n} + ggxx$$

$$ff - 2fgx\,\mathrm{coſ}\frac{4\pi - \omega}{n} + ggxx$$

$$ff - 2fgx\,\mathrm{coſ}\frac{4\pi + \omega}{n} + ggxx \quad \&c.$$

quorum tot ſunt ſumendi, donec eorum numerus fiat $= n$.

422. Si igitur proponatur iſta fractio in ſuas fractiones ſimplices reſoluenda:

$$\frac{x^{m-1}}{f^{2n} - 2f^n g^n x^n\,\mathrm{coſ}\,\omega + g^{2n} x^{2n}}$$

quoniam denominatoris factor trinomialis quicunque continetur in hac forma: $ff - 2fgx\,\mathrm{coſ}\,\phi + ggxx$

 exi-

exiſtente $\phi = \frac{2k\pi \pm \omega}{n}$, conſideretur iſta fractio:

$$\frac{x^m}{f^{2n}x - 2f^ng^nx^{n+1}\,\mathrm{coſ}\,\omega + g^{2n}x^{2n+1}}$$

illi aequalis, ac ponatur numerator $x^m = P$ ac denominator $f^{2n}x - 2f^ng^nx^{n+1}\,\mathrm{coſ}\,\omega + g^{2n}x^{2n+1} = Q$: erit $\frac{dQ}{dx} = f^{2n} - 2(n+1)f^ng^nx^n\,\mathrm{coſ}\,\omega + (2n+1)g^{2n}x^{2n}$.

Hinc ponendo $x^n = \frac{f^n}{g^n}\,\mathrm{coſ}\,n\phi$; erit:

$\mathfrak{P} = \frac{f^m}{g^m}\,\mathrm{coſ}\,m\phi$ ſeu $\mathfrak{P} = \frac{f^m}{g^m}\,\mathrm{coſ}\,\frac{m(2k\pi \pm \omega)}{n}$ &

$\mathfrak{Q} = f^{2n}[1 - 2(n+1)\,\mathrm{coſ}\,\omega\,\mathrm{coſ}\,n\phi + (2n+1)\,\mathrm{coſ}\,2n\phi]$.

Cum autem ſit $\mathrm{coſ}\,n\phi = \mathrm{coſ}\,\omega$, erit $\mathrm{coſ}\,2n\phi = 2\,\mathrm{cſ}\,\omega^2 - 1$; ideoque $\mathfrak{Q} = f^{2n}(-2n + 2n\,\mathrm{coſ}\,\omega^2) = -2nf^{2n}\,\mathrm{ſin}\,\omega^2$.

Deinde poſito $x^n = \frac{f^n}{g^n}\,\mathrm{ſin}\,n\phi$, fiet:

$$\mathfrak{p} = \frac{f^m}{g^m}\,\mathrm{ſin}\,m\phi = \frac{f^m}{g^m}\,\mathrm{ſin}\,\frac{m(2k\pi \pm \omega)}{n} \quad \&$$

$$\mathfrak{q} = -f^{2n}[2(n+1)\,\mathrm{coſ}\,\omega\,\mathrm{ſin}\,n\phi - (2n+1)\,\mathrm{ſin}\,2n\phi]$$

ob $\mathrm{ſin}\,2n\phi = 2\,\mathrm{ſin}\,n\phi\,\mathrm{coſ}\,n\phi = 2\,\mathrm{coſ}\,\omega\,\mathrm{ſin}\,n\phi$; erit $\mathfrak{q} = 2nf^{2n}\,\mathrm{coſ}\,\omega\,\mathrm{ſin}\,n\phi$. Cum autem ſit $n\phi = 2k\pi \pm \omega$, erit $\mathrm{ſin}\,n\phi = \pm\,\mathrm{ſin}\,\omega$ & $\mathfrak{q} = \pm 2nf^{2n}\,\mathrm{ſin}\,\omega.\,\mathrm{coſ}\,\omega$.

His inuentis erit: $\mathfrak{Q}^2 + \mathfrak{q}^2 = 4n^2f^{4n}\,\mathrm{ſin}\,\omega^2$

$$\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q} = \frac{2nf^{m+2n}}{g^m}(\pm\,\mathrm{cſ}\,m\phi.\,\mathrm{ſin}\,\omega.\,\mathrm{cſ}.\,\omega + \mathrm{ſin}\,m\phi.\,\mathrm{ſin}\,\omega)^2$$

ſiue $\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q} = \pm\frac{2nf^{m+2n}}{g^m}\,\mathrm{ſin}\,\omega.\,\mathrm{coſ}\,(m\phi \mp \omega)$ ſeu

$$\mathfrak{P}\mathfrak{q}-\mathfrak{p}\mathfrak{Q}=\pm\frac{2nf^{m+2n}}{g^m}\,\text{ſin}\,\omega.\text{coſ}\frac{2km\pi\pm(m-n)\omega}{n}$$

$$\mathfrak{P}\mathfrak{Q}+\mathfrak{p}\mathfrak{q}=\frac{2nf^{m+2n}}{g^m}(-\text{coſ}\,m\phi\,\text{ſin}\,\omega^2\pm\text{ſin}\,m\phi.\text{ſin}\,\omega\,\text{cſ}\,\omega)$$

$$\mathfrak{P}\mathfrak{Q}+\mathfrak{p}\mathfrak{q}=\pm\frac{2nf^{m+2n}}{g^m}\,\text{ſin}\,\omega.\,\text{ſin}\,(m\phi\mp\omega)\qquad\text{ſeu}$$

$$\mathfrak{P}\mathfrak{Q}+\mathfrak{p}\mathfrak{q}=\pm\frac{2nf^{m+2n}}{g^m}\,\text{ſin}\,\omega.\text{ſin}\frac{2km\pi\pm(m-n)\omega}{n}.$$

Hinc ex denominatoris factore: $ff-2fgx\,\text{coſ}\frac{2k\pi\pm\omega}{n}+ggxx$ naſcitur iſta fractio ſimplex:

$$\frac{\pm f\,\text{ſin}\frac{2k\pi\pm\omega}{n}\,\text{coſ}\frac{2km\pi\pm(m-n)\omega}{n}\pm\text{ſin}\frac{2km\pi\pm(m-n)\omega}{n}\left(gx-f\,\text{cſ}\frac{2k\pi\pm\omega}{n}\right)}{nf^{2n-m}g^{m-1}\,\text{ſin}\,\omega\left(ff-2fgx\,\text{coſ}\frac{2k\pi\pm\omega}{n}+ggxx\right)}$$

ſeu

$$\frac{\pm gx\,\text{ſin}\frac{2km\pi\pm(m-n)\omega}{n}\pm f\,\text{ſin}\frac{2k(m-1)\pi\pm(m-n-1)\omega}{n}}{nf^{2n-m}g^{m-1}\,\text{ſin}\,\omega\left(ff-2fgx\,\text{coſ}\frac{2k\pi\pm\omega}{n}+ggxx\right)}.$$

EXEMPLUM.

Reſoluere hanc fractionem $\frac{x^{m-1}}{f^{2n}-2f^ng^nx^n\,\text{coſ}\,\omega+g^{2n}x^{2n}}$ *in ſuas fractiones ſimplices.*

Iſtae fractiones ſimplices quaeſitae ergo erunt:

$$\frac{+f\,\text{ſin}\frac{\omega}{n}\,\text{coſ}\frac{(m-n)\omega}{n}+\text{ſin}\frac{(m-n)\omega}{n}\left(gx-f\,\text{coſ}\frac{\omega}{n}\right)}{nf^{2n-m}g^{m-1}\,\text{ſin}\,\omega\left(ff-2fgx\,\text{coſ}\frac{\omega}{n}+ggxx\right)}$$

$-f$

$$-\frac{f\sin\frac{2\pi-\omega}{n}\cos\frac{2m\pi-(m-n)\omega}{n}-\sin\frac{2m\pi-(m-n)\omega}{n}\left(gx-f\cos\frac{2\pi-\omega}{n}\right)}{nf^{2n-m}g^{m-1}\sin\omega\left(ff-2fgx\cos\frac{2\pi-\omega}{n}+ggxx\right)}$$

$$+\frac{f\sin\frac{2\pi+\omega}{n}\cos\frac{2m\pi+(m-n)\omega}{n}+\sin\frac{2m\pi+(m-n)\omega}{n}\left(gx-f\cos\frac{2\pi+\omega}{n}\right)}{nf^{2n-m}g^{m-1}\sin\omega\left(ff-2fgx\cos\frac{2\pi+\omega}{n}+ggxx\right)}$$

$$-\frac{f\sin\frac{4\pi-\omega}{n}\cos\frac{4m\pi-(m-n)\omega}{n}-\sin\frac{4m\pi-(m-n)\omega}{n}\left(gx-f\cos\frac{4x-\omega}{n}\right)}{nf^{2n-m}g^{m-1}\sin\omega\left(ff-2fgx\cos\frac{4\pi-\omega}{n}+ggxx\right)}$$

$$+\frac{f\sin\frac{4\pi+\omega}{n}\cos\frac{4m\pi+(m-n)\omega}{n}+\sin\frac{4m\pi+(m-n)\omega}{n}\left(gx-f\cos\frac{4\pi+\omega}{n}\right)}{nf^{2n-m}g^{m-1}\sin\omega\left(ff-2fgx\cos\frac{4\pi+\omega}{n}+ggxx\right)}$$

&c.

ſicque eousque erit progrediendum, quoad harum fractionum numerus fuerit n. Si m fuerit numerus vel maior quam $2n-1$ vel numerus negatiuus, priori caſu partes integrae, poſteriori vero fractiones inſuper ſunt adiiciendae, quae modo ante expoſito facile inueniuntur.

BEROLINI

EX OFFICINA MICHAELIS.

c.j.